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A geometric characterization of the nucleolus of the assignment game

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Abstract

Maschler et al. (1979) provide a geometrical characterization for the intersection of the kernel and the core of a coalitional game, showing that those allocations that lie in both sets are always the midpoint of certain bargaining range between each pair of players. In the case of the assignment game, this means that the kernel can be determined as those core allocations where the maximum amount, that can be transferred without getting outside the core, from one agent to his/her optimally matched partner equals the maximum amount that he/she can receive from this partner, also remaining inside the core (Rochford, 1984). We now prove that the nucleolus of the assignment game can be characterized by requiring this bisection property be satisfied not only for optimally matched pairs but also for optimally matched coalitions.

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1. Introduction

A two-sided assignment market consists of two disjoint sets of agents, let us say buyers and sellers or firms and workers, and a non-negative real number associated with each possible partnership between two agents of different sectors, that represents the potential profit of forming that pairing. Assuming transferable utility to share the profits of these partnerships, Shapley and Shubik (1972) introduce the assignment game to model this situation in a coalitional form where only individual coalitions and mixed-pair coalitions are relevant. They show that the core of this game is non-empty and consists of those individually rational allocations that are efficient and satisfy pairwise stability, that is, no buyer-seller pair can form a partnership and produce more than the sum of their payoffs. The core of the assignment game has been widely studied in the literature and, since it very rarely reduces to only one point, it becomes necessary to make a selection inside the core.

An outstanding element of the core for arbitrary coalitional games is the nucleolus (Schmeidler, 1969), which is the unique individually rational and efficient allocation that lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions. This definition can be interpreted as in Maschler et al. (1979), saying that the nucleolus is fair in the sense that it is the result of an arbitrator's desire to minimize the dissatisfaction of the most dissatisfied coalition.

Solymosi and Raghavan (1994) provide an algorithm that computes the nucleolus of an assignment game, based on the fact (already pointed out by Huberman, 1980) that for assignment games, only one-player coalitions and mixed-pair coalitions play a role in the calculation of the nucleolus.[1]

The kernel is another solution concept for arbitrary coalitional games. It was introduced by Davis and Maschler (1965) and it always contains the nucleolus. It is shown in Maschler et al. (1979) that for two games with the same core the intersection of the kernel and the core also coincides. In the same paper, a geometric characterization of those allocations in the intersection of the core and the kernel of a game is given. It is shown there that an outcome that lies in both the kernel and the core is always the midpoint of a certain bargaining range between each pair of players. Each endpoint of this range is in the boundary of the core, representing a maximum demand by one player, in that the other player can find a coalition to support him in resisting any greater demand. This view of the kernel gives it an intuitive interpretation as a "fair division" scheme. However, a similar geometric characterization of the locus of the nucleolus inside the core is not possible for arbitrary games, since there are games with the same core but different nucleolus.[2] Nevertheless, it is known from Núñez (2004) that two assignment games with the same core have the same nucleolus.[3] This suggests the possibility of characterizing the locus of the nucleolus in the core of the assignment game.

The kernel of the assignment game is always included in the core (Granot, 1995; Driessen, 1998). Taking this into account, Driessen (1999) applies to the assignment game the geometric interpretation of the kernel given in Maschler et al. (1979): given

[1]See also Raghavan and Sudhölter (2006) for examples of application of this algorithm.

[2]See Maschler et al. (1979) page 335.

[3]An analysis of different assignment games with the same core can be found in Martínez-de-Albéniz et al. (2011a) and Martínez-de-Albéniz et al. (2011b).

an allocation in the core of the assignment game and an optimally matched pair, one can consider the maximum amount that can be transferred from one member of the pair to her/his partner, the payoff to the remaining agents being unaltered, without getting outside the core. In a kernel element, and for each optimal pair, the transfers of both partners are balanced, that meaning that the kernel element is at a midpoint with respect to certain ranges of the core. The aim of the present paper is to determine which other bisection conditions in terms of transfers are necessary to individualize the nucleolus of the assignment game. Under the assumption that there are as many buyers as sellers we consider, for each core allocation and for each subset of buyers, what is the maximum equal payoff that each of them can transfer to his optimal partner without leaving the core. When this maximum transfer equals the maximum transfer of the coalition of partners we say that the initial core allocation satisfies the bisection property with respect to this coalition of buyers. Then, the nucleolus of the assignment game is characterized as the unique core allocation that has the bisection property with respect to all coalitions.

The paper is organized as follows. Section 2 includes the preliminaries about coalitional games and assignment games. Section 3 contains the geometric characterization of the nucleolus, although its proof is consigned to the appendix. Section 4 concludes.

2. Definitions and notations

Let $N = \{1, 2, ..., n\}$ denote a finite set of players, and 2^N the set of all possible coalitions or subsets of N. The cardinality of coalition S is denoted by $|S|$. Given two coalitions S and T, $S \subseteq T$ denotes inclusion while $S \subset T$ denotes strict inclusion.

