An Alternative Approach to the Existence of Sunspot Equilibria

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Abstract

This paper offers an alternative approach to the existence of sunspot equilibria. The economy has a single perishable good and fiat money within an overlapping generations framework with two possible extraneous events. The analysis uses the dynamic adjustment of market prices during voluntary trade to establish paths of current spot prices whose limit points are rational expectations equilibria. The paper shows that there is a sub-set of paths whose limit points are self-fulfilling if they are perfectly correlated with extraneous events. It is this sub-set that constitutes sunspot equilibria. Two implications of this approach follow: (a) the likelihood of this existence is generally low; and (b) adding an asset (or a commodity) to the economy invalidates the demonstration of existence.

KEYWORDS: Sunspot equilibria, extrinsic uncertainty, dynamic stability.

JEL Classification Numbers: D00, D50, D80.

1   Introduction

The sunspot literature offers no clear understanding of sunspots themselves; it is clear, however, that sunspots render the economic environment uncertain. Put differently, the source of price randomness in the economy is extraneous uncertainty. In these terms, the randomness of prices is a consequence of agents' beliefs with respect to sunspot events. By definition a sunspot equilibrium is a rational expectations equilibrium that is perfectly correlated with extraneous events. Accordingly, sunspot equilibria must have existed whenever agents' subjective beliefs of future prices are subsequently proven correct.

The sunspot literature (for a survey see Chaippori and Guesnerie (1991) and the references thereof; see also Evans and Hankapohja (2001), pp. 287-313) contains diverse and sometimes apparently unrelated strands. This paper focuses on one of these: sunspot equilibria in infinite horizon sequential models. This literature utilizes a two period overlapping generations framework in an

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economy with a representative agent whose inter-temporal budget comprises a perishable good and fiat money (this framework will also form the basis for our discussion). Chaippori and Guesnerie (1991) observe that this strand of the sunspot literature is particularly interesting for two reasons. Within the overlapping generations framework (a) the infinite horizon is better adapted than a finite horizon to study expectations formation; since a finite horizon, with expectations linked to some final period, is problematic; and (b) agents of different generations trade and deliver goods while their life times overlap, therefore it makes no sense to insure away sunspot events.

The key issue here is not the obvious fact that agents form expectations based on subjective beliefs; rather, it is the *self-fulfilment* of those beliefs. In the literature, certain classes of models were designed to establish the existence of sunspot equilibria (e.g., Gottardi and Kajii (1999)). In particular, a special relationship between inter-temporal income effects and substitution effects was imposed to establish the existence of market clearing prices in an environment of extraneous uncertainty.

An alternative approach to the existence of sunspot equilibria, based on dynamic stability considerations, is suggested here. The economic framework resembles that of the abovementioned literature (specifically, Spear (1984)). An overlapping generations model with two periods forms the basis for discussion. Assume that each period (generation) is of a finite length; and that during each period current prices adjust and plans are revised. Demographically, the economy comprises a representative individual (agent) of the young generation and a representative individual of the old generation, or any replica of the two. During the first period of his life the individual plans his inter-temporal consumption, using a von Neumann-Morgenstern expected utility function.

Assume an extrinsic random process in the form of a stationary Markov chain with two possible states, sunspot and non-sunspot. The young individual who plans for the future believes that the market clearing price is perfectly correlated with the state of the sunspot. This paper investigates whether the young individual's beliefs are self-fulfilling; and therefore, whether extraneous stochastic equilibria exist. This paper follows Radner (e.g., 1982) in establishing the existence of a sequence of temporary equilibria where the life times of the two representative individuals overlap; though the source of uncertainty is here extrinsic rather than intrinsic.

Given the Markov chain of future subjective beliefs, the existence of sunspot equilibria is predicated upon two observations:

1. The known (Spear (1984)) observation that future consumption of the young individual is determined solely by the state of future sunspots, irrespective of the state of current sunspots. Clearly, this observation requires restricting the individual' utilities, a special relationship between inter-temporal income and substitution effects should be imposed.

2. There is voluntary trade that is always feasible. The paper shows that when young and old representative individuals trade in the direction of

their excess demands, there is trade that is always consistent with price taking optimizing behaviour. Hence, current spot prices are dynamically stable. A set of time paths of dynamically stable current spot prices will then be formed.

