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Estimation of Forward-Looking Relationships in Closed Form: An Application to the New Keynesian Phillips Curve*

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Abstract

We illustrate the importance of placing model-consistent restrictions on expectations in the estimation of forward-looking Euler equations. In two-stage limited-information settings where first-stage estimates are used to proxy for expectations, parameter estimates can differ substantially, depending on whether these restrictions are imposed or not. This is shown in an application to the New Keynesian Phillips Curve (NKPC), first in a Monte Carlo exercise, and then on actual data. The closed-form (CF) estimates require by construction that expectations of inflation be model-consistent at all points in time, while the difference-equation (DE) estimates impose no model discipline on expectations. Between those two polar extremes there is a wide range of alternative DE specifications, based on the same dynamic relationship, that explicitly impose model restrictions on expectations for a finite number of periods. In our application, these last estimates quickly converge to the CF estimates, and illustrate that the DE estimates in Cogley and Sbordone (2008) are not robust to imposing modest model requirements on expectations. In particular, our estimates show that the NKPC is not purely forward-looking, and thus that time-varying trend inflation is insufficient to explain inflation persistence.

JEL Classification: E12, E31, E52

Keywords: closed form, model-consistent expectations, New Keynesian Phillips curve, forward-looking Euler equation, time-varying trend inflation.

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1 Introduction

Modern macroeconomic models have an important forward-looking dimension, with current variables depending, among other factors, on expected future economic conditions. Estimation of these intertemporal relationships is often carried out with the relationships expressed in Euler-equation form. In this form, a variable typically depends on its expected future, its past, and its driving process. For example, an aggregate supply relationship in Euler form can be expressed as a hybrid Phillips curve where inflation is a function of expected future (next-period) inflation, past inflation, and marginal costs (the driving process). In a limited-information estimation framework, the crucial issue is how to use the available data to properly identify the role of the forward-looking elements. Consider a two-stage estimation procedure where expectations are constructed first by means of a simple unrestricted linear projection on past data, and then substituted into the dynamic relationship of interest. At this point, it is important to distinguish between two polar approaches to estimation. At one extreme, expectations are left unconstrained. In particular, the projections are taken as the only information available on expectations and the econometrician directly estimates an Euler equation (in practice, a difference equation). At the other extreme, the evolution of expectations is itself constrained to obey the Euler equation. This latter approach restricts the expected value of the variable of interest to be governed by the same Euler equation one period forward. Taking into account this fact introduces expectations of the variable of interest shifted two periods forward, which in turn are governed by the Euler equation two periods forward, and so on. This recursive procedure constrains the evolution of expectations to be model-consistent ad infinitum.[1] A particular application of this approach is the estimation of a model's closed form. A closed-form solution to the Euler equation embeds by construction all model-based restrictions on the evolution of expectations. For simplicity, we refer to model estimates based on a closed-form solution as CF estimates, and to those based on the unrestricted Euler equation as DE (difference equation) estimates.

The CF estimates are of particular interest because they can be thought of as the limiting case in which all model restrictions on the forward-looking part of the model are imposed at the estimation stage. However, one can think of intermediate cases between the DE and the CF estimates where model-consistency of expectations is imposed only for a finite number of periods.[2] In other words,

[1] Such a forward iteration requires determinacy of the dynamic system's equilibrium.

[2] Our terminology should not mask the fact that these intermediate estimates are difference equations themselves.

the iteration described above when imposing model discipline on expectations is not carried out ad infinitum as in the closed form. These intermediate cases are especially relevant as they allow one to assess the extent of model discipline that is needed to obtain estimates that are close to the CF specification's results. In addition, they are particularly convenient when the closed form of an Euler equation is difficult to compute.

In this paper we analyze the differences that arise from estimating a forward-looking relationship when different degrees of model discipline are imposed on expectations. Of particular interest is the contrast of results when the relationship is expressed in its DE form versus the CF representation. We show in Monte Carlo exercises and in an empirical application that in small samples (of the size typically available for macro time-series) DE and CF estimates can differ substantially. Moreover, in our analysis a very small amount of model discipline imposed on expectations is sufficient to yield estimates that are very close to their CF counterparts. In other words, the intermediate cases between the polar DE and CF specifications are much more congruent with the CF estimates than with the DE estimates, even if they do not rely on explicitly solving the DE form.

To explain the differences in the estimates, we start from a first-stage estimation where expectations of the variable of interest are generated by means of some forecasting rule. This forecasting rule is the same unconstrained reduced form for all specifications that we consider (DE, CF and all intermediate cases). It is estimated in the first stage as an unconstrained vector autoregression. Given the estimated forecasting rule, the second stage uses minimum-distance methods to estimate the deep parameters of the intertemporal relationship of interest. Since the first-stage estimates are the same across all specifications, this two-step estimation isolates the impact on the estimates from adding restrictions on the way expectations enter the relationship at hand.