A *cooperative game in coalitional form* (a game) is a pair (N, v), where $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is the characteristic function which assigns to each coalition S the worth $v(S)$ it can attain.

Given a game (N, v), a payoff vector is $x \in \mathbb{R}^N$, where x_i stands for the payoff to player $i \in N$. The restriction of x to a coalition S is denoted by $x_{|S}$. An *imputation* is a payoff vector x that is efficient, $\sum_{i\in N} x_i = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations of a game (N, v) is denoted by $I(v)$, and when $I(v) \neq \emptyset$ the game is said to be *essential*. The *excess* of a coalition S at an imputation $x \in I(v)$ is $e^v(S, x) = v(S) - \sum_{i\in S} x_i$.

A *solution concept* defined on the set of games with player set N is a rule that assigns to each such game a subset of efficient payoff vectors. The best known set-solution concept for coalitional games is the core. The *core* of a game (N, v), denoted by $C(v)$, is the set of payoff vectors that are efficient and coalitionally rational, that is, $\sum_{i\in S} x_i \geq v(S)$ for all $S \subseteq N$. A game with a non-empty core is called *balanced*. Given a balanced game, a well known single–valued core selection is *the nucleolus* (Schmeidler, 1969).

Let us define the vector $\theta(x) \in \mathbf{R}^{2^n-2}$ of excesses of all coalitions (different from the grand coalition and the empty set) at x, arranged in a nonincreasing order. Then, the nucleolus of the game (N, v) is the imputation $\eta(v)$ which minimizes $\theta(x)$ with respect to the lexicographic order over the set of imputations: $\theta(\eta(v)) \leq_{Lex} \theta(x)$ for all $x \in I(v)$. This means that, for all $x \in I(v)$, either $\theta(\eta(v)) = \theta(x)$ or $\theta(\eta(v))_1 < \theta(x)_1$ or there exists $k \in \{1, 2, \ldots, 2^n - 3\}$ such that $\theta(\eta(v))_i = \theta(x)_i$ for all $1 \leq i \leq k$ and $\theta(\eta(v))_{k+1} < \theta(x)_{k+1}$.

The *kernel* (Davis and Maschler, 1965) is another set-solution concept for cooperative games. For *zero–monotonic games*,[4] as it is the case of assignment games, the kernel can be described by $\mathcal{K}(v) = \{z \in I(v) \mid s^v_{ij}(z) = s^v_{ji}(z) \text{ for all } i, j \in N, i \neq j\}$, where $s^v_{ij}(z) = \max\{e^v(S,z) | S \subseteq N, i \in S, j \notin S\}$. We will just write $s_{ij}(z)$ when no confusion regarding the game v can arise.

2.1. The assignment model

A *two-sided assignment market* (M, M', A) is defined by a finite set of buyers M, a finite set of sellers M', and a nonnegative matrix $A = (a_{ij})_{(i,j)\in M\times M'}$. The real number a_{ij} represents the profit obtained by the mixed-pair $(i,j) \in M \times M'$ if they trade. Let us assume there are $|M| = m$ buyers and $|M'| = m'$ sellers, and $n = m+m'$ is the cardinality of $N = M \cup M'$.

A *matching* $\mu \subseteq M\times M'$ between M and M' is a bijection from $M_0 \subseteq M$ to $M'_0 \subseteq M'$, such that $|M_0| = |M'_0| = \min\{|M|, |M'|\}$. We write $(i,j) \in \mu$ as well as $j = \mu(i)$ or $i = \mu^{-1}(j)$. The set of all matchings is denoted by $\mathcal{M}(M, M')$. If $m = m'$, the assignment market is said to be square.

A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the assignment market (M, M', A) if for all $\mu' \in \mathcal{M}(M, M')$ we have $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$, and we denote the set of optimal matchings by $\mathcal{M}^*_A(M, M')$.

Shapley and Shubik (1972) associate to any assignment market (M, M', A) a cooperative game in coalitional form, with player set $N = M\cup M'$ and characteristic function w_A, defined by: for $S \subseteq M$ and $T \subseteq M'$, $w_A(S\cup T) = \max\left\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\right\}$, $\mathcal{M}(S,T)$ being the set of matchings between S and T. The core of the assignment game is always non-empty, and it is enough to impose coalitional rationality for one-player coalitions and mixed-pair coalitions:

$$C(w_A) = \left\{(u,v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(N), \\ u_i + v_j \geq a_{ij}, \text{ for all } (i,j) \in M \times M' \end{array}\right\}, \tag{1}$$

where $\mathbb{R}_+$ stands for the set of non-negative real numbers. It follows from (1) that, if μ is an optimal matching, unassigned agents receive null payoff and, moreover,

$$\text{if } (i,j) \in \mu, \text{ then } u_i + v_j = a_{ij}. \tag{2}$$

Since the assignment game has a non-empty core, its nucleolus always lies in the core. Moreover, it can be deduced from Huberman (1980) that only individual coalitions and mixed-pair coalitions need to be taken into account in the computation of the nucleolus of an assignment game. Solymosi and Raghavan (1994) provide an algorithm to compute the nucleolus of the assignment game.

