Security and Potential Level Preferences with Thresholds

Ulrich Schmidt[1] & Alexander Zimper[2]

Working Paper Number 47

[1] Lehrstuhl fur Finanzmarkttheorie, University of Hannover

[2] School of Economics, University of Cape Town

Security and Potential Level Preferences with Thresholds*

Ulrich Schmidt[†] Alexander Zimper[‡]

Revised Version: March 19, 2007

Abstract

The security level models of Gilboa (1988) and of Jaffray (1988) as well as the security and potential level model of Cohen (1992) and Essid (1997) successfully accommodate classical Allais paradoxes while they offer an interesting explanation for their occurrence. However, experimental data suggest a systematic violation of these models when lotteries with low probabilities of bad or good outcomes are involved. In our opinion, one promising candidate for the explanation of these violations is the assumption of thresholds in the perception of security and potential levels. The present paper develops an axiomatic model that allows for such thresholds, so that the derived representation of preferences can accommodate the observed violations of the original security and potential level models.

Keywords: Allais paradoxes, Security Level, Potential Level, Thresholds
JEL Classification Number: D81

*The authors want to thank Martin Hellwig, Peter Wakker, Itzhak Gilboa, Craig Fox, Martin Peterson, and Lennart Sjöberg for their suggestions and comments. Furthermore, we want to thank two anonymous referees and John Miyamoto whose comments helped a lot to improve our paper.

†Lehrstuhl für Finanzmarkttheorie, University of Hannover, Königsworther Platz 1, 30167 Hannover, Germany. Email: u.schmidt@mbox.vwl.uni-hannover.de

‡Corresponding author. School of Economics, University of Cape Town, Private Bag, Rondebosch, 7701, Cape Town, South Africa. Email: zimper@bigfoot.com

1 Introduction

In a well-known study on the psychology of decision making under risk, Lopes (1987) concluded that a decision maker takes into account three different factors while evaluating lotteries: What is the expected utility of this lottery? What is the worst outcome I can end up with by choosing this lottery (i.e., what is the security level of this lottery)? What is the best outcome I can end up with (i.e., what is the potential level)? This conclusion motivated Cohen (1992) to develop a three-criteria decision model which generalizes expected utility by allowing for security level and potential level (=SL-PL) considerations. An extension of this model has been provided by Essid (1997). Earlier models of Gilboa (1988) and Jaffray (1988) are very similar to Cohen's model but restrict attention to the security level alone. All three approaches explain Allais paradoxes (cf. Allais (1979)) by discontinuities of preferences resulting from the different security and potential levels of the lotteries involved. More recently, Chateauneuf et al. (2004), building upon earlier work of Dow and Werlang (1994) and Eichberger and Kelsey (1999), have integrated Cohen's ideas in a model of decision making under uncertainty.

The accommodation of Allais paradoxes by the SL-PL models is in our view intuitively very appealing. However, SL-PL models exhibit two major problems. First, they perform descriptively rather poorly when they are confronted with experimental data that go beyond the classical Allais paradoxes. Since SL-PL models reduce to expected utility preferences in the interior of the Marschak-Machina triangle, typical violations of expected utility theory elicited in the interior of the Marschak-Machina triangle can not be accommodated by the original SL-PL models. Of course a model which could explain all possible violations of expected utility would be so general that it has no behavioral implications. However, one of our goals is to show that a slight generalization of the original SL-PL models already improves their empirical performance substantially. A second and somewhat more fundamental problem can be characterized as follows: in real life there is always an (arbitrarily) small chance of immediate death and also a tiny chance of finding a suitcase on the street containing a huge cash amount of say ten billion dollars. Thus, it may be argued that in all decision problems death is always the security level while the amount of ten billion dollars is the potential level. If the security and potential levels are, however, identical in all lotteries, SL-PL models simply reduce to expected utility theory. So it may be criticized that for real world applications there is no difference between expected utility and SL-PL models.

It turns out that both problems can be solved by a slight generalization of the original SL-PL models with respect to their assumption that security and potential considerations refer exclusively to the worst, respectively best, outcome in the support of a lottery, regardless of how small their probability actually is. Consequently, our model extends existing SL-PL models by so-called *thresholds* so that security or potential considerations become only relevant when the probabilities of bad, respectively good, outcomes are not below some perceptual threshold level. Extreme outcomes with probabilities beyond these thresholds will be disregarded in our model. For example, a lottery may be still perceived as very secure as long as bad outcomes occur with very small probability. Accordingly, a lottery may be associated with a low potential when the probability of a high outcome is only small for this lottery. As our formal main result, we derive a representation theorem for a generalized SL-PL model which involves two additional parameters: a threshold probability ε up to which a decision maker perceives the probability of a bad outcome as rather insignificant and a threshold probability η up to which she perceives the probability of a good outcome as negligible. Good or bad outcomes beyond these thresholds are ignored for the calculation of utility.[1]

Empirical observations, which suggest that people often neglect very small probabilities (cf. Sjöberg (1999), (2000) and Stone, Yates, and Parker (1994)), can be regarded as further evidence in favor of thresholds: if the worst (respectively best) outcome has a very small probability, it seems unreasonable that people attach psychological importance to this outcome by regarding it as security (respectively potential) level and, at the same time, neglect its probability. Also the editing phase of prospect theory (Kahneman and Tversky, 1979) suggests that outcomes with very small probabilities are ignored. Moreover, Birnbaum and Navarrete's (1998, p. 52f.) "recipe" for generating violations of first order stochastic dominance in class-room experiments seemingly exploits difficulties of decision makers in discriminating between lotteries whose cumulative probability of bad outcomes falls below some threshold value.

An analogous concept to our notion of thresholds can be seen in the Value-at-Risk (VaR) which is defined as the worst loss for a given confidence level (mostly 99%). More precisely, for a confidence level of 99% the VaR of a lottery equals x if the cumulative probability of outcomes smaller than x is given by 1%. The VaR has recently become very popular as a risk measure and it seems reasonable to consider the VaR as security level which is perfectly consistent with our model but not compatible with the original SL-PL models.

The introduction of thresholds appears to us as a natural extension of SL-PL models, and it can successfully explain the most persistent choice patterns that are inconsistent with the original SL-PL models. Thus, the main contribution of this paper is to improve the descriptive power of SL-PL models by formalizing the idea that extremely bad, respectively good, outcomes only influence preferences when their cumulative, respectively

[1] Note that it is beyond the scope of our paper to develop a psychological theory that actually explains the existence of perception thresholds.

decumulative, probabilities exceed a threshold.

Clearly, besides SL-PL models there exist other alternatives to expected utility which can explain Allais paradoxes and further violations of expected utility. Nowadays, cumulative prospect theory (Tversky and Kahneman, 1992) seems to be most prominent. Schmidt (2000) has shown that the main idea of SL-PL models can be integrated in the framework of cumulative prospect theory and can be considered as a special case resulting from a particular shape of the probability weighting function. If we take this view - also supported by the paper of Chateauneuf et al. (2004) - SL-PL models have less descriptive flexibility since they are less general but, conversely, they have more concrete implications in theoretical applications. For instance, Zimper (2006) demonstrates that under SL-PL preferences *equilibria in beliefs* may fail to exist for finite strategic games whereas there always exist such equilibria when preferences allow for a continuous utility representation (cf. Crawford, 1990). As another theoretical application, the assumption of SL-PL preferences may avoid Rabin's (2000) *calibration-paradox*, which - originally established for EU preferences - also applies to various standard alternatives to EU theory that consider final-wealth levels as outcomes (cf. Safra and Segal (2005), (2006)). So, our model may provides a better trade off between parsimony and predictive power than the original SL-PL models. However, despite the appealing psychological foundation of our SL-PL model, it is of course finally an empirical question which class of models may provide a better organization of the data.