Based on the above two observations and given two extraneous events (sunspot or non-sunspot), sunspot equilibria occur at the limit of some time paths within the above set of dynamically stable paths.[1]

The paper will also show that, in general, the set of sunspot equilibria is insignificant in size. Sunspot equilibria are consequently unlikely to occur and their existence is not, therefore, significant. Moreover, when one adds more assets or goods to the economy, the above existence result, in general, fails.[2]

The paper is structured as follows. Section 2 presents the monetary overlapping generations model that forms the basis of this discussion. Section 3 states the main results of the paper. In the remainder of the paper, two propositions are presented. The first suggests a method to demonstrate the existence of sunspot equilibria in a dynamic competitive equilibrium framework. The proof rests on dynamic stability considerations and requires the validation of a claim. The claim, presented in section 4, has two parts. The first part (4.1) derives the set of admissible current spot prices under voluntary trade between young and old individuals, where the criterion for admissibility is the feasibility of voluntary trade. The second part (4.2) establishes that, within this set, there exist specific time paths of current spot prices that satisfy the sufficiency conditions for dynamic stability.

Section 5 shows that sunspot equilibria exist, but in general, are unlikely to occur. It starts (5.1) with a proof of the abovementioned first proposition; namely that, when individuals' beliefs are extraneous, dynamic stability of current spot prices is sufficient for the existence of sunspot equilibria, provided the future consumption of the young individual is determined solely by the state of future sunspots and is independent of current sunspots.

The second proposition, (5.2), states that, in general, sunspot equilibria are unlikely to occur. It is followed by a corollary (5.3), showing that the inclusion of an extra asset (e.g., a bond) to the economy will generally render the first proposition invalid.

2   The Model

The overlapping generations model is now commonly used as a framework for the study of dynamics in real time. Such a pure exchange framework with

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[1] Note that the dynamic adjustment here is only a means to establish paths of current spot prices whose limit points are equilibrium (market clearing) prices. The equilibrium prices of these paths are effectively rational expectations equilibria; and within them there is a sub-set of paths whose limit points are self-fulfilling if they are perfectly correlated with extraneous events. It is this sub-set that constitutes sunspot equilibria.

[2] There has been an earlier attempt to expand the economy to N commodities and prove the existence of sunspot equilibria under specific conditions (Guesnerie (1986)).

constant population and without bequests is assumed here. There exist two identical individuals, one individual of each generation, each of whom lives for two generations (periods). Thus, in each period there exist two individuals $h = a, b$ (or any replica of them); one old ($h = b$) and one young ($h = a$). The old born in period $t - 1$ and the young born in the present period t (where $t = 1, 2, 3, ...$ are consecutive periods). Here it is assumed that t = 1 is the present period and $t = 2$ is the future period. There is one perishable consumption good. Consumption is denoted by c. The individual is endowed with a fixed quantity ω_1 of the consumption good in period 1 and with ω_2 in period 2. The young individual's inter-temporal utility is $u(c_1, c_2)$, where the subscripts 1 and 2 indicate the present period and the future period, respectively; u is smooth, monotone and concave function. Note that the superscript a is omitted here without obscuring clarity. The current price of the good is p and its expected future price is p^s, where s is the state of a future event. We assume that p and p^s belong to an interval of positive values $[\underline{\varepsilon}, \bar{\varepsilon}]$. A constant stock of fiat money M > 0 in the economy is initially held entirely by the old individual. The young saves by trading some of his current endowment for fiat money presently held by the old.

The economic activity takes place in the current spot markets of consumption and money. The decision problem of the young individual is to maximize his inter-temporal utility given his present and expected future budget constraints,

$$p(c_1 - \omega_1) + m_1^d \leq 0 \text{ and } p^s(c_2 - \omega_2) - m_1^d \leq 0 \tag{1}$$

Thus, at any date t the decision problem of the young is to choose his desired consumption c_1, his demand for money [3] m_1^d and his future consumption c_2. Maximization of the individual's inter-temporal utility subject to (1) yields his present and future commodity excess demands $z_1(\rho) \equiv c_1 - \omega_1, z_2(\rho) \equiv c_2 - \omega_2$ where $\rho \equiv p/p^s$; and his demand for money $m_1^d(p, p^s)$. Market excess demand for consumption is $z \equiv z_1 + z_2$. Clearly, ρ (or equivalently the individual's expected real interest rate $\rho - 1$) is the only parameter in the decision making of the young individual. Accordingly, the individual's demand for money is m_1^d $(p, p^s) \equiv -pz_1(\rho) \equiv p^s z_2(\rho)$ and hence Walras' law $\rho z_1(\rho) + z_2(\rho) \equiv 0$ holds for every $\rho > 0$.

Under extrinsic uncertainty this maximization assumes the following form.