As for the kernel of assignment games, it turns out that it is always included in the core, $\mathcal{K}(w_A) \subseteq C(w_A)$ (Granot, 1995; Driessen, 1998). Moreover, if $(u,v) \in C(w_A)$, then *(a)* $s_{ij}(z) = 0$ whenever $i,j \in M$ or $i,j \in M'$, and *(b)* if $i \in M$ and $j \in M'$, then $s_{ij}(z)$ is always attained at the excess of some individual coalition or mixed–pair coalition:

$$s_{ij}(u,v) = \max_{k\in M'\setminus\{j\}} \{-u_i, a_{ik} - u_i - v_k\}.$$

[4] A game (N, v) is zero-monotonic if for any pair of coalitions S, T, $S \subset T \subseteq N$ it holds $v(S) + \sum_{i\in T\setminus S} v(\{i\}) \leq v(T)$.

As a consequence, given $(u, v) \in C(w_A)$, we get that $(u, v) \in \mathcal{K}(w_A)$ if and only if $s_{ij}(u, v) = s_{ji}(u, v)$ for all (i, j) belonging to all the optimal matchings, since the remaining equalities hold trivially.

By adding dummy players, that is, null rows or columns in the assignment matrix, we can assume from now on, without loss of generality, that the number of sellers equals the number of buyers, since this does not modify the nucleolus payoff of the non-dummy agents.[5]

3. Characterization of the nucleolus

Given an arbitrary coalitional game (N, v), with any core allocation $z \in C(v)$ and any pair of different agents $i, j \in N$, there is associated a non-negative real number $\delta^v_{ij}(z)$ designating the largest amount that can be transferred from player i to player j with respect to the core allocation z while remaining in the core of the game (N, v):

$$\delta^v_{ij}(z) = \max\{\varepsilon \geq 0 \mid z - \varepsilon e^i + \varepsilon e^j \in C(v)\},$$

where, for all $i \in N$, $e^i \in \mathbb{R}^N$ is the vector defined by $e^i_i = 1$ and $e^i_k = 0$ for all $k \neq i$, $k \in N$. This critical number $\delta^v_{ij}(z)$ was introduced by Maschler et al. (1979). For any core element $z \in C(v)$, this number $\delta^v_{ij}(z)$ is related to the excess $s^v_{ij}(z)$ in the definition of the kernel by $\delta^v_{ij}(z) = -s^v_{ij}(z)$. They prove in the aforementioned paper that a bisection property characterizes those elements in the intersection of the kernel and the core: $z \in C(v) \cap \mathcal{K}(v)$ if and only if z is the midpoint of the core segment with extreme points $z - \delta^v_{ij}(z)e^i + \delta^v_{ij}(z)e^j$ and $z + \delta^v_{ji}(z)e^i - \delta^v_{ji}(z)e^j$, for all $i, j \in N$. In this section we introduce a stronger bisection property that characterizes the nucleolus of the assignment game.

Let (M, M', A) be an assignment market with as many buyers as sellers, that is, $|M| = |M'| = m$. For any $R \subseteq M$ or $R \subseteq M'$, the vector $e^R \in \mathbb{R}^m$ stands for $e^R_k = 1$ if $k \in R$ and $e^R_k = 0$ if $k \notin R$. Then, for each $S \subseteq M$ and $T \subseteq M'$, $S, T \neq \emptyset$, we define the largest amount that can be transferred from players in S to players in T with respect to the core allocation (u, v) while remaining in the core of w_A by

$$\delta^{w_A}_{S,T}(u, v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^S, v + \varepsilon e^T) \in C(w_A)\}. \tag{3}$$

Similarly,

$$\delta^{w_A}_{T,S}(u, v) = \max\{\varepsilon \geq 0 \mid (u + \varepsilon e^S, v - \varepsilon e^T) \in C(w_A)\}. \tag{4}$$

We write $\delta_{S,T}(u, v)$ and $\delta_{T,S}(u, v)$, respectively, if no confusion arises regarding the assignment game $(M \cup M', w_A)$.

Notice that if there exists an optimal matching $\mu \in \mathcal{M}^*_A(M, M')$ such that S and T do not correspond each other by this optimal matching ($\mu(S) \neq T$), then $\delta_{S,T}(u, v) = \delta_{T,S}(u, v) = 0$ for all $(u, v) \in C(w_A)$. The reason is that if there exists $i \in S$ such that $\mu(i) \notin T$ (and similarly for $j \in T$ such that $\mu^{-1}(j) \notin S$) we have that the payoff vector $(u', v') = (u - \varepsilon e^S, v + \varepsilon e^T)$ will lie outside the core for all $\varepsilon > 0$, since $u'_i + v'_{\mu(i)} = u_i - \varepsilon + v_{\mu(i)} \neq a_{i\mu(i)}$. This is why we will only consider transfers between coalitions that correspond by an optimal matching.