The paper proceeds as follows. The next section introduces the original SL-PL models and presents the typical experimental designs in which violations of these models have been observed. Section 3 introduces our proposal for a partition of a set of lotteries into subsets of different security and potential levels with respect to thresholds. In section 4 we introduce our axioms and state our representation theorem. In section 5 we conclude by demonstrating how the evidence against the original SL-PL models can be accommodated within our framework. All formal proofs are relegated to the appendix.

2 The Original SL-PL Models

In contrast to other alternatives to expected utility theory like models with the betweenness-property or rank dependent utility models with continuous weighting functions (see, e.g., Karni and Schmeidler (1991), Starmer (2000), Schmidt (2004), and Sugden (2004) for surveys), SL-PL models presume that discontinuities in the preferences describe best what is psychologically happening when decision makers commit Allais paradoxes. As an extension to expected utility, security and potential factors allow for jumps in the preferences so that a *secure* (respectively *high potential*) lottery now dominates all *insecure* (respectively *low potential*) lotteries that are sufficiently close in the sense of some mathematically defined neighborhood.

Consider a finite set $X = \{x_1, ..., x_n\}$ - interpreted as set of *outcomes* - and let $\triangle$ denote the set of all probability distributions, i.e., lotteries, over X where a lottery $p \in \triangle$ yields outcome x_k with probability p_k. Further, denote by x_m and x_M the worst and best outcomes of the lottery p. Then the evaluation of a lottery p in the model of Cohen (1992) and Essid (1997) is basically given by

$$U(p) = a(m, M) \sum_{k=1}^{n} p_k u(x_k) + b(m, M), \tag{1}$$

where $\sum_{k=1}^{n} p_k u(x_k)$ is the standard expected utility of p with respect to some utility indices $u : X \to \mathbb{R}$ while the real numbers $a(m, M)$ and $b(m, M)$ with $a(m, M) > 0$ depend on the given security level (m) and potential level (M) of p. Moreover, Cohen (1992) shows that such a utility representation satisfies monotonicity with respect to first order stochastic dominance, if and only if, for all $m, k, M, m', M' \in \{1, .., n\}$ with $m \leq m'$ and $M \leq M'$,

$$\min_{m \leq k \leq M} [a(m', M') - a(m, M)] u(x_k) + b(m', M') - b(m, M) \geq 0. \tag{2}$$

The models of Gilboa (1988) and Jaffray (1988) are similar but restrict attention to the security level m alone.

If $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ in (1) assume different values for different security- and potential levels, then the utility representation (1) is discontinuous at lotteries where the probability of the worst (best) outcome drops to zero since Cohen identifies the security (potential) level of a lottery with its worst (best) outcome. As a consequence, original SL-PL models are able to reflect jumps in the preferences that occur at the edges of the Marschak-Machina triangle. However, when we restrict attention to lotteries in the interior of the Marschak-Machina triangle all lotteries have identical security level (=the worst outcome in X) and potential level (=the best outcome in X), so that the utility representation (1) reduces to expected utility theory.

Based on a statistical analysis of experimental data of elicited preferences, Harless and Camerer (1994) investigate the question whether Allais paradoxes are persistently committed within the interior of the Marschak-Machina triangle or if they have to include lotteries at the edges. They conclude: "The conjecture that EU violations disappear in the interior appears to be false." In what follows, we present results from two studies reported in Harless and Camerer (1994), by which preferences over lotteries in the interior of the Marschak-Machina triangle were elicited. In our opinion, these data reveal very specific preference patterns by which expected utility theory - and therefore original SL-PL models - are significantly violated.

Problem 1. Consider the following pairs of lotteries Sk, Rk, $k \in \{1, ..., 5\}$, for which the respective probabilities of the outcomes \$0, $\$1 \cdot 10^6$, $\$5 \cdot 10^6$ are given as follows:

$$
\begin{array}{ll}
S1 = (.01, .98, .01) & R1 = (.02, .87, .11) \\
S2 = (.80, .19, .01) & R2 = (.81, .08, .11) \\
S3 = (.01, .19, .80) & R3 = (.02, .08, .90) \\
S4 = (.70, .19, .11) & R4 = (.71, .08, .21) \\
S5 = (.02, .87, .11) & R5 = (.03, .76, .21)
\end{array}
$$

Observe that all line segments $\overline{Sk, Rk}$, $k \in \{1, ..., 5\}$, lie in the interior of the Marschak-Machina triangle. Since the line segments $\overline{Sk, Rk}$ are parallel for all $k \in \{1, ..., 5\}$, the original SL-PL models - analogously to expected utility preferences - are therefore only consistent with strict preferences where the decision maker must either prefer Sk to Rk for all $k \in \{1, ..., 5\}$ or vice versa. But from 184 subjects confronted with this task, 17 subjects strictly preferred Sk to Rk for all $k \in \{1, ..., 5\}$ and 61 subjects strictly preferred Rk to Sk for all $k \in \{1, ..., 5\}$. Hence, a vast majority of elicited preferences is neither compatible with expected utility theory nor with the original SL-PL models. Most of the subjects (21 individuals) who did not comply to these preferences expressed the following preference pattern:

$$
\begin{aligned}
Sk \text{ preferred to } Rk \text{ for } k &= 1 \\
Rk \text{ preferred to } Sk \text{ for } k &\in \{2, ..., 5\}\,.
\end{aligned}
\tag{3}
$$

Is there some plausible reason why these individuals choose the pairs Sk, Rk for $k \in \{2, ..., 5\}$, in accordance with expected utility theory, whereas they strictly prefer $S1$ to $R1$, thereby violating expected utility theory as well as the original SL-PL models? Compared to lotteries Sk, $k \in \{2, ..., 5\}$, a peculiar feature of lottery $S1$ is the small probability of the worst and the best outcome. For this reason, we think that an appealing "explanation" of the occurrence of preferences (3) has to stress the role of small probabilities in decision making. In section 5, we demonstrate that our model can accommodate preferences (3) by stipulating the existence of some small threshold for the perception of security levels.

Problem 2. Now consider the following pairs of lotteries Sk, Rk, $k \in \{1, ..., 4\}$, for which the respective probabilities of outcomes \$0, \$3, \$6 are given as follows:

$$\begin{array}{ll} S1 = (.84, .14, .02) & R1 = (.89, .01, .10) \\ S2 = (.04, .94, .02) & R2 = (.09, .81, .10) \\ S3 = (.44, .14, .42) & R3 = (.49, .01, .5) \\ S4 = (.04, .14, .82) & R4 = (.09, .01, .9) \end{array}$$

As in problem 1, all line segments $\overline{Sk, Rk}$, $k \in \{1, ..., 4\}$, lie in the interior of the Marschak-Machina triangle and are parallel. Thus, the original SL-PL models as well as expected utility preferences can only accommodate the strict preference patterns where the decision maker must either prefer Sk to Rk for all $k \in \{1, ..., 4\}$ or vice versa. From 84 subjects, 10 subjects strictly preferred Sk to Rk for all $k \in \{1, ..., 4\}$ and 26 subjects strictly preferred Rk to Sk for all $k \in \{1, ..., 4\}$, so that a vast majority of subjects violates expected utility theory and the original SL-PL models. The most often reported preference pattern (10 individuals), violating expected utility theory and existing SL-PL models, is

$$\begin{aligned} Rk \text{ preferred to } Sk \text{ for } k &\in \{1, 2\} \\ Sk \text{ preferred to } Rk \text{ for } k &\in \{3, 4\}. \end{aligned} \tag{4}$$

Compared to the remaining lotteries, the lotteries $S1$ and $S2$ have the peculiar feature that the best outcome appears only with a very small probability. Again, we think that this negligible probability of a good outcome in lotteries $S1$ and $S2$ may be the reason why so many individuals prefer Rk to Sk for $k \in \{1, 2\}$ while they simultaneously prefer Sk to Rk for $k \in \{3, 4\}$ where the good outcome occurs with non-negligible probability. We will accommodate in section 5 the preferences (4) by assuming the existence of some small threshold for the perception of potential levels..

