PRODUCT COMPLEMENTARITY IN PRODUCTION: THE BY-PRODUCT CASE*

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The product-product relationship has been a traditional subject of most production economics and farm management courses for the past two decades. Although the traditional examples of product-product optimization have come primarily from the agricultural production sector (e.g., legume-corn rotations and crop-livestock combinations), the concept is useful in analyzing the organization of any multi-product firm–including those firms which produce externalities in the form of environmental degradation.

Three concepts or ideas usually are offered as giving rise to a positively sloped or complementary range on the product transformation surface–(1) one production process uses as an input a by-product of another production process, (2) one process uses quantities of a factor that are "surplus" to another, or (3) technical interaction (production function shifts) occurs. As Heady notes, the by-product idea is perhaps the most important of these concepts in the agricultural sector [6, p. 222]. However, this concept is (in our view) inadequately treated in most farm management and production economics texts.

The purpose of this paper is to propose a more useful framework for considering by-products than that traditionally provided. Following a brief critique of the traditional treatment of the by-product case, an alternative framework is proposed. The empirical and operational viability of the framework is demonstrated with a numerical example.

A CRITIQUE OF THE TRADITIONAL FORMAT

The usual treatment of by-products as one case of product-product complementarity is deficient as a conceptual basis for the important resource allocation problems posed by by-products. This deficiency, which reduces the pedagogical and empirical viability of the framework, arises for a number of reasons.

First, the traditional rate-of-product-transformation approach and its attendant mono-periodic and constant-outlay assumptions are ill-suited for the by-product case. For complementarity to arise from by-product, phenomena, the usual assumptions regarding fixity of the resource base and simultaneous production periods must be altered.[1] That is, if the production process of Y_1 uses a by-product of Y_2 production, then the resource base, X^0, is not fixed but variable, and production of Y_2 must logically precede Y_1 production in time.[2] Moreover, if the Y_1 process uses a by-product of Y_2 production, then each level of Y_2 defines a unique resource base. Product transformation curves derived under these conditions cannot be considered product transformation curves in the usual sense; the locus of product points defines output combinations attainable from a variable rather than from a fixed resource base.[3]

The classic example of legume-corn rotation violates the assumption of a fixed factor endowment as well as that of simultaneous production periods.

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[1] Throughout this paper, definitions and assumptions of multi-product, mono-periodic, production as specified in Carlson [4] and Pfouts [9] are adopted.

[2] As Bishop and Toussaint state: "Complementarity can occur only over a number of production periods, not within one given production period" [2, p. 119].

[3] This phenomenon has been referred to as "pseudo" complementarity because iso-outlay product transformation curves are not true iso-outlay curves [3, pp. 166-167].

Although the violation of these assumptions, perhaps, does not provide sufficient grounds for rejecting the traditional approach, it does (in our view) add an unnecessary element of pedagogical confusion. The traditional approach is much too simplistic for the by-product case.

More important, however, is that, by treating by-products in this manner, most of the important resource allocation and valuation implications are lost or at least serously disguised–resulting in an incomplete analysis of product-product relationships. Alternatively, a more thorough and empirically useful understanding of by-products can be obtained using concepts of inter-enterprise accounting and shadow pricing [1, pp. 73-76].

AN ALTERNATIVE FRAMEWORK FOR BY-PRODUCTS

The following framework for treatment of by-products is consistent with the usual assumptions of mono-periodic production theory and provides a basis for drawing resource-allocation implications for both positively and negatively priced by-products.

Let the production functions for Y_1 and Y_2 be given by

$$(1)\ Y_1 = Y_1\ (X_{11}, X_{21},...,X_{n1}),$$

$$(2)\ N_1 = N_1\ (Y_1), \text{ and}$$

$$(3)\ Y_2 = Y_2(X_{12}, X_{22},...,X_{m2}N)$$

where $X_{11},...,X_{n1}$ and $X_{12},...,X_{m2}$ represent inputs used in the production of Y_1 and Y_2, respectively, and N_1 is a by-product of Y_1 which affects (is used as an input in) the production of Y_2. Note in equation (3) that the variable N is listed as an input rather than N_1, indicating that Y_1 production need not be considered as the only means of providing N. That is, N in equation (3) is given by the identity

$$(4)\ N = N_1 + N_2$$

where N_2 represents N supplied from non-N_1 sources.[4]