[5] A detailed argument can be found in Núñez (2004).

Definition 1. *Let* (M, M', A) *be an assignment market,* $\mu \in \mathcal{M}_A^*(M, M')$ *and* $S \subseteq M$, $S \neq \emptyset$. *The core allocation* (u, v) has the S-bisection property *with respect to* μ *if and only if* $\delta_{S,\mu(S)}(u, v) = \delta_{\mu(S),S}(u, v)$.

Both for theoretical and practical purposes, it will be useful to have an explicit expression of the critical numbers $\delta_{S,T}(u, v)$ when $S \subseteq M$ and $\mu(S) = T$ by some optimal matching μ. Given $(u, v) \in C(w_A)$, if we want the allocation $(u', v') = (u - \varepsilon e^S, v + \varepsilon e^T)$ to remain in the core of the assignment game (see expression (1)), only the inequalities $u_i - \varepsilon \geq 0$ for all $i \in S$ and $u_i - \varepsilon + v_j \geq a_{ij}$ for all $i \in S$ and all $j \in M' \setminus T$ must hold. This means that, for all $(u, v) \in C(w_A)$, given a non-empty coalition $S \subseteq M$ and $\mu \in \mathcal{M}_A^*(M, M')$,

$$\delta_{S,\mu(S)}(u, v) = \min_{i \in S, j \in M' \setminus \mu(S)} \{u_i, u_i + v_j - a_{ij}\}. \tag{5}$$

and, similarly,

$$\delta_{\mu(S),S}(u, v) = \min_{i \in M \setminus S, j \in \mu(S)} \{v_j, u_i + v_j - a_{ij}\}. \tag{6}$$

At this point it is worth to remark that, by Maschler et al. (1979) and Driessen (1999), the kernel of the assignment game is the set of core allocations satisfying the $\{i\}$-bisection property for all $i \in M$. Since the nucleolus belongs to the kernel, it satisfies this property. What we state in the next theorem is that the nucleolus of the assignment game can be characterized by the S-bisection property, for all $S \subseteq M$, $S \neq \emptyset$, and with respect to any optimal matching μ.

Theorem 1. *Let* (M, M', A) *be a square assignment market. The nucleolus is the unique core allocation satisfying the* S*-bisection property, for all* $S \subseteq M$, $S \neq \emptyset$. *Formally, if* $(u, v) \in C(w_A)$ *and* $\mu \in \mathcal{M}_A^*(M, M')$, *then*

$$(u, v) = \eta(w_A) \textit{ if and only if } \delta_{s,\mu(S)}(u, v) = \delta_{\mu(S),S}(u, v) \textit{ for all } S \subseteq M, S \neq \emptyset.$$

The proof of the theorem is a bit technical and it is consigned to the appendix. The idea behind the proof is the following. Take an optimal matching μ and a coalition $S \cup \mu(S)$, and consider the core segment obtained from the nucleolus by making an equal transfer between agents in each pair $(i, \mu(i))$ with $i \in S$, the payoff of the other agents remaining fixed. We compute the excesses of all mixed-pair and individual coalitions at an arbitrary element of the segment. Most of these excesses are constant along the segment and thus do not play a role when computing the lexicographic minimum of the non-increasingly ordered vector of excesses over the segment. With the non-constant excesses we compute this lexicographic minimum and find it is the midpoint of the segment. Hence, the nucleolus has the S-bisection property.

Let us stress that Theorem 1 is useful to check if a given allocation in the core of an assignment game $(M \cup M', w_A)$ is in fact its nucleolus. Moreover, whenever several optimal matchings exist, the difficulty of the problem may be reduced. It is enough to check that the given allocation satisfies the S-bisection property for all coalitions $S \subseteq M$ that have the same image by all optimal matchings $\mu \in \mathcal{M}_A^*(M, M')$. Formally, if $\mathcal{S} = \{\, S \subseteq M \mid \mu(S) = \mu'(S) \text{ for all } \mu,\ \mu' \in \mathcal{M}_A^*(M, M')\}$, then

$$\eta(w_A) = \left\{(u, v) \in C(w_A) \,|\, \delta_{S,\,\mu(S)}(u, v) = \delta_{\mu(S),\,S}(u, v) \text{ for all } S \in \mathcal{S}\right\}. \tag{7}$$

Also, after Theorem 1 one may wonder if the nucleolus of the assignment game could be characterized by imposing the bisection property for some smaller subset of coalitions of M. The next example shows that imposing the bisection property for individual coalitions and the grand coalition is not enough to obtain the nucleolus.