Let us sum up. While SL-PL models may accommodate the occurrence of classical Allais paradoxes involving lotteries at the edges of the Marschak-Machina triangle, they can no longer explain deviations from expected utility theory when we move from the edges into the interior of the Marschak-Machina triangle. A closer examination of problem 1 and of problem 2 reveals that original SL-PL models are most persistently violated when lotteries are involved for which bad outcomes or good outcomes occur with rather small probability. That is, the typically observed violations of SL-PL models involve lotteries that are very close to the edges of the Marschak-Machina triangle but that do not actually belong to it.

Therefore, we think that the key for solving these systematic violations of SL-PL models is a departure from the assumption that a lottery is not secure, or is a high potential lottery, just because bad, respectively good, outcomes occur with positive probability. In contrast, our SL-PL model with thresholds will allow decisionmakers to perceive lotteries as secure (of low potential) when the bad (good) outcomes occurs only with sufficiently small probabilities.

3 Security and Potential Levels with Thresholds

The objective for our particular formalism of thresholds has been threefold. First, we wanted to keep the model as simple as possible. As a consequence we introduce only two new parameters to the original SL-PL models, a threshold for security levels and a threshold for potential levels, whereby the security level and the potential level of a lottery is then easily determined by its cumulative and decumulative distribution functions. More sophisticated SL-PL models with thresholds could be constructed, however, we are willingly trading off richness of the model for a simple formalism that captures well the basic idea.

Second, we formalize the idea that the decision maker is ignorant with respect to extreme outcomes (=tail outcomes) whose cumulated probabilities fall below the stipulated threshold values. For example, if there exists a security threshold of 0.02 for the preferences elicited in problem 1 (Section 2), then our decision maker is assumed to be indifferent between, e.g., the lotteries

$$S6 = (.01, .01, .98) \quad R6 = (0, .02, .98).$$

In this case, the decision maker of our model perceives the differences between the small probabilities of the worst and second worst outcomes as irrelevant to his comparison of both lotteries. Similarly, for a potential threshold of 0.02 we stipulate indifference between the lotteries

$$S7 = (.98, .01, .01) \quad R7 = (.98, .02, 0).$$

Both examples demonstrate that the preferences of our model violate monotonicity with respect to first-order stochastic dominance (FOSD) in its strict version. In contrast, the original SL-PL models, which do not consider thresholds, do not violate this fundamental requirement for rational decision makers. Although the empirical literature has observed violations of monotonicity (see, e.g., Birnbaum and Navarrete, 1998), our third objective is to restrict the violation of monotonicity with respect to FOSD in our model to the exclusive case where the decision maker is ignorant to the probabilities of extreme outcomes whenever their total probability is below the thresholds.

Denote by I_k the degenerate lottery that yields outcome x_k with probability one. Compound lotteries are supposed to reduce to lotteries in the standard way and we alternatively write $p_1 I_1 + ... + p_n I_n$ for the lottery p.

Let $F[p](x_k)$ denote the cumulative and $D[p](x_k)$ the decumulative distribution function of lottery p evaluated at outcome x_k. For $\varepsilon, \eta \in (0,1)$ with $\varepsilon + \eta < 1$ and m, M with $1 \leq m \leq M \leq n$ define $\triangle(m, M, \varepsilon, \eta)$ as the set of all lotteries $p \in \triangle$ such that

$$F[p](x_{m-1}) < \varepsilon \text{ AND } F[p](x_m) \geq \varepsilon \text{ AND } D[p](x_{M+1}) < \eta \text{ AND } D[p](x_M) \geq \eta.$$

Observation: *For any thresholds $\varepsilon, \eta \in (0,1)$ with $\varepsilon + \eta < 1$, the collection of sets, $(\triangle(m, M, \varepsilon, \eta))_{m \in \{1,...,n\}, M \geq m}$, is a partition of $\triangle$ into non-empty convex cells.*

We call $\triangle(m, M, \varepsilon, \eta)$ a *SL,PL-subset* and we say that a lottery $p \in \triangle(m, M, \varepsilon, \eta)$ has *security level* m and *potential level* M. The threshold-value ε for security levels guarantees that worse outcomes than x_m occur for a lottery of security level m with probability less than ε. Accordingly, better outcomes than x_M occur for a lottery of potential level M with probability less than η. For the sake of notational convenience, we henceforth simply write $\triangle(m, M)$ instead of $\triangle(m, M, \varepsilon, \eta)$; (though the reader should keep in mind that all sets $\triangle(m, M)$ are defined with respect to the fixed perception thresholds ε and η).

4 Axiomatic Analysis

Under the assumption that unlikely realizations of bad or good outcomes do not influence the perceived security- and potential level of a lottery, we present in this section our formal axioms and derive a corresponding representation theorem.

A1. Weak Ordering: *There exists a transitive and complete preference relation* $\succsim$ *on* $\triangle$ *such that* $I_{k+1} \succ I_k$ *for* $k \in \{1, ..., n-1\}$.

The following axiom formulates our second objective, namely, that the probabilities of tail outcomes are ignored if their cumulated probability is below the threshold for the security level (lower tail) respectively below the perception threshold for the potential level (upper tail). In particular, we assume that strictly worse outcomes than x_m are irrelevant for the perception of a lottery's security level as long as they occur with a probability strictly smaller than ε. Accordingly, strictly better outcomes than x_M are supposed to be irrelevant for the perception of a lottery's potential level as long as they occur with a probability strictly smaller than η.

A2. Perception Indifference: *If* $p, q \in \triangle(m, M)$ *such that*

$$\begin{aligned} F[p](x_m) &= F[q](x_m) \quad \text{AND} \quad D[p](x_M) = D[q](x_M), \\ p_k &= q_k \text{ for all } k \text{ with } m < k < M, \end{aligned}$$

then $p \sim q$.

Now define the set

$$\mathcal{P}(m, M) = \{p \in \triangle(m, M) \mid F[p](x_{m-1}) = 0 \quad \text{AND} \quad D[p](x_{M+1}) = 0\},$$

which collects all the lotteries of $\triangle(m, M)$ that do not contain strictly worse, respectively strictly better, outcomes than x_m, respectively x_M, in their support.

A3. $\mathcal{P}$-Restricted Archimedean Axiom: *For all* $p, q, r \in \mathcal{P}(m, M)$ *such that* $p \succ r \succ q$, *there exist* $\lambda, \mu \in (0, 1)$ *such that*

$$\lambda p + (1 - \lambda) q \succ r \succ \mu p + (1 - \mu) q.$$

A4. $\mathcal{P}$-Subset Restricted Independence: *For all* $p, q, r \in \mathcal{P}(m, M)$, $p \succsim q$ *if and only if*

$$\lambda p + (1 - \lambda) r \succsim \lambda q + (1 - \lambda) r$$

for all $\lambda \in (0, 1)$.

Recall the definition of first-order stochastic dominance (=FOSD): a lottery p dominates a lottery q w.r.t. FOSD, i.e., $p \succeq_{FOSD} q$, if and only if $F[p](x) \leq F[q](x)$ for all $x \in X$. Moreover, if additionally $F[p](x) < F[q](x)$ for some $x \in X$ we say that p strictly dominates a lottery q w.r.t. FOSD and we write $p \succ_{FOSD} q$.

Monotonicity w.r.t. FOSD - in its strict version - would require that $p \succ q$ whenever $p \succ_{FOSD} q$. However, axiom A2 implies the existence of lotteries $p, q \in \triangle(m, M)$ with $m \neq 1$ and $M \neq n$ such that $p \sim q$ while $p \succ_{FOSD} q$. This violation of monotonicity w.r.t. FOSD in its strict version is a natural consequence of our assumption that the probabilities of tail outcomes are ignored if their probability falls below the thresholds. Nevertheless, we want to impose monotonicity w.r.t. FOSD - even in its strict version - as a valid axiom whenever we compare lotteries of different subsets $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$ where the ignorance towards small probabilities of extreme outcomes has no effect.