2.1 Extrinsic Uncertainty

The theory of extrinsic uncertainty is based on the behavioural proposition that individuals form subjective beliefs about future economic outcomes (e.g., the future price) in an environment whose structure (i.e., endowments and preferences) is fully certain. Thus, all individuals share a common subjective belief on the origins of uncertainties that influence future economic outcomes.

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[3] Although money is not an argument in the inter-temporal utility function, it is obviously an argument in the indirect utility function.

The sunspot literature assumes a random process of extrinsic events in the economy. Specifically, it assumes that the young individual knows that the random process of extrinsic events follows an exogenous stationary Markov process π with two natural events, sunspot (θ) and non-sunspot (δ). Thus,

$$\pi = \left[\begin{array}{cc} \pi_{\theta\theta} & \pi_{\theta\delta} \\ \pi_{\delta\theta} & \pi_{\delta\delta} \end{array} \right]$$

where $\pi_{s\theta} + \pi_{s\delta} = 1$ for $s = \theta, \delta$. It is further assumed that the young individual believes in stationary and perfect correlations between market clearing price and natural events (both, sunspots or non-sunspots). The young individual therefore associates the sunspot with a price $p^\theta = \phi(\theta)$ (where ϕ is a function that translates the young individual's forecast (belief) into a price), and associates a non-sunspot natural event with a price $p^\delta = \phi(\delta)$.

The notion behind the existence of sunspot equilibrium is the following:

1. When the young individual's beliefs, as expressed by the prices $\phi(\theta)$ and $\phi(\delta)$, are self-fulfilling then these beliefs define a rational expectations equilibrium. If in addition,

2. these beliefs are indeed stochastic (i.e., $\phi(\theta) \neq \phi(\delta)$), then $\phi(\theta)$ and $\phi(\delta)$ constitute a sunspot equilibrium.

Given the Markov process π, the young individual's objective function is to maximize his expected utility $\pi_{s\theta} u \left(c_1^s, c_2^{s\theta} \right) + \pi_{s\delta} u \left(c_1^s, c_2^{s\delta} \right)^4$, $s = \theta, \delta$ subject to his inter-temporal budget constraint.

$$\begin{array}{rcll} m_1^{d(s)} & \leq & p^s \omega_1^s & s = \theta, \delta \\ p^{\overline{s}} c_2^{s\overline{s}} & \leq & m_1^{d(s)} & s, \overline{s} \in \{\theta, \delta\} \end{array} \qquad (2)$$

where $m_1^{d(s)}$ is the young individual's money demand in state s; $c_2^{s\overline{s}}$ is the second period consumption given that the current state is $\overline{s}$ and the previous state was s (the state where the individual was born); and that prices are assumed to depend only on states θ and δ.

Solving the above maximization problem yields the current demand for consumption c_1^s and the expected demand for consumption $c_2^{s\overline{s}}, s, \overline{s} \in \{\theta, \delta\}$.

3 The main results of the paper

The basis for analysis will be the von Neumann-Morgenstern expected utility of the young individual, where uncertainty is merely a basis for constructing

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[4] This von Neumann-Morgenstern expected utility function is suited to addressing the issue of extraneous uncertainty in the sense that sunspot activity has no *direct* effect on the individual's well-being. That is, the individual is interested only in payoffs and their respective probabilities; and the only effect that the usual outcome of sunspot activity has on u is through its effect on the actual value of c. Moreover, the separable form of the expected utility function implies that the consumption plan is inter-temporally consistent (on this issue of consistency see Donaldson and Selden (1981); Cass and Shell (1983)).

lotteries. The underlying utility, u, in the von Neumann-Morgenstern expected utility is independent of sunspot activity. Consequently, for a given today's (current) event, the constrained maximization (2) can be separated into two constrained maximization problems: a deterministic constrained maximization when the future event is sunspot and a deterministic constrained maximization when the future event is non-sunspot. The two outcomes can then be combined by the probabilities within π.

Based on the Markov process π, there are four combinations of prices that should be examined when the existence of sunspot equilibria is concerned. The two obvious cases are (i) today's event is sunspot and tomorrow's event is also sunspot; and (ii) today's event is non-sunspot and tomorrow's event is also non-sunspot. The two other cases are (iii) today's event is sunspot and tomorrow's event is non-sunspot; and (iv) today's event is non-sunspot and tomorrow's event is sunspot. The existence of a sunspot equilibrium depends on whether current spot market equilibrium prices (see definition 1 below) exist in each of the four cases.