Example 1. *Let it be the assignment market with set of buyers $M = \{1,2,3\}$, set of sellers $M' = \{4,5,6\}$ and defined by matrix*

$$A = \begin{pmatrix} 7 & 6 & 3 \\ 5 & 4 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$

There are two optimal matchings: $\mu_1 = \{(1,4),(2,5),(3,6)\}$ and $\mu_2 = \{(1,5),(2,4),(3,6)\}$. Thus, to check that a given allocation is the nucleolus we only need to verify that it satisfies the bisection property for those $S \subseteq M$ such that $\mu_1(S) = \mu_2(S)$. In this case, $\eta(w_A) = (3.5, 1.5, 0.5; 3.5, 2.5, 0.5)$ since it satisfies the bisection property with respect to coalitions $\{3\}$, $\{1,2\}$ and $\{1,2,3\}$. Moreover, if we consider the core element $(u,v) = (3,1,0.5;4,3,0.5)$ we realize that $\delta_{\{3\},\{6\}}(u,v) = 0.5 = \delta_{\{6\},\{3\}}(u,v)$, $\delta_{M,M'}(u,v) = 0.5 = \delta_{M',M}(u,v)$, but $\delta_{\{1,2\},\{4,5\}}(u,v) = 0.5$ while $\delta_{\{4,5\},\{1,2\}}(u,v) = 1.5$.

4. Concluding remarks

The kernel of an assignment game is a subset of the core and it is characterized as the set of core allocations that satisfy the bisection property for each pair in an optimal matching. This means that a kernel element is the midpoint of all maximal core segments obtained by transfers between a pair of optimally matched partners. It is also known (Granot and Granot, 1992) that the kernel of an assignment game need not be a convex set, so finding a geometric characterization of one particular kernel point (the nucleolus) seems an interesting problem. The presented characterization of the nucleolus is a strengthening of the bisection property of kernel points, it only needs to be required for some additional line segments inside the core, each obtained by transferring the same payoffs from a group of players in one side of the market to their optimally matched partners. This geometric characterization is specially useful to check if a given core element is the nucleolus of the assignment game.

Appendix. Proof of Theorem 1

Proof. We first prove that the nucleolus satisfies the S-bisection property for all $S \subseteq M$, $S \neq \emptyset$. Let us denote (for short) by $\eta = (\eta_{|M}, \eta_{|M'})$ the nucleolus $\eta(w_A)$, and let us fix a coalition $S \subseteq M$, $S \neq \emptyset$. We now consider the core segment $[\eta_S^-, \eta_S^+]$ that can be obtained from η by means of doing equal transfers from agents in S to agents in $\mu(S)$ (and reciprocally). By (3) and (4), the extreme points of the segment are

$$\begin{aligned} \eta_S^- &= \left(\eta_{|M} - \delta_{S,\mu(S)}(\eta)e^S, \eta_{|M'} + \delta_{S,\mu(S)}(\eta)e^{\mu(S)}\right) \text{ and} \\ \eta_S^+ &= \left(\eta_{|M} + \delta_{\mu(S),S}(\eta)e^S, \eta_{|M'} - \delta_{\mu(S),S}(\eta)e^{\mu(S)}\right). \end{aligned}$$

For simplicity of notation we will omit the subscript and write η^- and η^+.

Let $K = \delta_{S,\mu(S)}(\eta) + \delta_{\mu(S),S}(\eta)$, then the segment $[\eta^-, \eta^+]$ can be described as the set of those payoff vectors $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ for which there exists $\varepsilon_{(u,v)} \in [0, K]$ such that

$$\begin{array}{ll} u_i = \eta_i^+ - \varepsilon_{(u,v)} \text{ for all } i \in S, & u_i = \eta_i^+ \text{ for all } i \in M \setminus S \\ v_j = \eta_j^+ + \varepsilon_{(u,v)} \text{ for all } j \in \mu(S), & v_j = \eta_j^+ \text{ for all } j \in M' \setminus \mu(S). \end{array} \tag{8}$$

Note, from the definition of η^+ and η^-, that $\varepsilon_{\eta^+} = 0$ and $\varepsilon_{\eta^-} = K$. Moreover, the nucleolus η is obtained taking $\varepsilon_\eta = \delta_{\mu(S),S}$.

Since by definition the vector of ordered excesses (with respect to individual and mixed-pair coalitions) of the nucleolus, $\theta(\eta)$, satisfies $\theta(\eta) \leq_L \theta(u, v)$ for all $(u, v) \in C(w_A)$, we have, in particular, that $\theta(\eta)$ lexicographically minimizes the vector of excesses $\theta(u, v)$ over $[\eta^-, \eta^+]$. We will see that η satisfies the equation $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$ or, equivalently, $\varepsilon_\eta = \frac{K}{2}$.