A5. Restricted Monotonicity with respect to FOSD: *For all $p \in \mathcal{P}(m, M)$ and all $p' \in \mathcal{P}(m', M')$, if $p \succeq_{FOSD} p'$ then $p \succeq p'$; and if $p \succ_{FOSD} p'$ then $p \succ p'$.*

A6. Reinforced Weak Independence: *For all $p, q \in \mathcal{P}(m, M)$ and all $p', q' \in \mathcal{P}(m', M')$,*

$$\text{if } p \succsim p' \text{ and } q \succsim q' \text{ then } \lambda p + (1-\lambda) q \succsim \lambda p' + (1-\lambda) q' \text{ for all } \lambda \in (0,1);$$
$$\text{if } p \sim p' \text{ and } q \sim q' \text{ then } \lambda p + (1-\lambda) q \sim \lambda p' + (1-\lambda) q' \text{ for all } \lambda \in (0,1).$$

We say that preferences *overlap* between two sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$, with $m < M$ and $m' < M'$, if there is a non-empty open set $O \subset \mathcal{P}(m, M)$ such that there exists for every $p \in O$ some $p' \in \mathcal{P}(m', M')$ such that $p \sim p'$. For the case $m = M$ and $m' < M'$ we say that preferences between $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$ *overlap* if $I_m \sim p'$ for some $p' \in \mathcal{P}(m', M')$.

A7. Connected Preferences: *There exists a sequence of pairs $\{(m, M)_i\}_{i=1,\ldots,\frac{n(n+1)}{2}}$ with $m \leq M$ such that $(m, M)_1 = (1, 1)$, $(m, M)_n = (n, n)$, $(m, M)_i \neq (m, M)_{i+1}$ for all $i \in \left\{1, \ldots, \frac{n(n+1)}{2}\right\}$, and preferences between $\mathcal{P}(m, M)_i$ and $\mathcal{P}(m, M)_{i+1}$ overlap for all $i \in \left\{2, \ldots, \frac{n(n+1)}{2} - 1\right\}$.*

The intuition for axiom (A7), "Connected Preferences", is that security- and potential level considerations should have no extreme effect on the evaluation of lotteries. Except for the extreme security and potential levels $(1,1)$ and (n,n), there exist by axiom (A7) for all security and potential levels (m,M) lotteries $p \in \mathcal{P}(m,M)$ such that the decision maker is indifferent between these lotteries and lotteries that have a different security and potential level (m',M'). As one implication of axiom (A7), we exclude preferences that are lexicographic with respect to the security or/and the potential level.

Representation Theorem: *Preferences that satisfy the axioms (A1)-(A7) are representable by $U : \triangle \to \mathbb{R}$ such that, for $p \in \triangle(m,M)$ with $m < M$,*

$$\begin{aligned} U(p) &= a(m,M)\left[F[p](x_m)\,u(x_m) + D[p](x_M)\,u(x_M) + \sum_{k=m+1}^{M-1} p_k u(x_k)\right] \\ &\quad + b(m,M), \end{aligned} \tag{5}$$

and for $p \in \triangle(m,M)$ with $m = M$,

$$U(p) = a(m,M)\,u(x_m) + b(m,M),$$

where $u : X \to \mathbb{R}$ is a strictly increasing function. The functions U and u are unique up to a positive affine transformation. The coefficients of the representation satisfy the following conditions:

(i) $a(m,M), b(m,M) \in \mathbb{R}$ *with* $a(m,M) > 0$;

(ii) *Axiom (A7) requires for any two sets $\mathcal{P}(m,M)$ and $\mathcal{P}(m',M')$ for which preferences overlap,*

$$\begin{aligned} & a(m,M)\left[\varepsilon u(x_m) + (1-\varepsilon)\,u(x_M)\right] + b(m,M) \\ > \; & a(m',M')\left[(1-\eta)\,u(x_{m'}) + \eta u(x_{M'})\right] + b(m',M') \end{aligned} \tag{6}$$

and

$$\begin{aligned} & a(m',M')\left[\varepsilon u(x_{m'}) + (1-\varepsilon)\,u(x_{M'})\right] + b(m',M') \\ > \; & a(m,M)\left[(1-\eta)\,u(x_m) + \eta u(x_M)\right] + b(m,M) \end{aligned} \tag{7}$$

if $m < M$ and $m' < M'$; and

$$\begin{aligned} & a(m',M')\left[\varepsilon u(x_{m'}) + (1-\varepsilon)\,u(x_{M'})\right] + b(m',M') \\ \geq \; & a(m,M)\,u(x_m) + b(m,M) \\ \geq \; & a(m',M')\left[(1-\eta)\,u(x_{m'}) + \eta u(x_{M'})\right] + b(m',M') \end{aligned} \tag{8}$$

if $m = M$ and $m' < M'$;

(iii) *Axiom (A5) requires*

for all $m, M \in \{1, ..., n\}$ *with* $m \leq M$ *and* $m > 1$,

$$\begin{aligned} \min_{\{k \in \mathbb{N} | m \leq k \leq M\}} \quad & (a(m,M) - a(m-1,M)) [\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_M)] \\ & + b(m,M) - b(m-1,M) \\ \geq \quad & a(m-1,M) \varepsilon (u(x_{m-1}) - u(x_m)), \end{aligned} \tag{9}$$

for all $m, M \in \{1, ..., n\}$ *with* $m \leq M$ *and* $M > 1$,

$$\begin{aligned} \min_{\{k \in \mathbb{N} | m \leq k \leq M-1\}} \quad & (a(m,M) - a(m,M-1)) [\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_{M-1})] \\ & + b(m,M) - b(m,M-1) \\ \geq \quad & a(m,M) \eta (u(x_{M-1}) - u(x_M)). \end{aligned} \tag{10}$$

Proof: The proof - relegated to the Appendix - proceeds in several steps. In the first step it is shown, by adopting an argument of Cohen (1992), that the axioms (A1), (A3), and (A4) admit for the construction of an expected utility functional which represents preferences on lotteries within a given set $\mathcal{P}(m, M)$. In a second step, the representation of preferences is extended from the set $\mathcal{P}(m, M)$ to the set $\triangle(m, M)$ by employing the indifference conditions of axiom (A2). Step 3 mentions a result by Cohen (1992), which extends, by axioms (A6) and (A7), the utility representation to the whole domain $\triangle$. Step 4 establishes that u is strictly increasing. In step 5 it is demonstrated that axiom (A7) requires the conditions (6) - (8) to hold. Finally, in step 6 the conditions (9) and (10) are derived under the assumption that axiom A5 is satisfied.

Remark. Although the original SL,PL-model of Cohen is not a special case of our model - we assume the existence of strictly positive perception thresholds for security and potential levels - Cohen's representation (1) obtains as the limiting case of our representation when we let the perception thresholds ε and η converge towards zero. Namely, for the limiting case of our utility representation (5) it obtains that

$$\begin{aligned} & \lim_{\varepsilon, \eta \to 0} a(m,M) \left[F[p](x_m) u(x_m) + D[p](x_M) u(x_M) + \sum_{k=m+1}^{M-1} p_k u(x_k) \right] + b(m,M) \\ = \quad & a(m,M) \sum_{k=m}^{M} p_k u(x_k) + b(m,M), \end{aligned}$$

since

$$\begin{aligned} \lim_{\varepsilon \to 0} F[p](x_m) &= p_m \\ \lim_{\eta \to 0} F[p](x_M) &= p_M. \end{aligned}$$

Moreover, our conditions (9) and (10) coincide in the limit with the condition (2) which ensures that monotonicity w.r.t. FOSD holds in the original SL,PL-model of Cohen.