When the first-stage reduced form is identical to the "true" data generating process the DE and the CF specifications (and all the intermediate cases) estimated in the second stage should be the same. Yet it is fair to assume that any unconstrained reduced form for the "true" process underlying actual data is bound to be, at best, an approximation. In this case, the DE and CF specifications (and all the intermediate ones) are not equivalent.[3] The way in which these specifications differ is that the DE form does not exploit model restrictions on expectations. Instead, all other specifications impose at least some model discipline on expectations. We show that these restrictions are equivalent to appending additional moment conditions to the DE specification. As

[3] More specifically, the estimates obtained from each specification will differ as long as the estimation is over-identified. This is the most relevant case in empirical applications.

long as the dynamic relationship we are estimating provides an accurate description of the data, placing these additional constraints results in more precise estimates. In this respect, our paper links the estimation problem of Euler equations to the literature on the gains in efficiency that result from adding model constraints to minimum distance estimators (Kodde, Palm, and Pfann, 1990, and Hayashi, 2000, ch. 3). This issue has already been explored in seminal work by Hansen and Sargent (1982). Our analytical approach, however, is novel in that it shows how model discipline can always be imposed through re-weighting of the very same cross-equation restrictions that characterize the unstructured minimum-distance problem for the DE case. Thus, the econometrician can require any degree of model consistency between (and including) the two polar cases without changing the dimension of the minimum-distance problem.

The empirical application that we consider, and which also informs the setup of our Monte Carlo exercise, is a New Keynesian Phillips curve (NKPC) estimated on U.S. data. The Monte Carlo results show that estimates embedding model-consistent restrictions on inflation expectations (with the CF estimates as a limiting case) are much more precise than the DE estimates and more robust to a form of misspecification affecting the indexation mechanism that appears to be empirically relevant. For the estimation on actual data, we use a NKPC with time-varying coefficients to account for changes in the inflation trend. Using this model, Cogley and Sbordone (2008) report DE estimates that imply that the NKPC is purely forward-looking, with no role for lagged inflation in explaining inflation dynamics.

Estimating a DE form of the NKPC that allows for two lags of inflation indexation (and hence nests Cogley and Sbordone's specification with a single lag) already produces drastically different estimates.[4] Nonetheless, even abstracting from this misspecification, our main empirical finding is that imposing model-consistent restrictions on inflation expectations produces estimates of the NKPC parameters that are far away from Cogley and Sbordone's estimates, both from a statistical and an economic standpoint. According to the CF estimates, lagged and expected future inflation enter the Euler equation form of the NKPC with rather similar weights. Another important dimension in which the DE and CF estimates differ is the frequency with which prices are readjusted optimally. In the DE specification this frequency is estimated at 3.9 months, while in the CF specification it is close to one year.[5] In addition, we show that a modest amount of model discipline

[4] In a quarterly model, allowing for two lags of inflation indexation implies that information about inflation covering the past six months is potentially relevant to characterize the indexation process.

[5] Our CF estimates find an important role for indexation to past inflation. In the presence of indexation, firms change prices every period (some are re-optimizing, while others are not). Hence the frequency of price re-optimization

on expectations is already enough to significantly alter the estimated model dynamics of inflation relative to the DE estimates. Specifically, imposing just four quarters of explicit model discipline on inflation expectations produces estimates that are very close to their CF counterparts.

Overall, our results underscore that U.S. inflation persistence within a widely used NKPC framework cannot be explained entirely by time-varying trend inflation and the persistence of the driving process. In particular, lagged inflation contributes importantly to inflation dynamics. These findings apply not only to the full 1960-to-2003 sample period, but also to the post-1983 sample.

There is now a large literature on estimating NKPC models.[6] The forward-looking component in the NKPC is usually derived from a micro-founded problem in which firms cannot reset prices optimally in every period (Calvo, 1983) or face convex adjustment costs (Rotemberg, 1982). Firms then take into account not only current market conditions, but also expected future conditions when setting prices optimally. This mechanism alone provides no role for lagged inflation in the NKPC. But in actual data, inflation can be highly persistent, and purely forward-looking versions of the NKPC often fit the data worse than "hybrid" versions where current inflation depends not just on expected future inflation, but also on its own past. The dependence on past inflation is frequently introduced through some ad-hoc pricing mechanism (for example, indexation or "rule-of-thumb" price setters). For many purposes this is unsatisfactory, as the mechanism lacks microfoundations, though sluggish nominal adjustment has been related to learning and information processing constraints. The work by Cogley and Sbordone (2008) explores the possibility that the persistence in the inflation process is due to a time-varying inflation target rather than to some ad-hoc element in firms' price-setting decisions.[7,8] There is considerable evidence that the Federal Reserve's inflation target has not remained constant over time (Ireland, 2007), and this raises the possibility that variations in the target are an important source of inflation persistence. The empirical findings in Cogley and Sbordone do indeed favor a purely forward-looking Phillips curve where inflation persistence results entirely from a time-varying inflation target. These findings, therefore, are consistent with a price-setting framework that does not rely on ad-hoc, backward-looking price adjustment.