To start with, time flow is introduced into each period. The length of the period is $[0,1]$ with $\tau\epsilon[0,1]$ added as a continuous time variable that runs through each time period τ. Thus,

$$\tau=0 \overbrace{}^{t} \tau=1 \quad \tau=0 \underbrace{}_{t+1} \tau=1$$
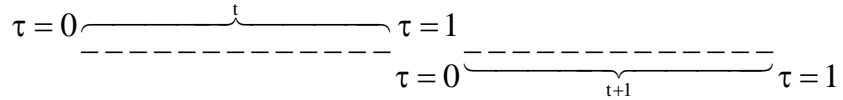
Dynamic stability of current spot prices will be our focal point of analysis. We accordingly call $\left\{p_t^\delta\left[\tau\mid p^\delta(0)\right]:p^s=p^\theta\right\}$ and $\left\{p_t^\theta\left[\tau\mid p^\theta(0)\right]:p^s=p^\delta\right\}$ the time (in τ) paths of current spot prices when today's event is, respectively, non-sunspot and sunspot; and while the future event is, respectively, sunspot and non-sunspot. The initial prices in each path are, respectively, $p^\delta(0)$ and $p^\theta(0)$.

We accordingly define current spot market equilibrium prices (in any time period t).

Definition 1 *Current spot market equilibrium prices* (in any time t) are two current spot prices, p_t^δ and p_t^θ, for which markets clear, i.e., $z_t^{a(s)}+z_t^{b(s)}=0, s=\theta,\delta$ (where $z_t^{a(s)}$ and $z_t^{b(s)}$ are, respectively, the young and the old individuals' current excess demands for consumption). In effect p_t^δ and p_t^θ are rational expectations equilibrium prices. Obviously, current spot prices are in disequilibrium if $z_t^{a(s)}+z_t^{b(s)}\neq0, s=\theta,\delta$.

Definition 2 *Time paths of current spot prices are* **dynamically stable** *in period t if, given two extraneous events, θ and δ, then[5],*

$$\begin{aligned}\left\{p_t^\delta\left[\tau\mid p^\delta(0)\right]:p^s=p^\theta\right\}&\underset{\tau\to1}{\longrightarrow}p_t^\delta\\\left\{p_t^\theta\left[\tau\mid p^\theta(0)\right]:p^s=p^\delta\right\}&\underset{\tau\to1}{\longrightarrow}p_t^\theta\end{aligned} \quad (3)$$

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[5] Strictly speaking, in (3) time τ should approach infinity. The imposition $\tau\to1$ is only an approximation to dynamic stability.

where p_t^δ and p_t^θ are current spot market equilibrium prices in period t.

We assume that each time path of prices in (3) is constructed such that it is continuous and unique in its respective initial price $\left(p^\delta\left(0\right), p^\theta\left(0\right)\right)$, and in its respective expected price $\left(p^\theta, p^\delta\right)$.

Accordingly, we define a sunspot equilbrium.

Definition 3 *A sunspot euqilibrium is current spot market equilibrium prices p^θ and p^δ, while $p^\theta \neq p^\delta$. That is[6],*

$$\begin{aligned}\left\{p_t^\delta\left[\tau \mid p^\delta\left(0\right)\right] : p^s = p^\theta\right\} &\xrightarrow[\tau \to 1]{} p^\delta \\ \left\{p_t^\theta\left[\tau \mid p^\theta\left(0\right)\right] : p^s = p^\delta\right\} &\xrightarrow[\tau \to 1]{} p^\theta\end{aligned} \tag{4}$$

and

$$c_2^{\theta s} = c_2^{\delta s} \qquad s = \theta, \delta \tag{5}$$

Definition 3 requires explanation. The **first** time path of prices in (4) states that if today's event is non-sunspot, then given p^θ as the price of a future sunspot event, there should be a time path of today's (current) prices whose limit point is p^δ (as appears in the **second** time path of prices in (4)). The **second** time path of prices in (4) is constructed similarly. That is, if today's event is sunspot, then given p^δ as the price of a future non-sunspot event, there should be a time path of today's prices whose limit point is p^θ (as appears in the **first** time path of prices in (4)). Equation (5) states that the future consumption depends only on the future natural event (sunspot or non-sunspot) and is independent of the current natural event (as originally been pointed out by Spear (1984)). By construction, (4) and (5) are necessary and sufficient for the existence of sunspot equilibria.[7]

The remainder of the paper presents two propositions and a corollary.