If the segment $[\eta^-, \eta^+]$ reduces to a single point, we are done. Otherwise, let us fix an arbitrary allocation $(u, v) \in [\eta^-, \eta^+]$ and analyze first the excesses of mixed-pair coalitions at (u, v):

- If $(i, j) \in S \times \mu(S)$, and taking (8) into account, there exists $\varepsilon_{(u,v)} \in [0, K]$ such that

$$e(\{i, j\}, (u, v)) = a_{ij} - u_i - v_j = a_{ij} - (\eta_i^+ - \varepsilon_{(u,v)}) - (\eta_j^+ + \varepsilon_{(u,v)}) = a_{ij} - \eta_i^+ - \eta_j^+ = e(\{i, j\}, \eta^+). \tag{9}$$

- Similarly, if $(i, j) \in (M \setminus S) \times (M' \setminus \mu(S))$, then

$$e(\{i, j\}, (u, v)) = a_{ij} - u_i - v_j = a_{ij} - \eta_i^+ - \eta_j^+ = e(\{i, j\}, \eta^+). \tag{10}$$

Since the excesses of the above coalitions are constant on $[\eta^-, \eta^+]$ they need not be considered in the lexicographically minimization of the vector of excesses $\theta(u, v)$ over the segment $[\eta^-, \eta^+]$. Thus, the relevant excesses of mixed-pair coalitions are those with either one agent in S and the other one in $M' \setminus \mu(S)$ or one agent in $M \setminus S$ and the other one in $\mu(S)$:

- If $(i, j) \in S \times (M' \setminus \mu(S))$, by (8) and the fact that $\eta_i^+ = \eta_i^- + K$ and $\eta_j^+ = \eta_j^-$, we have

$$e(\{i, j\}, (u, v)) = a_{ij} - u_i - v_j = a_{ij} - (\eta_i^+ - \varepsilon_{(u,v)}) - \eta_j^+ = a_{ij} - \eta_i^- - \eta_j^- - K + \varepsilon_{(u,v)} \leq -K + \varepsilon_{(u,v)}. \tag{11}$$

- Similarly, if $(i, j) \in (M \setminus S) \times \mu(S)$, then, by (8),

$$e(\{i, j\}, (u, v)) = a_{ij} - u_i - v_j = a_{ij} - \eta_i^+ - (\eta_j^+ + \varepsilon_{(u,v)}) \leq -\varepsilon_{(u,v)}. \tag{12}$$

Let us now analyze the excesses of individual coalitions at the allocation $(u, v) \in [\eta^-, \eta^+]$. Notice that if $i \in M \setminus S$, by (8) we have $e(\{i\}, (u, v)) = -\eta_i^+$ and similarly, if $j \in M' \setminus \mu(S)$ it holds $e(\{j\}, (u, v)) = -\eta_j^+$. Again, since the excesses of the above individual coalitions are constant on $[\eta^-, \eta^+]$ they need not be taken into account in the computation of the lexicographic minimum of the vector of ordered excesses over $[\eta^-, \eta^+]$. It remains to consider the excesses of individual coalitions at (u, v) with $i \in S$ or $j \in \mu(S)$:

- If $i \in S$, then by (8) and taking into account that $\eta_i^+ = \eta_i^- + K$, we have

$$e(\{i\},(u,v)) = -(\eta_i^+ - \varepsilon_{(u,v)}) = -\eta_i^- - K + \varepsilon_{(u,v)} \leq -K + \varepsilon_{(u,v)}. \quad (13)$$

- If $j \in \mu(S)$, by (8) we get

$$e(\{j\},(u,v)) = -(\eta_j^+ + \varepsilon_{(u,v)}) \leq -\varepsilon_{(u,v)}. \quad (14)$$

Now, by definition of η^+, there must be some core constraint that is tight at the extreme point η^+ and not tight at all other points of the segment $[\eta^-, \eta^+]$. If this core constraint were related to a coalition $\{i\}$ with $i \in S$, then $\eta_i^+ = 0$ would imply, by (8), $u_i = -\varepsilon_{(u,v)} \geq 0$ or, equivalently, $\varepsilon_{(u,v)} = 0$, for all $(u,v) \in [\eta^-, \eta^+]$, in contradiction with the assumption that $[\eta^-, \eta^+]$ is not a singleton. Also, if the constraint that is tight at η^+ is $\{i,j\}$ with $(i,j) \in S \times (M' \setminus \mu(S))$ we have, by the second equality in (11), that for all $(u,v) \in [\eta^-, \eta^+]$, $e(\{i,j\},(u,v)) = \varepsilon_{(u,v)}$ and since excesses at core allocations are always non-positive we obtain $\varepsilon_{(u,v)} = 0$ for all $(u,v) \in [\eta^-, \eta^+]$, which implies, as before, a contradiction. This means that either:

a) There exists $(i*, j*) \in (M \setminus S) \times \mu(S)$ such that $\eta_{i*}^+ + \eta_{j*}^+ = a_{i*j*}$, and then for all $(u,v) \in [\eta^-, \eta^+]$, and taking (12) and (14) into account, we have

$$e(\{i*, j*\},(u,v)) = a_{i*,j*} - \eta_{i*}^+ - (\eta_{j*}^+ + \varepsilon_{(u,v)}) = -\varepsilon_{(u,v)} \geq e(T,(u,v)), \quad (15)$$

for all $T = \{i,j\}$ with $(i,j) \in (M \setminus S) \times \mu(S)$ and all $T = \{j\}$ with $j \in \mu(S)$.