5 Accommodating the Experimental Evidence

We have focused on our simple concept of security and potential preferences with thresholds, with only two parameters more than Cohen's original model, because we wanted to obtain a model which is as simple as possible while it can solve the two major problems concerning the original SL-PL models mentioned in the introduction. Our formalism of thresholds presented in section 3 is clearly a very idealized concept and, therefore, it seems unreasonable to expect that this concept could capture all empirical choice patterns which may be associated with the existence of thresholds in a decision maker's evaluation of lotteries. Nevertheless, we believe that our SL-PL model with thresholds offers indeed a convincing explanation within the reasoning of SL-PL models for the two most persistent violations of expected utility theory when lotteries are involved that lie in the interior of the Marschak-Machina triangle but are very close to its edges:

Lotteries with only small probabilities of bad outcomes may be perceived as comparably favorable (problem 1).

Lotteries with only small probabilities of good outcomes may be perceived as comparably inferior (problem 2).

In the remainder of this section, we demonstrate that our model of SL-PL preferences with thresholds can indeed accommodate the observed preference patterns of the two problems presented in section 2 which violate the original SL-PL models.

Problem 1. In a first step, we specify our utility representation, so that we obtain the desired preference pattern (3) for an appropriately chosen security threshold ε. In a second step, we show that this utility representation satisfies monotonicity with respect to first order stochastic dominance.

For

$$\begin{aligned} x_m &\in \{\$0\} \text{ and } x_M \in \left\{\$0, \$1 \cdot 10^6, \$5 \cdot 10^6\right\} \\ x_{m'} &\in \left\{\$1 \cdot 10^6, \$5 \cdot 10^6\right\} \text{ and } x_{M'} \in \left\{\$1 \cdot 10^6, \$5 \cdot 10^6\right\} \text{ with } x_{m'} \leq x_{M'} \end{aligned} \tag{11}$$

define

$$\begin{aligned} a(m, M) &= b(m', M') = 1 \\ a(m', M') &= b(m, M) = 0.8 \end{aligned}$$

and

$$\begin{aligned} u\left(\$0\right) &= 0 \\ u\left(\$1\cdot 10^6\right) &= 0.25 \\ u\left(\$5\cdot 10^6\right) &= 1. \end{aligned}$$

Furthermore, assume the existence of some security threshold ε such that $0.01 < \varepsilon \leq 0.02$, so that the two lotteries $S1$ and $S3$ are associated with the high security level for outcome $\$1 \cdot 10^6$ whereas the remaining lotteries $S2$, $S4$, $S5$, and Rk, for $k \in \{1, .., 5\}$ are associated with the low security level for outcome \$0. Computing utilities then gives the preference pattern (3):

$$\begin{array}{ll} U(S1) = 1.206 & U(R1) = 1.1275 \\ U(S2) = 0.8575 & U(R2) = 0.93 \\ U(S3) = 1.68 & U(R3) = 1.72 \\ U(S4) = 0.9575 & U(R4) = 1.03 \\ U(S5) = 1.1275 & U(R5) = 1.2 \end{array}$$

Since the r.h.s. of (9) is strictly less than zero, condition (9) is satisfied if

$$\min_{\{k\in\mathbb{N}|m\leq k\leq M\}} \left(a(m,M) - a(m-1,M)\right)\left[\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_M)\right]$$
$$\geq b(m-1,M) - b(m,M),$$

where $m > 1$. This inequality becomes for $m = 2$

$$\min_{\{k\in\mathbb{N}|m\leq k\leq M\}} \varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_M) \leq 1,$$

and for $m = 3$, $0 \geq 0$. Both inequalities are obviously satisfied, so that condition (9) holds for the above utility representation.

Similarly, condition (10) is satisfied if

$$\min_{\{k\in\mathbb{N}|m\leq k\leq M-1\}} \left(a(m,M) - a(m,M-1)\right)\left[\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_{M-1})\right]$$
$$\geq \quad b(m,M-1) - b(m,M),$$

with $M > 1$. For $M = 3$ we have $0 \geq 0$, and for $M = 2$ we have

$$\min_{\{k\in\mathbb{N}|m\leq k\leq M-1\}} \varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_M) \leq 1.$$

Again, both inequalities are satisfied, so that condition (10) also holds for the above utility representation.

Problem 2. Now assume the existence of some potential threshold η such that $0.02 < \eta \leq 0.1$. Furthermore, suppose that the parameters of our utility representation are given as follows:

$$\begin{aligned} u(\$0) &= 0 \\ u(\$3) &= 0.75 \\ u(\$6) &= 1 \end{aligned}$$

and

$$\begin{aligned} a(m, M) &= a(m', M') = 1 \\ b(m, M) &= 0 \\ b(m', M') &= 0.1 \end{aligned}$$

for

$$\begin{aligned} x_m &\in \{\$0, \$3\} \text{ and } x_M \in \{\$0, \$3\} \text{ with } x_m \leq x_M \\ x_{m'} &\in \{\$0, \$3, \$6\} \text{ and } x_{M'} \in \{\$6\}. \end{aligned} \tag{12}$$

Thus, only the two lotteries $S1$ and $S2$ are associated with the low potential level for outcome \$3 whereas the remaining lotteries are associated with the high potential level for outcome \$6. Computing utilities then gives the preference pattern (4):

$$\begin{array}{ll} U(S1) = 0.12 & U(R1) = 0.2075 \\ U(S2) = 0.72 & U(R2) = 0.8075 \\ U(S3) = 0.625 & U(R3) = 0.6075 \\ U(S4) = 1.025 & U(R4) = 1.0075 \end{array}$$

Finally, it can be immediately verified that this utility representation satisfies the conditions (9) and (10).

Appendix: Proof of the Representation Theorem

Step 1. We prove the following lemma:

Lemma 1: *If preferences satisfy the axioms (A1), (A3) and (A4), then there exists a function* $u(\cdot,\mathcal{P}(m,M)):X\rightarrow\mathbb{R}$ *- unique up to some positive affine transformation - such that for all* $p,q\in\mathcal{P}(m,M)$,

$$p\succsim q\Leftrightarrow\sum_{k=m}^{M}p_{k}u(x_{k},\mathcal{P}(m,M))\geq\sum_{k=m}^{M}q_{k}u(x_{k},\mathcal{P}(m,M)) \tag{13}$$

Proof: Our proof of lemma 1 proceeds along the lines of Cohen's (1992) proof of proposition 1 (cf. pp. 122-125) whereby we have to consider subsets $\mathcal{P}(m,M)$ instead of Cohen's definition of security- and potential level subsets.

(i) Consider lotteries

$$\begin{aligned} r &:= (1-\lambda)I_{m}+\lambda I_{M}\in\mathcal{P}(m,M)\\ r' &:= (1-\lambda')I_{m}+\lambda' I_{M}\in\mathcal{P}(m,M)\end{aligned}$$

with $\lambda'\neq\lambda$. Furthermore, assume that $\lambda,\lambda'\geq\eta$ so that $r,r'\in\mathcal{P}(m,M)$. It is well known (cf. Fishburn, 1988) that the axioms (A1), (A3) and (A4) imply the existence of a linear utility V_0 on $\mathcal{P}(m,M)$ which is unique up to some positive affine transformation. As a consequence, there exists a unique utility U_0 - resulting from an positive affine transformation of V_0 - on $\mathcal{P}(m,M)$ such that

$$\begin{aligned} U_{0}(r) &= \lambda\\ U_{0}(r') &= \lambda'.\end{aligned}$$

Let

$$\delta:=\left(\varepsilon+\frac{1-\eta-\varepsilon}{2}\right)I_{m}+\left(\eta+\frac{1-\eta-\varepsilon}{2}\right)I_{M}\in\mathcal{P}(m,M)$$

and define, for some $k\in\{m,...,M\}$, the lottery

$$r^{k}(\alpha):=\alpha I_{k}+(1-\alpha)\delta.$$

Furthermore, suppose that α satisfies

$$(1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) \geq \varepsilon \text{ AND } (1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) \geq \eta \tag{14a}$$
$$\text{if } k \in \{m+1,...,M-1\};$$
$$\alpha+(1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) \geq \varepsilon \text{ AND } (1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) \geq \eta \tag{14b}$$
$$\text{if } k = m;$$
$$(1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) \geq \varepsilon \text{ AND } \alpha+(1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) \geq \eta \tag{14c}$$
$$\text{if } k = M,$$

implying that $r^k(\alpha) \in \mathcal{P}(m,M)$. Observe that, by our assumption $\eta+\varepsilon<1$, there exists some non-empty closed interval $\mathfrak{I} \subset [0,1]$ such that all $\alpha \in \mathfrak{I}$ satisfy (14a) - (14c).