Proposition 4 *When (5) is satisfied, there exist prices $p_t^\theta = p^\theta$ and $p_t^\delta = p^\delta, p^\theta \neq p^\delta$, that maintain (4). That is, sunspot equilibrium exists.*

Proposition 4 states that there exist two expected prices, p^δ and p^θ, such that, the limit of the first (second) time path of prices in (4) is the expected price in the second (first) time path. That is, the limit of each time path of prices, given an extraneous event (sunspot or non-sunspot), constitutes a sunspot equilibrium.

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[6]It is important to distinguish between the pair of prices $\left(p_t^\delta, p_t^\delta\right)$ in definition 2 and the pair of prices $\left(p^\delta, p^\theta\right)$ in definition 3. In definition 2 the limit prices p_t^δ and p_t^θ need not, in general, coincide with the respective prices p^δ and p^θ that appear in the Markov process π. The limit prices in definition 3, however, do coincide with those that appear in π.

[7]The condition of equation (4) states that the current young individual will clear his future excess demands (when he is old) at prices p^δ and p^θ **provided** the future young individual will be willing to provide these excess demands at these prices. The condition of equation (5) states that the future young individual will **indeed** clear his excess demands at the above prices. Hence, p^δ and p^θ are sunspot equilibrium prices.

The proof of proposition 4 is based on dynamic considerations at disequilibrium of current spot prices. Accordingly we assume that at disequilibrium, current spot prices adjust according to market excess demands, thus,

$$\dot{p}_t^s(\tau) = H\left[z_t^s(\tau)\right] \qquad \begin{array}{l} s = \theta, \delta \\ \forall \tau \in [0,1] \end{array} \qquad (6)$$

where $z_t^s(\tau) \equiv z_t^{a(s)}(\tau) + z_t^{b(s)}(\tau)$ is the current spot market excess demand for the good, where H is a sign preserving continuous function and where a dot is the operation $\frac{d}{d\tau}$.

Although it is sufficient to use (6) as a means for analysis of dynamic stability, here, nevertheless, we utilize voluntary trade (as expressed in definition 5 below) along the price adjustment process. The aim of the introduction of voluntary trade at disequilibrium prices is two fold: (a) voluntary trade is a manifestation of the optimizing behaviour of the individuals; as such, voluntary trade between the young and the old individuals is an essential part of the overlapping generations model; and (b) the introduction of voluntary trade during the adjustment of prices (6), explains clearly (as is shown in section 5.3) why dynamic stability, in general, fails when we depart from the stylized framework of the economy.

Some definitions are now in order.

Definition 5 [8] *A trade between the young and the old individuals is called **feasible**, if at disequilibrium prices, $p_t^s(\tau)$, the young individual's trade, $\dot{\omega}_t^{a(s)}(\tau)\,d\tau$, satisfies the following conditions simultaneously:*

*[a] Trade is **voluntary**: that is, the good is exchanged in the direction of its excess demand,*

$$\dot{\omega}_t^{a(s)}(\tau)\,z_t^{a(s)}(\tau) > 0 \qquad \begin{array}{l} s = \theta, \delta \\ \forall \tau \in [0,1] \end{array}$$

for $z_t^{a(s)}(\tau) \neq 0$.

*[b]Trade between the young and the old **price taking** individuals is carried out on the basis of **quid pro quo**: the nominal value of the young individual's receipts should equal the nominal value of payments. That is,*

$$p_t^s(\tau)\,\dot{\omega}_t^{a(s)}(\tau) = \dot{m}_t^{d(s)}(\tau) \qquad \begin{array}{l} s = \theta, \delta \\ \forall \tau \in [0,1] \end{array}$$

Definition 6 *Time paths of current spot prices $\left\{p_t^\delta\left[\tau \mid p^\delta(0)\right] : p^s = p^\theta\right\}$ and $\left\{p_t^\theta\left[\tau \mid p^\theta(0)\right] : p^s = p^\delta\right\}$ are **admissible** if each maintains definition 5 of feasible trade while (6) takes place.*

The proof of proposition 4 requires the validation of the following claim.

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[8] Trade that maintains the conditions of definition 5, while prices change according to (6), is sometimes termed the Hahn process. A sufficient condition for dynamic stability is then that individuals' indirect utilities decline monotonically at disequilibrium prices (e.g., Arrow and Hahn (1971), pp. 337-345).

Claim 7 *There exist admissible time paths of current spot prices*

$$\left\{p_t^\delta\left[\tau \mid p^\delta\left(0\right)\right] : p^s = p^\theta\right\} \ and \ \left\{p_t^\theta\left[\tau \mid p^\theta\left(0\right)\right] : p^s = p^\delta\right\}$$

that satisfy the sufficiency conditions for dynamic stability as in definition 2. (This will be shown in section IV.)