If dN_1/dY_1 and $\partial Y_2/\partial N$ are of like sign, then an increase in the production of Y_1 enhances the production of Y_2. (In the case where both are positive, as in most farm management texts, we can think of Y_1 as a legume, N as nitrogen, and Y_2 as corn.) However, if $\partial N_1/\partial Y_1$ and $\partial Y_2/\partial N$ differ in sign, then an increase in the production of Y_1 impedes the production of Y_2. Consequently, if dN_1/dY_1 and $\partial Y_2/\partial N$ have the same (opposite) sign, then in considering the optimal level of factors in the production of Y_1, an upward (downward) adjustment in the marginal value productivities of the X_{i1}'s must be made to reflect the value (disvalue) of the production of Y_1 in the production of Y_2 (see first-order conditions later this section).

The Profit Equations

Since N_1 is jointly produced with Y_1, the profit equation for Y_1 and N_1 may be represented as

$$(5)\ \pi_1 = P_1 Y_1 + \gamma N_1 - \sum_{i=1}^{n} r_i X_{i1}$$

where P_1 is the price of Y_1, γ is the shadow price of N_1, and r_i denotes the price of the i^{th} factor. The appropriate shadow price for γ is the marginal value productivity of N in Y_2, i.e., $P_2\ \partial Y_2/\partial N$, assuming Y_2 is sold in a perfectly competitive market. In this manner, the value of N_1 in the production of Y_2 is properly credited (debited in the case of a negatively priced by-product) to the Y_1 enterprise.

Similarly, the profit equation for Y_2 may be represented as

$$(6)\quad \pi_2 = P_2 Y_2 - \sum_{i=1}^{m} r_i X_{i2} - \gamma N_1 - \delta N_2$$

where P_2 is the price of Y_2 and δ is the price (marginal factor cost under perfect competition) of N_2–the price of non-N_1 sources of N. In this manner, the cost of N from whatever source is properly debited to the Y_2 enterprise.

Thus the multi-product profit equation $(\pi_1 + \pi_2)$ is given as

$$(7)\quad \pi = P_1 Y_1 + P_2 Y_2 - \sum_{i=1}^{n} r_i X_{i1} - \sum_{i=1}^{m} r_i X_{i2} - \delta N_2$$

as the value of N_1 credited to the Y_1 account is offset by a corresponding debit to the Y_2 account.

The First-Order Conditions

Substituting for N_2 from equation (4) and the production functions, equations (1) and (3), into the

[4] Non-N_1 sources of N in the Legume-corn rotation example could represent commercial sources of nitrogen. In a negatively priced by-product case, e.g., pollution, N_2 would represent N contributed by processes other than from Y_1 production.

joint profit equation, equation (7), we obtain[5]

$$(8)\quad \pi = P_1 Y_1(X_{11},...,X_{n1}) + P_2 Y_2(X_{12},...,X_{m2}, N) - \sum_{i=1}^{n} r_i X_{i1} - \sum_{i=1}^{m} r_i X_{i2} - \delta N + \delta N_1 [Y_1(X_{11},...,X_{n1})].$$

First partial derivatives of equation (8) are

$$(9)\quad \frac{\partial \pi}{\partial X_{i1}} = P_1 \frac{\partial Y_1}{\partial X_{i1}} - r_i + \delta \frac{dN_1}{dY_1} \frac{\partial Y_1}{\partial X_{i1}} \text{ for } i = 1,..., n,$$

$$(10)\quad \frac{\partial \pi}{\partial X_{i2}} = P_2 \frac{\partial Y_2}{\partial X_{i2}} - r_i \qquad \text{for } i = 1,..., m \text{ and}$$

$$(11)\quad \frac{\partial \pi}{\partial N} = P_2 \frac{\partial Y_2}{\partial N} - \delta.$$

Setting equations (9), (10), and (11) equal to zero and solving the system of m + n + 1 equations simultaneously, optimal levels for X_{i1}, X_{i2} and N are determined (assuming satisfaction of second-order conditions). Thus, optimal levels of Y_1 and Y_2 are determined. Furthermore, since Y_1 is determined, N_1 is determined from equation (2), and finally N_2 is determined from equation (4).