Now fix

$$\begin{aligned} u(x_m, \mathcal{P}(m,M)) &= 0 \\ u(x_M, \mathcal{P}(m,M)) &= 1, \end{aligned}$$

so that, e.g.,

$$U_0(\delta) = \frac{1+\eta-\varepsilon}{2}.$$

Define the function $u_\alpha : \{x_m,...,x_M\} \to \mathbb{R}$ such that

$$u_\alpha(x_k) = \frac{U_0\left(r^k(\alpha)\right) - (1-\alpha)U_0(\delta)}{\alpha}$$

and consider $\alpha, \alpha' \in \mathfrak{I}$ implying that $r^k(\alpha), r^k(\alpha') \in \mathcal{P}(m,M)$. W.l.o.g let $\alpha < \alpha'$ and observe that

$$r^k(\alpha) = \frac{\alpha}{\alpha'}r^k(\alpha') + \left(1-\frac{\alpha}{\alpha'}\right)\delta.$$

Since $r^k(\alpha), r^k(\alpha'), \delta \in \mathcal{P}(m,M)$, we have, by linearity of U_0 on $\mathcal{P}(m,M)$,

$$\begin{aligned} U_0\left(r^k(\alpha)\right) &= \frac{\alpha}{\alpha'}U_0\left(r^k(\alpha')\right) + \left(1-\frac{\alpha}{\alpha'}\right)U_0(\delta) \Leftrightarrow \\ \frac{U_0\left(r^k(\alpha)\right)-(1-\alpha)U_0(\delta)}{\alpha} &= \frac{U_0\left(r^k(\alpha')\right)-(1-\alpha')U_0(\delta)}{\alpha'}. \end{aligned}$$

Thus, for any $r^k(\alpha), r^k(\alpha') \in \mathcal{P}(m,M)$, we have $u_\alpha(x_k) = u_{\alpha'}(x_k)$ for $x_k \in \{x_m,...,x_M\}$ so that we can write

$$U_0(\alpha I_k + (1-\alpha)\delta) = \alpha u(x_k, \mathcal{P}(m,M)) + (1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right)$$

where $u(x_k, \mathcal{P}(m,M)) = u_\alpha(x_k)$.

(ii) Notice that every $p \in \mathcal{P}(m, M)$ can be written as

$$p = q_m I_m + ... + q_M I_M + (1-\alpha)\delta$$

where

$$\begin{aligned}
(1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) + q_M &= p_M \\
(1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) + q_m &= p_m \\
q_k &= p_k \text{ for } k \in \{m+1, ..., M-1\}
\end{aligned}$$

for some α which must satisfy

$$(1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) + q_m \geq \varepsilon \text{ AND } (1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) + q_M \geq \eta. \tag{15}$$

Notice that $\sum_{k=m}^{M} \frac{q_k}{\alpha} = 1$ so that we can rewrite $p \in \mathcal{P}(m, M)$ as

$$p = \sum_{k=m}^{M} \frac{q_k}{\alpha}\left[\alpha I_k + (1-\alpha)\delta\right].$$

Observe that whenever α satisfies condition (15), the conditions (14a) - (14c) must also be satisfied for all $\alpha I_k + (1-\alpha)\delta$ such that $m \leq k \leq M$ since $q_M, q_m \leq \alpha$. That is, whenever $p \in \mathcal{P}(m, M)$ we also have $\alpha I_k + (1-\alpha)\delta \in \mathcal{P}(m, M)$ for all k such that $m \leq k \leq M$. By the existence of the linear utility U_0 on $\mathcal{P}(m, M)$, we therefore obtain

$$\begin{aligned}
U_0(p) &= \sum_{k=m}^{M} \frac{q_k}{\alpha} U_0(\alpha I_k + (1-\alpha)\delta) \\
&= \sum_{k=m}^{M} \frac{q_k}{\alpha} \alpha u(x_k, \mathcal{P}(m, M)) + \sum_{k=m}^{M} \frac{q_k}{\alpha}(1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) \\
&= \sum_{k=m}^{M} \frac{q_k}{\alpha} \alpha u(x_k, \mathcal{P}(m, M)) + (1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) \\
&= \sum_{k=m+1}^{M-1} q_k u(x_k, \mathcal{P}(m, M)) + \left((1-\alpha)\left(\frac{1-\eta+\varepsilon}{2}\right) + q_m\right) u(x_m, \mathcal{P}(m, M)) \\
&\quad + \left((1-\alpha)\left(\frac{1+\eta-\varepsilon}{2}\right) + q_M\right) u(x_M, \mathcal{P}(m, M)) \\
&= \sum_{k=m}^{M} p_k u(x_k, \mathcal{P}(m, M)),
\end{aligned}$$

whereby we have exploited the fact that

$$\begin{aligned} u\left(x_m, \mathcal{P}\left(m, M\right)\right) &= 0 \\ u\left(x_M, \mathcal{P}\left(m, M\right)\right) &= 1. \end{aligned}$$

Finally, since V_0 is unique up to some positive affine transformation so must be $u\left(\cdot, \mathcal{P}\left(m, M\right)\right)$.□

Step 2. By step 1, we have derived a utility representation - satisfying axioms (A1), (A3), (A4) - for lotteries within a given set $\mathcal{P}\left(m, M\right)$. By axiom (A2), every lottery $p \in \triangle\left(m, M\right)$ is indifferent to the lottery $p' \in \mathcal{P}\left(m, M\right)$ where

$$\begin{aligned} F\left[p\right]\left(x_m\right) &= p'_m \text{ and } D\left[p\right]\left(x_M\right) = p'_M, \\ p_k &= p'_k \text{ for all } k \text{ with } m < k < M \end{aligned}$$

if $m < M$, and $p' = I_m$ if $m = M$. Thus, we can extend the utility representation to lotteries within a given set $\triangle\left(m, M\right)$:

Lemma 2: *If preferences satisfy the axioms (A1)-(A4), then there exists a function $u\left(\cdot, \triangle\left(m, M\right)\right) : X \rightarrow \mathbb{R}$ - unique up to some positive affine transformation - such that, for all $p, q \in \triangle\left(m, M\right)$ with $m < M$, $p \succsim q$ if and only if*

$$\begin{aligned} & F\left[p\right]\left(x_m\right) u\left(x_m, \triangle\left(m, M\right)\right) + D\left[p\right]\left(x_M\right) u\left(x_M, \triangle\left(m, M\right)\right) \\ & + \sum_{k=m+1}^{M-1} p_k u\left(x_k, \triangle\left(m, M\right)\right) \\ \geq \; & F\left[q\right]\left(x_m\right) u\left(x_m, \triangle\left(m, M\right)\right) + D\left[q\right]\left(x_M\right) u\left(x_M, \triangle\left(m, M\right)\right) \\ & + \sum_{k=m+1}^{M-1} q_k u\left(x_k, \triangle\left(m, M\right)\right). \end{aligned}$$

Step 3. We leave it to the reader to verify that the proof of proposition 6 in Cohen (1992, p. 115) together with lemma 2 immediately establishes the following lemma. (Notice that the corresponding proof employs the axioms A6 and A7.)