A purely forward-looking NKPC has important implications for inflation dynamics. As long

in this case cannot be directly compared to micro-evidence based on price changes alone.

[6] See, among others, Galí and Gertler (1999), Galí, Gertler, and López-Salido (2005), Rudd and Whelan (2005, 2006), and Sbordone (2002).

[7] The usual ad-hoc assumption of Calvo (1983) pricing notwithstanding.

[8] Kozicki and Tinsley (2002) is the first study to explicitly consider time-varying trend inflation when estimating a NKPC. Cogley and Sbordone (2008), however, provide a full derivation of the NKPC with time-varying inflation from the firms' optimization problem, and their empirical exercise is tightly linked to the theoretical model.

as the inflation target is not moving, inflation is as persistent as its driving process. For example, consider a situation in which real marginal costs drop below their steady-state level and are expected to revert to the steady state in one year. Inflation then drops immediately and returns to its target level in one year, in sync with real marginal costs.[9] The same is true for a markup shock: a one-period markup shock has only a one-period effect on inflation. Instead, when inflation is not purely forward-looking, the adjustment of inflation to movements in real marginal costs or to markup shocks is slower. A one-period negative markup shock, for example, results in lower current inflation and, given the dependence of inflation on its own past, this lowers inflation in the next period. Indeed, inflation converges only asymptotically to the target, despite the one-time shock. These differences in inflation dynamics can have substantial implications for the design of optimal monetary policy.[10] In the current U.S. economic environment in which real marginal costs are well below normal and are expected to increase only gradually, the difference in the projected path for inflation can be large. In this situation, the extent of monetary policy accommodation is critically dependent on the degree of backward-looking behavior that characterizes inflation. In the purely forward-looking NKPC, the decline in inflation may well be modest, but when inflation is not purely forward-looking the pull of depressed marginal costs on inflation can be significant.[11]

The rest of the paper proceeds as follows. In section 2 we describe the DE, CF, and intermediate specifications in the context of a simple NKPC model and discuss the two-stage estimation procedure. We then provide an explanation for the gain in efficiency from estimating specifications that impose model-consistent constraints on expectations and provide some Monte Carlo evidence. In section 3 we consider a NKPC model that allows for time-varying trend inflation, and we compare estimates from the different specifications using actual U.S. data. Section 4 offers some concluding remarks.

2 A Simple New Keynesian Phillips Curve Framework: Estimation Methodology and Monte Carlo Simulations

In order to illustrate the main points of the paper, we consider in this section a conventional NKPC relationship with fixed coefficients. This setup, therefore, does not allow for a time-varying

[9] The NKPC model in Cogley and Sbordone (2008) features terms other than real marginal costs as additional driving processes, but their role in explaining inflation dynamics is estimated to be very small.

[10] See, for example, Benigno and López-Salido (2006) and Steinsson (2003).

[11] See Fuhrer and Olivei (2010).

inflation target. We do so for simplicity of exposition, as our main results do not hinge on these specifics. We consider the same NKPC specification as in Christiano, Eichenbaum, and Evans (2005). In this framework, firms that do not change optimally their price in a given period through the Calvo (1983) random drawing can still update their current price. The updating follows an indexation mechanism based on the aggregate inflation rate in the last period, and the degree to which indexation occurs is governed by the parameter $\rho \in [0, 1]$, with $\rho = 0$ denoting absence of indexation (and thus no mechanical updating) and $\rho = 1$ denoting full indexation. The latter case yields an NKPC relationship that depends almost as much on expected future inflation as on lagged inflation. In this setup, the difference equation (DE) specification of the NKPC takes the following form[12]

$$\pi_t = \rho\pi_{t-1} + \beta(E_t\pi_{t+1} - \rho\pi_t) + \zeta mc_t + u_t. \tag{1}$$