Implications and insights provided by the first-order conditions are numerous; we discuss here some of the more obvious but perhaps most important ones. First, as one would expect, in determining the optimal factor levels in Y_1 production (X_{i1}'s), the marginal value productivity of the X_{i1}'s in the production of Y_1 (MVP_{i1}) must be augmented to reflect the value (disvalue when $\delta < 0$) of the by-product N_1. For example, if N represents nitrogen used in the production of corn (Y_2), then the value of nitrogen (N_1) supplied by legume production (Y_1) is reflected in an upward shift of the MVP_{i1}'s by $\delta\ \partial N_1/\partial X_{i1}$, (i.e., by $\delta \frac{dN_1}{dY_1} \frac{\partial Y_1}{\partial X_{i1}}$ in (9)) yielding greater equilibrium levels for the X_{i1}'s. Alternatively, if the marginal productivity of N in the production of Y_2 is negative (e.g., pollution), then the shadow price of N would be negative ($\delta < 0$) and the MVP_{i1}'s would be adjusted downward accordingly, yielding lower equilibrium levels for the X_{i1}'s, ceteris paribus.

Several implications are provided in terms of changing product and factor prices (P_1, P_2, r_i, δ). For example, if δ increases (as has been the case recently for nitrogen), then $\delta\ \partial N_1/\partial X_{i1}$ increases, which implies that X_{i1}'s increase, ceteris paribus. That is, we would expect more nitrogen to be supplied via legume-corn rotations and less from purchased sources. It does not follow that total nitrogen (N) utilized in Y_2 would be reduced, although it could be.

When by-products are appropriately accounted for, the impact of primary product price changes depends on changes in the value of the by-product. If P_1 falls relative to P_2, the resulting decrease in Y_1 would be less than if δ were zero. However, if a decrease in P_1 is accompanied by an increase in δ, it is possible that Y_1, and thus N, would increase and Y_2 would decrease. That is, if the price of the legume drops and the price of nitrogen increases, ceteris paribus, it is possible for legume production to increase and corn production to decrease.

In the preceding discussion, the by-products could be defined as intrafirm, interproduct externalities. The by-product evolving from the production of one product affects the production of another product produced by the same firm. In this case, the costs and benefits of the products are exclusive to the firm, and the problem is merely an accounting one. However, when by-products affect the production process of another firm, the classic issue of appropriately pricing the externality arises.[6] For example, assume that Y_1 is produced by an upstream firm whose by-product N_1 adversely affects the downstream producer of Y_2. There are a number of ways in which this externality can be internalized [8]. If a basin-wide firm is created, then the joint profit equation, (8), under the control of one

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[5] The model, equation (8), and resulting first-order conditions could be more completely specified by adding a temporal dimension. However, for our purposes this simplified form of the model is sufficient. Empirically, optimal input-output levels and sequencing of production activities could be handled in a poly-periodic programming framework.

[6] It should be noted here that the basic production relationships and resulting implications for the externalities case were identified by Castle in his article, "The Market Mechanism, Externalities, and Land Economics" [5]. This paper complements Castle's work by providing a more detailed specification and development of the underlying model.

manager forces the decision maker to consider the detrimental impact of Y_1 production on Y_2; i.e., management will adjust the X_{i1} levels according to equation (9).

If environmental property-rights in the stream are assigned to either the producer of Y_1 or Y_2, then a bargaining solution will yield a market price for pollution rights equal to δ, and again the externality is internalized. Alternatively, if the property rights are held by the government, then a pollution control agency could administer a system of bribes or charges (with the appropriate fee being δ) or set up discharge standards consistent with equilibrium conditions of equations (9), (10), and (11).

A NUMERICAL EXAMPLE

The legume-corn rotation problem is the example most often used of a by-product giving rise to enterprise complementarity. Thus, this problem is especially relevant for demonstrating the empirical and operational viability of the model presented in the preceding section. To maintain the simplicity of the model, we shall assume an alternate year corn-legume rotation.