Lemma 3: *If preferences satisfy the axioms (A1)-(A4) and (A6),(A7) then any utility representation $U : \triangle \rightarrow \mathbb{R}$ must satisfy, for $p \in \triangle\left(m, M\right)$ with $m < M$,*

$$\begin{aligned} U\left(p\right) &= a\left(m, M\right)\left[F\left[p\right]\left(x_m\right) u\left(x_m\right) + D\left[p\right]\left(x_M\right) u\left(x_M\right) + \sum_{k=m+1}^{M-1} p_k u\left(x_k\right)\right] \\ & \quad + b\left(m, M\right), \end{aligned}$$

and for $p \in \triangle(m, M)$ *with* $m = M$,

$$U(p) = a(m, M) u(x_m) + b(m, M),$$

where $u : X \to \mathbb{R}$ *is unique up to some positive affine transformation and* $a(m, M), b(m, M) \in \mathbb{R}$ *with* $a(m, M) > 0$.

Step 4. We establish that if the axioms (A1)-(A6) hold, then $u : X \to \mathbb{R}$ is strictly increasing.

Observe that

$$\begin{aligned} p &:= \varepsilon I_1 + (1 - \varepsilon - \eta) I_k + \eta I_n \in \mathcal{P}(1, n) \\ q &:= \varepsilon I_1 + (1 - \varepsilon - \eta) I_{k+1} + \eta I_n \in \mathcal{P}(1, n) \end{aligned}$$

for $1 \leq k \leq n - 1$ so that, by axiom A5,

$$p \succ q.$$

Thus, by lemma 3,

$$\begin{aligned} a(m, M) \sum_{k=1}^{M} p_k u(x_k) + b(m, M) &> a(m, M) \sum_{k=1}^{M} q_k u(x_k) + b(m, M) \Leftrightarrow \\ u(x_{k+1}) &> u(x_k), \end{aligned}$$

which proves the claim. □

Step 5. We demonstrate that our axiom (A7) implies the conditions (6) - (8). Consider at first the case $m < M$ and $m' < M'$. Then there are overlapping preferences between the two sets $\mathcal{P}(m, M)$and $\mathcal{P}(m', M')$ if and only if

$$\max_{\{p \in \mathcal{P}(m,M)\}} U(p) > \min_{\{p' \in \mathcal{P}(m',M')\}} U(p') \tag{16}$$

$$\max_{\{p' \in \mathcal{P}(m',M')\}} U(p') > \min_{\{p \in \mathcal{P}(m,M)\}} U(p). \tag{17}$$

Observe that

$$\begin{aligned} \max_{\{p \in \mathcal{P}(m,M)\}} a(m, M) \sum_{k=m}^{M} p_k u(x_k) + b(m, M) &= a(m, M) [\varepsilon u(x_m) + (1 - \varepsilon) u(x_M)] + b(m, M), \\ \min_{\{p \in \mathcal{P}(m,M)\}} a(m, M) \sum_{k=m}^{M} p_k u(x_k) + b(m, M) &= a(m, M) [(1 - \eta) u(x_m) + \eta u(x_M)] + b(m, M) \end{aligned}$$

so that (16) is equivalent to (6) and (17) is equivalent to (7).

Consider now the case that $m = M$ and $m' < M'$. Then preferences between the two sets $\mathcal{P}(m, M)$and $\mathcal{P}(m', M')$ overlap if and only if

$$\max_{\{p' \in \mathcal{P}(m',M')\}} U(p') \geq U(I_m) \geq \min_{\{p' \in \mathcal{P}(m',M')\}} U(p'),$$

i.e.,

$$\begin{aligned} & a(m', M') [\varepsilon u(x_{m'}) + (1-\varepsilon) u(x_{M'})] + b(m', M') \\ \geq \; & a(m, M) u(x_m) + b(m, M) \\ \geq \; & a(m', M') [(1-\eta) u(x_{m'}) + \eta u(x_{M'})] + b(m', M'), \end{aligned}$$

which is condition (8).

Finally observe that, by Axiom A5, there cannot be overlapping preferences between sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$ where $m = M$ and $m' = M'$.□

Step 6. The strictly increasing function $u : X \to \mathbb{R}$ ensures that monotonicity w.r.t. FOSD is satisfied for all lotteries within a given set $\mathcal{P}(m, M)$. It remains to be shown that monotonicity w.r.t. FOSD is also satisfied for all lotteries from different sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$, i.e., axiom A5, if and only if the conditions (9) and (10) are satisfied.

At first consider any two sets $\mathcal{P}(m, M)$and $\mathcal{P}(m-1, M)$. Given a strictly increasing function $u : X \to \mathbb{R}$, axiom A5 is satisfied for preferences on $\mathcal{P}(m, M) \cup \mathcal{P}(m-1, M)$ if and only if, for all $p \in \mathcal{P}(m, M)$,

$$\begin{aligned} & a(m, M) \sum_{k=m}^{M} p_k u(x_k) + b(m, M) \qquad (18) \\ \geq \; & \sup_{\{p' \in \mathcal{P}(m-1,M) | p \geq_{FOSD} p'\}} a(m-1, M) \sum_{k=m-1}^{M} p'_k u(x_k) + b(m-1, M). \end{aligned}$$

By construction,

$$\begin{aligned} & \sup_{\{p' \in \mathcal{P}(m-1,M) | p \geq_{FOSD} p'\}} a(m-1, M) \sum_{k=m-1}^{M} p'_k u(x_k) + b(m-1, M) \qquad (19) \\ = \; & a(m-1, M) \left[\varepsilon u(x_{m-1}) + (p_m - \varepsilon) u(x_m) + \sum_{k=m+1}^{M} p_k u(x_k) \right] + b(m-1, M) \end{aligned}$$

since the lottery $\varepsilon I_{m-1} + (p_m - \varepsilon) I_m + ... + p_M I_M \in \mathcal{P}(m-1, M)$ gives the greatest utility of all lotteries in $\mathcal{P}(m-1, M)$ that are dominated w.r.t. FOSD by lottery p.

Combining (18) and (19) gives

$$\begin{aligned} & (a(m,M) - a(m-1,M)) \sum_{k=m}^{M} p_k u(x_k) + b(m,M) - b(m-1,M) \\ \geq \quad & a(m-1,M)\, \varepsilon\, (u(x_{m-1}) - u(x_m)) . \end{aligned} \tag{20}$$

Since every lottery $p \in \mathcal{P}(m,M)$ can be represented as a convex combination of lotteries

$$\varepsilon u(x_m) + (1-\varepsilon-\eta)\, u(x_k) + \eta u(x_M) \in \mathcal{P}(m,M)$$

with $m \leq k \leq M$, inequality (20) is satisfied for all $p \in \mathcal{P}(m,M)$ if and only if

$$\begin{aligned} \min_{\{k \in \mathbb{N} | m \leq k \leq M\}} \quad & (a(m,M) - a(m-1,M)) \left[\varepsilon u(x_m) + (1-\varepsilon-\eta)\, u(x_k) + \eta u(x_M)\right] \\ & + b(m,M) - b(m-1,M) \\ \geq \quad & a(m-1,M)\, \varepsilon\, (u(x_{m-1}) - u(x_m)) . \end{aligned}$$

This proves condition (9).