In equation (1), π denotes inflation and mc real marginal costs, while E_t is the expectations operator conditional on the available information at time t. The parameter $\beta < 1$ is a discount factor, while $\zeta > 0$ is a function of the model's structural parameters, with $\zeta = (1-\alpha)(1-\alpha\beta)/(\alpha+\alpha\theta\omega)$. In this expression, $(1 - \alpha)$ denotes the firms' probability of adjusting prices optimally each period, $\theta > 1$ is the elasticity of substitution across goods, and $\omega > 0$ is the elasticity of firms' marginal costs to their own output (a measure of the degree of strategic complementarity in pricing decisions across firms). The unpredictable error term u is assumed to be i.i.d., and can be thought of as capturing potential misspecifications in the relationship or shocks to firms' desired mark-up. Rearranging (1) gives the following expression for period t inflation

$$\pi_t = \frac{\rho}{1+\beta\rho}\pi_{t-1} + \frac{\beta}{1+\beta\rho}E_t\pi_{t+1} + \frac{\zeta}{1+\beta\rho}mc_t + \widetilde{u}_t. \tag{2}$$

From either (1) or (2), it is possible to obtain a closed-form representation of inflation, conditional on the expected discounted path of real marginal costs. Since the relationship in (2) holds in every period, the one-period-ahead discounted inflation expectations can be written as

$$\beta E_t\pi_{t+1} = \frac{\beta\rho}{1+\beta\rho}\pi_t + \frac{\beta^2}{1+\beta\rho}E_t\pi_{t+2} + \frac{\zeta\beta}{1+\beta\rho}E_t mc_{t+1}. \tag{3}$$

Similarly, the two-period-ahead discounted inflation expectations are

$$\beta^2 E_t\pi_{t+2} = \frac{\beta^2\rho}{1+\beta\rho}E_t\pi_{t+1} + \frac{\beta^3}{1+\beta\rho}E_t\pi_{t+3} + \frac{\zeta\beta^2}{1+\beta\rho}E_t mc_{t+2},$$

[12] See Woodford (2003) and Christiano, Eichenbaum, and Evans (2005) for a derivation.

and so on. Substituting iteratively these expressions into (2) or, equivalently, summing the left-hand and the right-hand sides of these expressions from time t onward, we obtain the closed form (CF) representation of the NKPC[13]

$$\pi_t = \rho\pi_{t-1} + \zeta \sum_{i=0}^{\infty} \beta^i E_t mc_{t+i} + u_t. \tag{4}$$

The difference between equations (1) and (4) is that the CF representation explicitly incorporates model-consistent expectations about future inflation, whereas in (1) expectations about future inflation — the second term on the right-hand side of equation (2) — are left unconstrained.

2.1 Estimating the NKPC Structural Parameters

The ultimate goal of the estimation procedure is to provide inference about the NKPC structural parameters α, β, θ, ρ, and ω (or a subset of these parameters), which we collect in the vector $\boldsymbol{\psi}$. The estimation procedure exploits cross-equations restrictions between the NKPC structural parameters and the parameters of a reduced-form VAR.[14] Estimation of rational expectations models is based on taking to the data parameter restrictions established by the structural relationships. While there is a wide variety of estimation methods in the literature, we focus on the two-step minimum-distance setup in Cogley and Sbordone (2008), which has a long tradition in the estimation of expectational Euler equations (for a general discussion see Newey and McFadden, 1994).

Consider a (column) vector of variables $\mathbf{x}$ that includes, possibly among others, inflation and real marginal costs. We assume that the law of motion for $\mathbf{x}$ can be represented by a reduced-form VAR of order p. Defining the vector $\mathbf{z}_t = (\mathbf{x}'_t, \mathbf{x}'_{t-1}, ..., \mathbf{x}'_{t-p+1})'$, it is possible to rewrite the VAR(p) in first-order form as

$$\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\varepsilon}_{z,t}, \tag{5}$$

where $\mathbf{A}$ is a square matrix of coefficients with all roots inside the unit circle, and $\boldsymbol{\varepsilon}_z$ is a vector of i.i.d. residuals.[15] For simplicity and without loss of generality, we are omitting constants.[16] In what follows, we assume that the solution to the NKPC model for the variables in $\mathbf{x}$ has a reduced-form representation that is captured by (5). This relationship is then used to form expectations

[13] The closed form representation of inflation can also be obtained from equation (1) by forward iteration of $E_t(\pi_{t+j} - \rho\pi_{t+j-1})$, $j \geq 1$.

[14] The estimation procedure follows Sbordone (2002) and Cogley and Sbordone (2008).

[15] If $\mathbf{x}_t$ contains n variables, then $\mathbf{z}_t$ is a vector of size $n \cdot p$. Hence the matrix $\mathbf{A}$ is $(n \cdot p) \times (n \cdot p)$, with the VAR coefficients in the first n rows.