The following are numerical counterparts to equations (1), (2), and (3):

1. The response function for alfalfa is given by
   (1a) $Y_1 = 2.2837 + .0097X_{11} - .00002X_{11}^2$
   where Y_1 denotes tons of alfalfa and X_{11} denotes pounds of phosphorus applied [7, p. 516].[7]
2. The nitrogen fixation (by-product) relationship is assumed to be
   (2a) $N_1 = 50Y_1$
   where N_1 denotes pounds of nitrogen provided by alfalfa production [10, p. 4].
3. The response function for continuous corn is given by
   (3a) $Y_2 = 52.65 + .714N - .0016N^2$
   where N denotes nitrogen (from any source) [7, p. 478].[8]

From these three equations the multi-product profit equation may be expressed as

(9a) $$\pi = P_1(2.2837 + .0097X_{11} - .00002X_{11}^2) + P_2(52.65 + .714N - .0016N^2) - r_1X_{11} - \delta N + 50\delta(2.2837 + .0097X_{11} - .00002X_{11}^2).$$

Taking first partials of (9a) with respect to X_{11} and N yields

(10a) $$\frac{\partial\pi}{\partial X_{11}} = P_1(.0097 - .00004X_{11}) - r_1 + 50\delta(.0097 - .00004X_{11}) \text{ and}$$

(12a) $$\frac{\partial\pi}{\partial N} = P_2(.714 - .0032N) - \delta.$$

Setting (10a) and (12a) equal to zero, optimal levels of X_{11} and N are

$$X_{11} = -\frac{r_1}{.00004(P_1 + 50\delta)} + 242.5 \text{ and}$$

$$N = -\frac{\delta}{.0032P_2} + 223.1.$$

Since we know the optimal levels of X_{11} and N (P_1, P_2, r_1 and δ are parameters), we can determine the optimal levels of N_1 and N_2 from equations (2a) and (4), respectively,

If we assume an alfalfa price (P_1) of $25 per ton, a corn price (P_2) of $3 per bushel, a phosphorus price (r_1) of 15 cents per pound, and a nitrogen price (δ) of 10 cents per pound, then the optimal quantity of phosphorus applied to alfalfa is 117 lbs. per acre. The optimal level of nitrogen on corn is 213 lbs. per acre, of which 56 lbs. would be in the form of direct fertilizer application and 157 lbs. are from nitrogen fixation. If the price of nitrogen doubles, ceteris paribus, then the optimal application of phosphorus increases to 135 lbs., and the optimal level of nitrogen on corn drops to 202 lbs., of which 161 lbs., would be from nitrogen fixation--an increase of 4 lbs. per acre from this source.

If no consideration were given to the nitrogen by-product of legume production, then (for this example) the profit maximizing level of phosphorus application would be 92 lbs. per acre (assuming the above prices). That is, an application of 25 and 43

[7]The production function (1a) is Heady's equation (14.35) with K set equal to 10. The response functions were reduced to functions of one independent variable for computational and expositional simplicity.

[8]The production function (3a) is Heady's equation (14.2) with P set at 160.

lbs. less than would be applied if the value of the nitrogen were accounted for, assuming 10-cent and 20-cent nitrogen, respectively.

Numerous other implications could be demonstrated, even in this most elementary example, by postulating other changes in the product and factor price parameters–implications (we might add) that have intuitive appeal and pedagogical value. Most of these important resource allocation implications are seriously disguised, if not lost, in the traditional production possiblities treatment.

CONCLUSIONS

Most firms in our economic system are multi-product production units, many of which have by-product dimensions. Yet multi-product production theory and the by-product case in particular, receive limited attention in production economics and farm management texts. Furthermore, the traditional treatment of by-products is not easily understood by our students, and empirical application has been limited. This is unfortunate because, as our social and economic systems become more cognizant of negative-valued products (externalities) and provide incentives for firms to internalize the social costs of these products, new emphasis on the product-product concepts will occur.

The alternative of considering by-products in terms of inter-enterprise accounting seems preferable in terms of its consistency with established enterprise budgeting procedures and because of its explanatory power and general applicability to a host of important resource allocation and valuation problems. The application of inter-enterprise accounting procedures to such multi-product resource allocation problems as crop (feed)- livestock enterprise combinations has been well accepted by economists. Fundamentally, the by-product phenomenon as exemplified by the production and use of nitrogen in the legume-corn rotation is not different, albeit more complex, since the by-product cannot be marketed other than through corn or some other corn.

By using the by-product approach presented here, the benefit (cost) of the by-product is explicitly recognized, and implications of changes in input, as well as product price for resource allocation and acquisition of inputs from alternative sources, are easily identified. Furthermore, this framework offers considerable empirical viability as demonstrated in the numerical example herein.

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