b) Or there exists $j* \in \mu(S)$ with $\eta_{j*}^+ = 0$, and then for all $(u,v) \in [\eta^-, \eta^+]$, again taking (12) and (14) into account, we have

$$e(\{j*\},(u,v)) = -\varepsilon_{(u,v)} \geq e(T,(u,v)), \quad (16)$$

for all $T = \{i,j\}$ with $(i,j) \in (M \setminus S) \times \mu(S)$ and all $T = \{j\}$ with $j \in \mu(S)$.

Similarly, there must be some core constraint that is tight at η^- and not tight at all other points of $[\eta^-, \eta^+]$. If this core constraint were related to a coalition $\{j\}$ with $j \in \mu(S)$, then $\eta_j^- = 0$ would imply, by (8), $v_j = \eta_j^+ + \varepsilon_{(u,v)} = \eta_j^- - K + \varepsilon_{(u,v)} = -K + \varepsilon_{(u,v)}$. Since $\varepsilon_{(u,v)} \in [0, K]$ and $v_j \geq 0$, we have $v_j = 0$ for all $(u,v) \in [\eta^-, \eta^+]$. Also, if the constraint that is tight at η^- is $\{i,j\}$ with $(i,j) \in (M \setminus S) \times \mu(S)$ we have by (12), and the fact that $\eta_i^- = \eta_i^+$ and $\eta_j^- = \eta_j^+ + K$, $e(\{i,j\},(u,v)) = a_{ij} - \eta_i^+ - (\eta_j^+ + \varepsilon_{(u,v)}) = a_{ij} - \eta_i^- - (\eta_j^- - K + \varepsilon_{(u,v)}) = K - \varepsilon_{(u,v)} \leq -\varepsilon_{(u,v)}$, for all $(u,v) \in [\eta^-, \eta^+]$, which implies $K = 0$ or, equivalently, the reduction of the segment $[\eta^-, \eta^+]$ to only one point, in contradiction with our assumption. This means that either:

c) There exists $(i*, j*) \in S \times (M' \setminus \mu(S))$ such that $\eta_{i*}^- + \eta_{j*}^- = a_{i*j*}$, and then for all $(u,v) \in [\eta^-, \eta^+]$, and taking (11) and (13) into account, we have

$$e(\{i*, j*\},(u,v)) = -K + \varepsilon_{(u,v)} \geq e(T,(u,v)), \quad (17)$$

for all $T = \{i,j\}$ with $(i,j) \in S \times (M' \setminus \mu(S))$ and all $T = \{i\}$ with $i \in S$.

d) Or there exists $i* \in S$ with $\eta^-_{i*} = 0$, and then for all $(u,v) \in [\eta^-, \eta^+]$, again taking (11) and (13) into account, we have

$$e(\{i*\}, (u,v)) = -K + \varepsilon_{(u,v)} \geq e(T, (u,v)), \tag{18}$$

for all $T = \{i,j\}$ with $(i,j) \in S \times (M' \setminus \mu(S))$ and all $T = \{i\}$ with $i \in S$.

To sum up, let us denote by $\mathcal{C}$ the set of coalitions that are to be taken into account for the lexicographic minimization of the vector of ordered excesses over the segment $[\eta^-, \eta^+]$. That is

$$\mathcal{C} = \{\{i\} \mid i \in S\} \cup \{\{j\} \mid j \in \mu(S)\} \cup \{\{i,j\} \mid (i,j) \in (S \times (M' \setminus \mu(S))) \cup ((M \setminus S) \times \mu(S))\}.$$

Then, for all $(u,v) \in [\eta^-, \eta^+]$ we have

$$\max_{S \in \mathcal{C}} e(S, (u,v)) = \max\{-\varepsilon_{(u,v)}, -K + \varepsilon_{(u,v)}\}$$

and thus

$$\min_{(u,v) \in [\eta^-, \eta^+]} \max_{S \in \mathcal{C}} e(S, (u,v))$$

is attained at the point $(u,v) \in [\eta^-, \eta^+]$ such that $-\varepsilon_{(u,v)} = -K + \varepsilon_{(u,v)}$, that is $\varepsilon_{(u,v)} = \frac{K}{2}$. Since the nucleolus lexicographically minimizes the vector of excesses over the segment $[\eta^-, \eta^+]$ we deduce that $\varepsilon_\eta = \frac{K}{2}$ and thus, since $\varepsilon_\eta = \delta_{\mu(S),S}(\eta)$, we have $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$, which proves the S-bisection property of the nucleolus with respect to the arbitrary coalition $S \subseteq M$.