[16] The intercepts play a central role in the NKPC, with time-varying trend inflation considered in section 3. In the present setup, they are immaterial.

about the variables of interest, inflation and real marginal costs. We can express the conditional expectation of a variable $y_{t+k} \in \mathbf{x}_{t+k}$ at time t as

$$E_t y_{t+k} = \mathbf{e}_y' \mathbf{A}^k \mathbf{z}_t, \tag{6}$$

where the vector $\mathbf{e}_y'$ selects variable y_t in $\mathbf{z}_t$. Consider then taking expectations as of $t-1$ of the NKPC written in the DE form (1) using the forecasting rule (6). We have

$$\mathbf{e}_\pi' \mathbf{A} \mathbf{z}_{t-1} = -\beta\rho \mathbf{e}_\pi' \mathbf{A} \mathbf{z}_{t-1} + \rho \mathbf{e}_\pi' \mathbf{I} \mathbf{z}_{t-1} + \beta \mathbf{e}_\pi' \mathbf{A}^2 \mathbf{z}_{t-1} + \zeta \mathbf{e}_{mc}' \mathbf{A} \mathbf{z}_{t-1}, \tag{7}$$

with $\mathbf{I}$ denoting an identity matrix that conforms with $\mathbf{A}$. The left-hand side of (7) is the expectation of inflation from the reduced-form VAR. The right-hand side is the expectation of inflation based on the NKPC model. Equation (7) says that if the NKPC in (1) is the true data generating process for inflation, the reduced-form forecast and the NKPC-based forecast for inflation must be the same. Imposing that (7) holds for all realizations of $\mathbf{z}$ yields a vector of non-linear restrictions involving the VAR coefficients matrix $\mathbf{A}$ and the NKPC structural parameters $\boldsymbol{\psi}$:

$$\mathbf{e}_\pi' \mathbf{A} = -\beta\rho \mathbf{e}_\pi' \mathbf{A} + \rho \mathbf{e}_\pi' \mathbf{I} + \beta \mathbf{e}_\pi' \mathbf{A}^2 + \zeta \mathbf{e}_{mc}' \mathbf{A} \equiv \mathbf{g}^D(\mathbf{A}, \boldsymbol{\psi}), \tag{8}$$

or

$$\mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \equiv \mathbf{e}_\pi' \mathbf{A} - \mathbf{g}^D(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}', \tag{9}$$

where $\underline{0}$ is a column vector of zeros with same size as $\mathbf{e}_\pi$, and the superscript D indicates that the expressions correspond to the DE specification.

The estimation procedure consists of two steps. The first step estimates the law of motion for $\mathbf{x}$ from an unrestricted reduced-form VAR as in (5). This yields an estimated coefficients matrix $\widehat{\mathbf{A}}$. Given this estimate, the second step involves searching for values of the NKPC parameters $\boldsymbol{\psi}$ that minimize the squared deviation of $\mathbf{g}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi})$ from $\mathbf{e}_\pi' \widehat{\mathbf{A}}$, that is[17]

$$\widehat{\boldsymbol{\psi}}^D \equiv \arg\min_{\boldsymbol{\psi}} \mathbf{F}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi}) \cdot \mathbf{F}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi})'. \tag{10}$$

The same reasoning applies to the NKPC written in closed form, equation (4). In this case, time $t-1$ expectations of the NKPC conditional on the forecasting rule (6) are

$$\mathbf{e}_\pi' \mathbf{A} \mathbf{z}_{t-1} = \rho \mathbf{e}_\pi' \mathbf{I} \mathbf{z}_{t-1} + \zeta \mathbf{e}_{mc}' (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{A} \mathbf{z}_{t-1}, \tag{11}$$

[17] See Gouriéroux, Monfort, and Trognon (1985), and Gouriéroux and Monfort (1995, Ch. 9) for a discussion of asymptotic least-squares (ALS). Efficient ALS estimates would require computation of an optimal weighting matrix and consistency of $\widehat{\mathbf{A}}$.

and the vector of non-linear restrictions involving the VAR coefficients matrix $\mathbf{A}$ and the NKPC parameters $\boldsymbol{\psi}$ takes the form

$$\mathbf{e}'_\pi \mathbf{A} = \rho \mathbf{e}'_\pi \mathbf{I} + \zeta \mathbf{e}'_{mc} (\mathbf{I} - \beta \mathbf{A})^{-1} \mathbf{A} \equiv \mathbf{g}^C(\mathbf{A}, \boldsymbol{\psi}), \tag{12}$$

or

$$\mathbf{F}^C(\mathbf{A}, \boldsymbol{\psi}) \equiv \mathbf{e}'_\pi \mathbf{A} - \mathbf{g}^C(\mathbf{A},\boldsymbol{\psi}) = \underline{0}'. \tag{13}$$

where the superscript C indicates that the expressions correspond to the CF specification. The first step of the estimation procedure remains the same as before, while the second step involves searching for values of the NKPC structural parameters $\boldsymbol{\psi}$ that minimize the squared deviation of $\mathbf{g}^C(\widehat{\mathbf{A}}, \boldsymbol{\psi})$ from $\mathbf{e}'_\pi \widehat{\mathbf{A}}$, that is

$$\widehat{\boldsymbol{\psi}}^C \equiv \arg\min_{\boldsymbol{\psi}} \mathbf{F}^C(\widehat{\mathbf{A}}, \boldsymbol{\psi}) \cdot \mathbf{F}^C(\widehat{\mathbf{A}}, \boldsymbol{\psi})'. \tag{14}$$