Proposition 4 will then be presented in section 5.1. We conclude section 5.1 with the following proposition 8 and a corollary.

Proposition 8 *In general, sunspot equilibria are unlikely to occur. (This will be presented in section 5.2.)*

Proposition 8 states that although sunspot equilibria exist in this economy, they are unlikely to occur when π is not stationary.

Corollary 9 *Adding an extra asset to the economy renders, in general, proposition 4 invalid. (This will be presented in section 5.3.)*

4   Dynamic stability of admissible time paths of current spot prices

To prove claim 7 this section first derives the set of admissible time paths of current spot prices (as in definition 6). It then shows that the admissible time paths of current spot prices $\left\{p_t^\delta\left[\tau \mid p^\delta\left(0\right)\right] : p^s = p^\theta\right\}$ and $\left\{p_t^\theta\left[\tau \mid p^\theta\left(0\right)\right] : p^s = p^\delta\right\}$ satisfy sufficient conditions for dynamic stability as in definition 2. In proving claim 7 the discussion will be based on the young individual's optimization behaviour. (The behaviour of the old individual is simple, he wants to spend all his money for consumption at the prevailing prices.)

4.1   Admissibility of time paths of current spot prices

Assume that today's event is a non-sunspot (if today's event is sunspot the treatment is similar). The constrained maximization of the young individual's expected utility will involve constrained maximization of two separate problems: (i) constrained maximization of a deterministic utility in a fully certain environment given a sunspot future event with price p^θ; and (ii) constrained maximization of a deterministic utility in a fully certain environment given a non-sunspot future event with price p^δ. In this section the admissibility of time paths of prices in case (i) is considered; the admissibility of time paths of prices in case (ii) is then obvious. Accordingly, throughout the discussion here it is assumed that while a sunspot future price p^θ is considered, the current spot price $p_t^\delta\left(\tau\right)$ adjusts to clear the markets.

The young individual observes a sunspot event (with a price p^θ) and then considers an (initial) current spot price $p_t^\delta\left(0\right)$ that might clear the market. He conducts his deterministic constrained utility maximization, knowing that the future natural event is sunspot. (Henceforth notation will be upset slightly by adding the subscripts c_1, c_2, and m to the letter z to indicate excess demand in

each respective market.) The existence of current spot market equilibrium prices depends on the question: at the initial price $p_t^\delta(0)$, does the quantity $z_t^{b(\delta)}(0)_{c_2}$ (i.e., the initial old individual's excess demand for consumption) coincide with $-z_t^{a(\delta)}(0)_{c_1}$ (i.e., the initial young individual's excess supply of consumption)?

Clearly, at the arbitrary initial price $p_t^\delta(0)$ the coincidence of the above two initial quantities does not hold in general. Therefore the initial price will have to adjust according to (6). At the new price $p_t^\delta(0 + d\tau)$ there will be new quantities $c_2^{b(\delta)}(0 + d\tau)$ and $c_1^{a(\delta)}(0 + d\tau)$.

Figure 1 addresses the issue of the admissibility of the time paths of current spot prices using the inter-temporal budget constraint of the young individual (see also Abraham (2005)).

Given an expected price p^θ, point A corresponds to a target point of the young individual's demands at price $p_t^\delta(0)$; and point D corresponds to his target point of demands at price $p_t^\delta(0 + d\tau)$. By construction, at price $p_t^\delta(0 + d\tau)$, point A is a current spot market disequilibrium position, from which the young individual should move to the new target point D. Clearly, the movement from A to D depends on whether the old individual would be willing to carry out such a trade.

At price $p_t^\delta(0 + d\tau)$, points A and D are each cut by three lines. Each line is a geometric place of **constant** value. At point D, the line $z_t^a(0 + d\tau)_{c_2} = 0$ is individual $a's$ excess demand for future consumption; and by construction the value of this excess demand is zero. At point D, the excess demands for c_1 and m are also zero. At point A, the line $z_t^{a(\delta)}(0)_{c_2} = \hat{c}_2$ is an excess demand for c_2 whose constant value is $\hat{c}_2 < 0$. The constant excess demands for c_1 and m, at point A, are positive ($\hat{c}_1 > 0$) and negative ($\hat{m} < 0$), respectively.

At price $p_t^\delta(0 + d\tau)$, the signs of the young individual's excess demands at points A and D in Figure 1 are set in Table 1.