Consider now any two sets $\mathcal{P}(m,M)$and $\mathcal{P}(m,M-1)$. Given a strictly increasing function $u : X \to \mathbb{R}$, axiom A5 is satisfied for preferences on $\mathcal{P}(m,M) \cup \mathcal{P}(m,M-1)$ if and only if, for all $p' \in \mathcal{P}(m,M-1)$,

$$\inf_{\{p \in \mathcal{P}(m,M) | p \geq_{FOSD} p'\}} a(m,M) \sum_{k=m}^{M} p_k u(x_k) + b(m,M) \geq a(m,M-1) \sum_{k=m}^{M-1} p'_k u(x_k) + b(m,M-1) . \tag{21}$$

By construction,

$$\begin{aligned} & \inf_{\{p \in \mathcal{P}(m,M) | p \geq_{FOSD} p'\}} a(m,M) \sum_{k=m}^{M} p_k u(x_k) + b(m,M) \\ = \quad & a(m,M) \left[\sum_{k=m}^{M} p'_k u(x_k) + \left(p'_{M-1} - \eta\right) u(x_{M-1}) + \eta u(x_M) \right] + b(m,M) \end{aligned} \tag{22}$$

since the lottery $p'_m I_m + ... + \left(p'_{M-1} - \eta\right) I_{M-1} + \eta I_M \in \mathcal{P}(m,M)$ gives the smallest utility of all lotteries in $\mathcal{P}(m,M)$ that dominate lottery $p' \in \mathcal{P}(m,M-1)$ w.r.t. FOSD. Combining (21) and (22) gives

$$\begin{aligned} & (a(m,M) - a(m,M-1)) \sum_{k=m}^{M-1} p'_k u(x_k) + b(m,M) - b(m-1,M) \\ \geq \quad & a(m,M)\, \eta\, (u(x_{M-1}) - u(x_M)) . \end{aligned} \tag{23}$$

Since every lottery $p' \in \mathcal{P}(m, M-1)$ can be represented as a convex combination of lotteries

$$\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_{M-1}) \in \mathcal{P}(m, M-1)$$

with $m \leq k \leq M-1$, inequality (23) is satisfied for all $p' \in \mathcal{P}(m, M-1)$ if and only if

$$\begin{aligned} \min_{\{k\in\mathbb{N}|m\leq k\leq M-1\}} \quad & (a(m,M) - a(m,M-1))\left[\varepsilon u(x_m) + (1-\varepsilon-\eta) u(x_k) + \eta u(x_{M-1})\right] \\ & + b(m,M) - b(m,M-1) \\ \geq \quad & a(m,M)\,\eta\left(u(x_{M-1}) - u(x_M)\right). \end{aligned}$$

Finally, notice that monotonicity w.r.t. FOSD between arbitrary sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m', M')$ is satisfied if and only if monotonicity w.r.t. FOSD between

(i) sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m-1, M)$ for all $m, M \in \{1, ..., n\}$ with $m \leq M$ and $m > 1$ and

(ii) sets $\mathcal{P}(m, M)$ and $\mathcal{P}(m, M-1)$ for all $m, M \in \{1, ..., n\}$ with $m \leq M$ and $M > 1$,

is satisfied, since $p \geq_{FOSD} p'$ implies $p \in \mathcal{P}(m, M)$ and $p' \in \mathcal{P}(m', M')$ where $m' \leq m$ and $M' \leq M$.

□□

References

Allais, M. (1979), “The Foundation of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School”. Part II in Allais, M., and O. Hagen [eds.], *Expected Utility Hypotheses and the Allais Paradox*, D. Reidel: Dordrecht etc.

Birnbaum, M. H., and J.B. Navarrete (1998), “Testing Descriptive Utility Theories: Violations of Stochastic Dominance and Cumulative Independence”, *Journal of Risk and Uncertainty* **17**, 49-78.

Chateauneuf, A., Eichberger, J., and S. Grant (2004), “Choice under Uncertainty with the Best and Worst in Mind: Neo-additive Capacities”, *Working Paper* **03-10**, Sonderforschungsbereich 504 Universität Mannheim.

Cohen, M. (1992), “Security Level, Potential Level, Expected Utility: A Three-Criteria Decision Model Under Risk”, *Theory and Decision* **33**, 101-104.

Crawford, V.P. (1990), ”Equilibrium without Independence”, *Journal of Economic Theory* **50**, 127-154.

Dow, J., and S.C.R. Werlang (1994), “Nash Equilibrium under Uncertainty: Breaking down Backward Induction”, *Journal of Economic Theory* **64**, 305-324.

Eichberger, J., and D. Kelsey (1999), “E-Capacities and the Ellsberg Paradox”, *Theory and Decision* **46**, 107-140.

Essid, S. (1997), “Choice under Risk with Certainty and Potential effects: A General Axiomatic Model”, *Mathematical Social Sciences* **34**, 223-247.

Fishburn, P. C. (1988), *Nonlinear Preferences and Utility Theory under Risk*, The John Hopkins University Press: Baltimore, London.

Gilboa, I. (1988), “A Combination of Expected Utility and Maxmin Decision Criteria”, *Journal of Mathematical Psychology* **32**, 405-420.

Harless, D.W., and C.F. Camerer (1994), “The Predictive Utility of Generalized Expected Utility Theories”, *Econometrica* **62**, 1251-1289.

Jaffray, J.-Y. (1988), “Choice under Risk and the Security Factor: An Axiomatic Model”, *Theory and Decision* **24**, 169-200.

Kahneman, D., and A. Tversky (1979), “Prospect Theory: An Analysis of Decision under Risk”, *Econometrica* **47**, 263-291.

Karni, E., and D. Schmeidler (1991), “Utility Theory with Uncertainty” in: Hildenbrand, W. and H. Sonnenschein [eds.], *Handbook of Mathematical Economics*, Vol. IV, North-Holland: Amsterdam etc., 1763-1831.

Lopes, L. L. (1987), “Between Hope and Fear: The Psychology of Risk” in: Berkowitz, L. [ed.], *Advances In Experimental Social Psychology*, Vol. 20, New York: Academic Press, 255-295.

Rabin, M. (2000), “Risk Aversion and Expected-Utility Theory: A Calibration Theorem”, *Econometrica* **68**, 1281-1292.

Safra, Z., and U. Segal (2005), “Are Universal Preferences Possible? Calibration Results for Non-Expected Utility Theories”, mimeo.

Safra, Z., and U. Segal (2006), “Calibration Results for Non-Expected Utility Theory”, mimeo.

Sjöberg, L. (1999), “Consequences of Perceived Risk: Demand for Mitigation”, *Journal of Risk Research* **2**, 129-149.

Sjöberg, L. (2000), “Consequences Matter, ’Risk‘ Is Marginal”, *Journal of Risk Research* **3**, 287-295.

Schmidt, U. (2000), “The Certainty Effect and Boundary Effects with Transformed Probabilities”, *Economics Letters* **67**, 29-33.

Schmidt, U. (2004), “Alternatives to Expected Utility: Some Formal Theories”, in: P.J. Hammond, S. Barberá, and C. Seidl [eds.], *Handbook of Utility Theory* Vol. II, Kluwer, Boston, chapter 15.

Starmer, C. (2000), “Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk”, *Journal of Economic Literature* **38**, 332-382.

Stone, E. R., J. F. Yates, and A. M. Parker (1994), “Risk Communication: Absolute versus Relative Expressions of Low-Probability Risks”, *Organizational Behavior and Human Decision Processes* **60**, 387-408.

Sugden, R. (2004), “Alternatives to Expected Utility: Foundation”, in: P.J. Hammond, S. Barberá, and C. Seidl [eds.], *Handbook of Utility Theory* Vol. II, Kluwer, Boston, chapter 14.

Tversky, A., and D. Kahneman (1992), “Advances in Prospect Theory: Cumulative Representation of Uncertainty”, *Journal of Risk and Uncertainty* **5**, 297-323.

Zimper, A. (2006), “Strategic Games with Security- and Potential Level Players”, *Theory and Decision*, forthcoming.