To conclude the proof we must see that a core allocation different from the nucleolus fails to satisfy the S-bisection property for some coalition $S \subseteq M, S \neq \emptyset$. Let us consider $z \in C(w_A)$ such that $z \neq \eta$. Then, either there exists $i \in M$ such that $z_i > \eta_i$ or there exists $i \in M$ such that $z_i < \eta_i$. In the first case, there exists a non-empty coalition $S \subseteq M$ such that $z_i > \eta_i$ for all $i \in S$ and $z_i \leq \eta_i$ for all $i \in M \setminus S$. As a consequence, it follows from (2) that $z_j < \eta_j$ for all $j \in \mu(S)$ and $z_j \geq \eta_j$ for all $j \in M' \setminus \mu(S)$. Then, making use of expressions (5) and (6),

$$\begin{aligned}
\delta_{S,\mu(S)}(z) &= \min_{i \in S, j \in M' \setminus \mu(S)} \{z_i, z_i + z_j - a_{ij}\} > \min_{i \in S, j \in M' \setminus \mu(S)} \{\eta_i, \eta_i + \eta_j - a_{ij}\} \\
&= \min_{j \in \mu(S), i \in M \setminus S} \{\eta_j, \eta_i + \eta_j - a_{ij}\} > \min_{j \in \mu(S), i \in M \setminus S} \{z_j, z_i + z_j - a_{ij}\} = \delta_{\mu(S),S}(z),
\end{aligned}$$

where the second equality follows from the fact that $\delta_{S,\mu(S)}(\eta) = \delta_{\mu(S),S}(\eta)$. Then, $\delta_{S,\mu(S)}(z) > \delta_{\mu(S),S}(z)$ implies that z does not satisfies the S-bisection property.

The proof in the second case, that is when there exists $i \in M$ such that $z_i < \eta_i$, is analogous. □

References

Davis, M. and M.Maschler (1965) "The kernel of a cooperative game" *Naval Research Logistics Quarterly* **12**, 223–259.

Driessen, T.S.H. (1998) "A note on the inclusion of the kernel in the core of the bilateral assignment game" *International Journal of Game Theory* **27**, 301–303.

Driessen, T.S.H. (1999) "Pairwise-bargained consistency and game theory: the case of a two-sided firm" in *Topics in Mathematical Economics and Game Theory: Essays in honor of Robert J. Aumann* by Myrna H. Wooders Ed., Fields Institute Communications Series **23**, 65–82. The Fields Institute for Research in Mathematical Sciences, Toronto, Canada.

Granot, D. (1995) "On a new bargaining set for cooperative games" Working Paper No. 95 MSC-005. Faculty of Commerce and Business Administration, The University of British Columbia, Vancouver, B.C., Canada.

Granot, D. and F. Granot (1992) "On some network flow games" *Mathematics of Operations Research* **17**, 792–841.

Huberman, G. (1980) "The nucleolus and the essential coalitions" in *Analysis and Optimization of Systems, Lecture Notes in Control and Information Systems* **28**, 417–422.

Maschler, M., Peleg, B. and L.S. Shapley (1979) "Geometric properties of the kernel, nucleolus and related solution concepts" *Mathematics of Operations Research* **4**, 303–338.

Núñez, M. (2004) "A note on the nucleolus and the kernel of the assignment game" *International Journal of Game Theory* **33**, 55–65.

Martínez-de-Albéniz, F., Núñez, M. and C. Rafels (2011a) "Assignment markets that are uniquely determined by their core" *European Journal of Operational Research* **212**, 529–534.

Martínez-de-Albéniz, F., Núñez, M. and C. Rafels (2011b) "Assignment markets with the same core" *Games and Economic Behavior* **73**, 553–563.

Raghavan, T.E.S. and P. Sudhölter (2006) "On Assignment Games" in *Advances in Dynamic Games. Annals of the International Society of Dynamic Games* A. Haurie et al. Eds. Springer, 179–193.

Rochford, S.C. (1984) "Symmetrically pairwise-bargained allocations in an assignment market" *Journal of Economic Theory* **34**, 262–281.

Shapley, L.S. and M. Shubik (1972) "The Assignment Game I: The Core" *International Journal of Game Theory* **1**, 111–130.

Schmeidler, D. (1969) "The nucleolus of a characteristic function game" *SIAM Journal of Applied Mathematics* **17**, 1163–1170.

Solymosi, T. and T.E.S. Raghavan (1994) "An algorithm for finding the nucleolus of assignment games" *International Journal of Game Theory* **23**, 119–143.