To summarize, the minimum-distance problems in (10) and (14) are both based on a system of implicit equations $\mathbf{F}^i(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}'$, for $i \in \{D, C\}$. Each system consists of k equations and l unknowns, where l is equal to the size of the parameter vector $\boldsymbol{\psi}$, and k is the size of the square VAR matrix $\mathbf{A}$.[18] This system of equations provides the basis for the estimation method. Hence, when we replace $\mathbf{A}$ with its approximation $\widehat{\mathbf{A}}$ in (10) and (14), the goal is to choose the estimate of $\boldsymbol{\psi}$ that makes the vector $\mathbf{F}^i(\widehat{\mathbf{A}}, \boldsymbol{\psi})$ as close as possible to zero. The minimization problems in (10) and (14) differ, even if the NKPC is the "true" data generating process for inflation and $\widehat{\mathbf{A}}$ is the same in both minimizations. The reason for this difference is that $\mathbf{A}$ is estimated with sampling error. Then, as long as the system of equations is over-identified ($k > l$), the estimates $\widehat{\boldsymbol{\psi}}^D$ and $\widehat{\boldsymbol{\psi}}^C$ are also going to be different. It is only in the very special case of exact identification ($k = l$) that $\widehat{\boldsymbol{\psi}}^D = \widehat{\boldsymbol{\psi}}^C$, regardless of $\widehat{\mathbf{A}}$. With an over-identified system, $\mathbf{g}^D(\mathbf{A},\boldsymbol{\psi})$ equals $\mathbf{g}^C(\mathbf{A},\boldsymbol{\psi})$ only when $\mathbf{A}$ is known, in which case there exists a vector $\boldsymbol{\psi}$ such that (8) and (12) hold exactly. Then it does not matter which specification of the NKPC is being estimated, since inflation forecasts generated from the reduced-form VAR with the true matrix $\mathbf{A}$ – the term $\beta \mathbf{e}'_\pi \mathbf{A}^2 \mathbf{z}_{t-1}$ on the right-hand side of (7) – are perfectly model-consistent.

To see how the CF specification imposes model-consistent constraints on expectations that are

[18] The number of equations in the VAR is given by n, and p is the order of the VAR. Then $k = n \cdot p$.

not imposed on the DE form, note that it is possible to write $\mathbf{g}^D(\widehat{\mathbf{A}},\boldsymbol{\psi})$ as follows

$$\begin{aligned}\mathbf{g}^D(\widehat{\mathbf{A}},\boldsymbol{\psi}) &\equiv -\beta\rho\mathbf{e}'_\pi\widehat{\mathbf{A}}+\rho\mathbf{e}'_\pi\mathbf{I}+\beta\mathbf{e}'_\pi\widehat{\mathbf{A}}^2+\zeta\mathbf{e}'_{mc}\widehat{\mathbf{A}} \\ &= \rho\mathbf{e}'_\pi\mathbf{I}+\zeta\mathbf{e}'_{mc}(\mathbf{I}-\beta\widehat{\mathbf{A}})^{-1}\widehat{\mathbf{A}}+\beta\mathbf{k}(\widehat{\mathbf{A}},\boldsymbol{\psi})\widehat{\mathbf{A}} \\ &= \mathbf{g}^C(\widehat{\mathbf{A}},\boldsymbol{\psi})+\beta\mathbf{k}(\widehat{\mathbf{A}},\boldsymbol{\psi})\widehat{\mathbf{A}}, \end{aligned} \tag{15}$$

where

$$\mathbf{k}(\widehat{\mathbf{A}},\boldsymbol{\psi})=\mathbf{e}'_\pi\widehat{\mathbf{A}}-\rho\mathbf{e}'_\pi\mathbf{I}-\zeta\mathbf{e}'_{mc}(\mathbf{I}-\beta\widehat{\mathbf{A}})^{-1}\widehat{\mathbf{A}}.$$

It is then apparent that for $\mathbf{g}^D(\widehat{\mathbf{A}},\boldsymbol{\psi})$ to equal $\mathbf{g}^C(\widehat{\mathbf{A}},\boldsymbol{\psi})$, the restrictions $\mathbf{k}(\widehat{\mathbf{A}},\boldsymbol{\psi}) = \underline{0}'$ must hold. These restrictions are model-consistent, and represent all of the constraints on inflation expectations' formation implied by the NKPC at any point in time. They need to be satisfied in order to obtain the closed-form of the NKPC (see the expression in 13), but they are not exploited in the difference equation form that yields $\mathbf{g}^D(\widehat{\mathbf{A}},\boldsymbol{\psi})$. In this latter case, the only information used to characterize expectations is the unrestricted linear projections obtained through $\widehat{\mathbf{A}}$, which does not take explicitly into account that the behavior of future inflation should also satisfy the NKPC relationship.