Table 1			
	c_1	m	c_2
A	$\hat{c}_1 > 0$	$\hat{m} < 0$	$\hat{c}_2 < 0$
D	0	0	0

If the young individual is a **price taking** optimizer, then at the given price $p_t^\delta(0 + d\tau)$ his trade with the old (to move from A towards D) is feasible if it concurrently satisfies conditions [a] and [b] of definition 5. The geometric interpretation of [a] and [b] is the following.

Condition [a] means that trade should be conducted only within the boundaries of the set $ACDF$ in Figure 1. Condition [b] means that trade should be conducted on the surface of the young individual's inter-temporal budget. Thus, trade that maintains conditions [a] and [b] of definition 5 is one that geometrically is confined to the surface of the set $ACDF$. The translation of feasible trade to quantities of trade means that for $\dot{p}_t^s(0)\,d\tau$ there exist trades $\alpha\dot{\omega}_1^{a(\delta)}(0)\,d\tau, 0 < \alpha \leq 1$, that are confined to the surface of the set $ACDF$. Thus, call, $\left\{\alpha\dot{\omega}_1^{a(\delta)}(0)\,d\tau, 0 < \alpha \leq 1 \mid ACDF\right\}$ (read: $\alpha\dot{\omega}^{a(\delta)}(0)\,d\tau, 0 < \alpha \leq 1$,

confined to the surface of the set $ACDF$) a set of **feasible trades**.

Clearly, a sequence of sets, whose geometric structure is similar to the set $ACDF$, ensues along the dynamic process (6), the requirement

$$\left\{ \alpha \dot{\omega}_1^{a(\delta)} (\tau) \, d\tau, 0 < \alpha \leq 1 \mid ACDF \right\}$$

is a sufficient restriction for the admissibility of the path of current spot prices $\left\{ p_t^\delta \left[\tau \mid p^\delta (0) \right] : p^s = p^\theta \right\}$. The admissible path $\left\{ p_t^\theta \left[\tau \mid p^\theta (0) \right] : p^s = p^\delta \right\}$ may be similarly constructed.

4.2 Dynamic stability

The above admissible time path of prices maintain sufficient conditions for dynamic stability of current spot prices as in definition 2. To demonstrate this refer to the table below.

Table 2			
	c_1	m	c_2
old $(h = b)$	$+$	$-$	
young $(h = a)$	0	0	0

Assume, without restricting generality, that Table 2 contains the signs of the old and the young individuals' excess demands after trade has been concluded at price $p_t^\delta (0 + d\tau)$. Note that in the c_1- column the positive sign means that the old individual has positive excess demand for current output; and the zero sign means that the young individual is satisfied with the level of his consumption at the prevailing prices.

To reduce the magnitudes of the excess demands in Table 2, the commodity price $p_t^\delta (\tau)$ should increase according to (6). The difficulty here is that any **finite** increase in $p_t^\delta (\tau)$ will upset the young individual's zero excess demands; consequently the sufficiency conditions for dynamic stability may not be satisfied. To overcome this problem, two conditions must hold along the admissible time path of current spot prices:

1. the change in $p_t^\delta (\tau)$, according to (6), must be *infinitesimal*; and,

2. while $p_t^\delta (\tau)$ changes, the young individual should *instantaneously* trade with the old while perpetually maintaining the zero value of his excess demands, $z_t^{a(\delta)} (\tau)_{c_2} = z_t^{a(\delta)} (\tau)_{c_1} = z_t^{a(\delta)} (\tau)_m = 0, \forall \tau \in [0, 1]$.

The outcome of conditions 1. and 2. is an admissible time path of current spot prices along which the young individual's direct utility coincides perpetually with his indirect utility. Clearly, conditions 1. and 2. above guarantee the maintenance of definion 2 of dynamic stability; and thus, sufficient conditions for dynamic stability of current spot prices are satisfied.

5   Sunspot equilibria exist but are unlikely to occur

This section will prove the existence of sunspot equilibria (as per proposition 4). It will then demonstrate two important qualifications: (a) despite their existence sunspot equilibria are, in general, unlikely to occur (as per proposition 8); and (b) the corollary, namely, adding an asset to this economy will generally render proposition 4 invalid.

5.1   Existence of sunspot equilibria

To prove proposition 4 it is shown that, whilst (5) is maintained, there exist two expected prices, p^θ and p^δ $\left(p^\theta \neq p^\delta\right)$, such that, the limit of the first (second) time path of prices in (4) is the expected price in the second (first) time path.