The question of interest, therefore, is how inference about $\boldsymbol{\psi}$, given the estimated $\widehat{\mathbf{A}}$, changes when in the second stage of the estimation process, instead of using the difference equation version of the NKPC, we use other specifications that place model-consistent restrictions on expectations. The CF specification is one of them, but we now turn to other specifications (of which the CF is a limiting case) that use a finite number of those restrictions and that conceptually fall in between the DE and CF formulations.

2.2 The Efficiency Gains From Imposing Model-Consistent Constraints On Inflation expectations

In this subsection we illustrate the effect of placing model-consistent constraints on inflation expectations when estimating the DE specification of the NKPC. Suppose that we are interested in estimating the NKPC in DE form, but we explicitly require that the same equation be valid for two consecutive periods. The two equations involved in the estimation, therefore, are the following

$$E_{t-1}\pi_t = E_{t-1}\left\{\rho\pi_{t-1}+\beta(\pi_{t+1}-\rho\pi_t)+\zeta mc_t\right\} \tag{16}$$

$$E_{t-1}\pi_{t+1} = E_{t-1}\left\{\rho\pi_t+\beta(\pi_{t+2}-\rho\pi_{t+1})+\zeta mc_{t+1}\right\}. \tag{17}$$

Combining these two relationships yields the following difference equation formulation

$$E_{t-1}\left\{\pi_t - \rho\pi_{t-1}\right\} = E_{t-1}\left\{\beta^2(\pi_{t+2} - \rho\pi_{t+1}) + \beta\zeta mc_{t+1} + \zeta mc_t\right\},$$

where we have restricted the behavior of $E_{t-1}\left\{\pi_{t+1} - \rho\pi_t\right\}$ to be model consistent. It is also possible to impose restrictions on $E_{t-1}\pi_{t+1}$ and $E_{t-1}\pi_t$ separately, and this still requires using the two equations above.[19]

Equations (16) and (17) can be translated into two sets of cross-equation restrictions, with each set containing k restrictions:

$$\begin{aligned} \mathbf{c}_0\left(\mathbf{A}, \boldsymbol{\psi}\right) &\equiv \mathbf{F}^D\left(\mathbf{A}, \boldsymbol{\psi}\right) = \underline{0}' && (18) \\ \mathbf{c}_1\left(\mathbf{A}, \boldsymbol{\psi}\right) &\equiv \mathbf{e}_\pi'\mathbf{A}^2 - \rho\mathbf{e}_\pi'\mathbf{A} + \beta\rho\mathbf{e}_\pi'\mathbf{A}^2 - \zeta\mathbf{e}_{MC}'\mathbf{A}^2 - \beta\mathbf{e}_\pi'\mathbf{A}^3 = \underline{0}'. && (19) \end{aligned}$$

Given the definitions in (8) and (9), it follows that

$$\mathbf{c}_1\left(A, \boldsymbol{\psi}\right) = \mathbf{c}_0\left(A, \boldsymbol{\psi}\right) \cdot \mathbf{A}. \tag{20}$$

In addition, since the square matrix $\mathbf{A}$ is full rank, equation (20) implies that the following must be true

$$\mathbf{c}_1\left(\mathbf{A}, \boldsymbol{\psi}\right) = \underline{0}' \Leftrightarrow \mathbf{c}_0\left(\mathbf{A}, \boldsymbol{\psi}\right) = \underline{0}'. \tag{21}$$

Because of (20) and (21), we do not need to estimate the model parameters taking into account all of the $2 \cdot k$ cross-equation conditions in (18) and (19). Those $2 \cdot k$ conditions are equivalent to the following k constraints:[20]

$$\begin{aligned} \mathbf{c}_1^*\left(\mathbf{A}, \boldsymbol{\psi}\right) &\equiv \mathbf{c}_0\left(\mathbf{A}, \boldsymbol{\psi}\right) + \beta\mathbf{c}_1\left(\mathbf{A}, \boldsymbol{\psi}\right) = \underline{0}' \\ &= \mathbf{F}^D\left(\mathbf{A}, \boldsymbol{\psi}\right) \cdot \left(\mathbf{I} + \beta\mathbf{A}\right) = \underline{0}'. && (22) \end{aligned}$$