Consider the time paths of prices in (3). In the first time path in (3), for each initial price $p^\delta(0)$ and for a given expected price $p^s = p^\theta$, there exists a limit price p_t^δ that is a current spot market equilibrium price (as has been shown in claim 7). In the second time path, for each initial price $p^\theta(0)$ and for a given expected price $p^s = p^\delta$ there exists a limit price p_t^θ that is a current spot market equilibrium price. Therefore, for a continuum of expected prices in the closed interval of positive values $[\underline{\varepsilon}, \bar{\varepsilon}]$ there exist (according to claim 7), respectively, a continuum of current spot market equilibrium prices $p_t^\delta \in [\underline{\varepsilon}, \bar{\varepsilon}]$ and $p_t^\theta \in [\underline{\varepsilon}, \bar{\varepsilon}]$. By virtue of the continuity of p_t^δ and p_t^θ in the closed interval $[\underline{\varepsilon}, \bar{\varepsilon}]$, there exists a fixed point $p_t^\delta \in [\underline{\varepsilon}, \bar{\varepsilon}]$ and $p_t^\theta \in [\underline{\varepsilon}, \bar{\varepsilon}]$ for which, respectively, $p_t^\delta = p^\delta$ and $p_t^\theta = p^\theta$ as in (4). Furthermore, because of the uniqueness of each time path in (3), $p^\delta \neq p^\theta$; i.e., sunspot equilibria exist in the economy.

5.2   Sunspot equilibria are unlikely to occur

Proposition 4 established that there exist prices $p^\theta, p^\delta \in [\underline{\varepsilon}, \bar{\varepsilon}]$ that are sunspot equilibria. Furthermore, since the Markov process π is stationary, these prices will perpetuate over time. Obviously, this result is a direct consequence of the stationary economy where the frequency of sunspot equilibria is an irrelevant issue.

For a 'proper' theory of sunspot equilibria one should establish that when π is non-stationary, all prices $p^\theta, p^\delta \in [\underline{\varepsilon}, \bar{\varepsilon}]$ are self fulfilling and therefore are sunspot equilibria; i.e., that the sets of rational expectations equilibria and sunspot equilibria coincide. The frequency of sunspot equilibria then becomes a relevant issue. It is known, however, that 'most' (in a topological sense (e.g. Guillemin and Pollack (1974), pp. 119-130)) continuous maps on compact sets are Lefschetz. Therefore, 'most' fixed points in proposition 4 are isolated for a non-stationary π. This implies that, unfortunately, sunspot equilibria are unlikely to occur despite their existence.

5.3 A corollary

Adding an extra asset (e.g., a bond g) to the economy will generally render proposition 4 invalid. To give effect to this claim assume that the economy has reached the following conceivable position of excess demands.

Table 3				
	c_1	m	g	c_2
old $(h = b)$	+	$0 \downarrow$	-	
young $(h = a)$	$0 \uparrow$	+	-	0

The only feasible trade is between c_1 and m as is shown (in Table 3) by the arrows in the respective columns of excess demands. Thus, based on conditions 1. and 2. in section 4.2, trade between the two individuals must be carried out such that the positive excess demands decline while maintaining the zero excess demands. In general, however, the maintenance of the zero excess demands requires here (unlike in the previous economy) a violation of the *quid pro quo*, expressed in condition [b] of definition 5. Based on the above, it is clear that, in general, there are no admissible paths of current spot prices that maintain (3). In general, therefore sunspot equilibria do not exist.

The introduction of money as a numeraire and as a medium of exchange would add the two regions ABC and DEF to the feasible set of trades in Figure 1. However, this would still leave the above impasse unresolved because exchanging money against other goods in a **direction opposite** to excess demand for money would result in a depletion of the money stock and a return to the above inconsistency. Moreover, an increase of the stock of money during the price adjustment process would still leave the problem unresolved because of the classical dichotomy – any increment in the stock of money would upset the absolute level of prices without adding any additional purchasing power to the individuals.

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Figure 1

$z_t^{a(\delta)}(0)_{c_2} = \hat{c}_2$

$z_t^{a(\delta)}(0 + d\tau)_{c_2} = 0$

c_1

$z_t^{a(\delta)}(0)_m = \hat{m}$

$z_t^{a(\delta)}(0 + d\tau)_m = 0$

B   A

$z_t^{a(\delta)}(0)_{c_1} = \hat{c}_1$

C

F

$z_t^{a(\delta)}(0 + d\tau)_{c_1} = 0$

D   E

c_2

m