The minimum-distance estimation of $\boldsymbol{\psi}$ in this case yields estimates

$$\widehat{\boldsymbol{\psi}}^{D(1)} \equiv \arg\min_{\boldsymbol{\psi}} \mathbf{F}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi}) \cdot (\mathbf{I} + \beta\widehat{\mathbf{A}}) \cdot (\mathbf{I} + \beta\widehat{\mathbf{A}})' \cdot \mathbf{F}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi})', \tag{23}$$

where the superscript $D(1)$ indicates that we are explicitly imposing that the DE formulation of the NKPC holds for one additional period.[21] The weighting matrix $(\mathbf{I} + \beta\widehat{\mathbf{A}}) \cdot (\mathbf{I} + \beta\widehat{\mathbf{A}})'$ in (23) forces

[19] This can be seen by noting that the two conditions in (16) and (17) can also be expressed as in (2) and (3) respectively, with expectations taken as of time $t-1$.

[20] The use of β facilitates linking the DE and CF problems in the discussion that follows in the main text. But the coefficient premultimplying $\mathbf{c}_1\left(\mathbf{A}, \boldsymbol{\psi}\right)$ in (22) does not need to equal β, although it needs to be positive.

[21] Therefore, $\widehat{\boldsymbol{\psi}}^{D(0)}$ corresponds to the DE estimate $\widehat{\boldsymbol{\psi}}^{D}$ from (10).

the minimum-distance estimation to penalize differently specific errors and interactions among the errors $\mathbf{F}^D(\widehat{\mathbf{A}}, \boldsymbol{\psi})$. The minimization problem in (23) imposes model discipline on inflation expectations by acknowledging that the NKPC difference equation (1) should also apply at $t+1$. In this way, the estimation takes into account that some specific violations of the cross-equation restrictions $\mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}'$ have greater consequences for inflation expectations than others. Considering this fact is especially important when estimating forward-looking models. In contrast, the DE problem in (10) disregards the relative importance of those errors and interactions. In essence, the DE specification is more agnostic about the way in which inflation expectations are formed and attaches the same importance to all cross-equation restrictions in $\mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}'$.

Of course, we may want to discipline expectations not only one period ahead, but also two periods ahead. Following the same logic that resulted in (22), the sets of cross-equation restrictions involved in this problem (a total of $3 \cdot k$ conditions) are now equivalent to the k conditions

$$\mathbf{c}_2^*(\mathbf{A}, \boldsymbol{\psi}) \equiv \mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \cdot \left(\mathbf{I} + \beta\mathbf{A} + \beta^2\mathbf{A}^2\right) = \underline{0}'.$$

More generally, we may want to impose model-consistent expectations for up to $j \geq 1$ periods ahead in time. By induction, the set of k cross-equation restrictions is then given by

$$\mathbf{c}_j^*(\mathbf{A}, \boldsymbol{\psi}) \equiv \mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \cdot \left(\mathbf{I} + ... + (\beta\mathbf{A})^j\right) = \underline{0}', \tag{24}$$

and the corresponding estimates are then

$$\widehat{\boldsymbol{\psi}}^{D(j)} \equiv \arg\min_{\boldsymbol{\psi}} \mathbf{c}_j^*(\widehat{\mathbf{A}}, \boldsymbol{\psi}) \cdot \mathbf{c}_j^*(\widehat{\mathbf{A}}, \boldsymbol{\psi})',$$

with $D(j)$ indicating model-consistent restrictions on expectations for j consecutive additional periods. It follows that, imposing model discipline on inflation expectations at *any* future point in time, the infinite number of model-consistent constraints on expectations is then equivalent to the k conditions:

$$\begin{aligned} \mathbf{c}_\infty^*(\mathbf{A}, \boldsymbol{\psi}) &\equiv \lim_{j\to\infty} \mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \cdot \left(\mathbf{I} + ... + (\beta\mathbf{A})^j\right) \\ &= \mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \cdot (\mathbf{I} - \beta\mathbf{A})^{-1} = \underline{0}'. \end{aligned} \tag{25}$$

Combining the definitions (8), (9), (12), and (13), it is possible to show that the DE and CF cross-equation restrictions are related by

$$\mathbf{F}^D(\mathbf{A}, \boldsymbol{\psi}) \cdot (\mathbf{I} - \beta\mathbf{A})^{-1} = \mathbf{F}^C(\mathbf{A}, \boldsymbol{\psi}).$$

The k cross-equation restrictions $\mathbf{c}_\infty^*(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}'$ are then the very same restrictions that need to hold for the closed-form NKPC:

$$\mathbf{c}_\infty^*(\mathbf{A}, \boldsymbol{\psi}) = \mathbf{F}^C(\mathbf{A}, \boldsymbol{\psi}) = \underline{0}'. \tag{26}$$

Hence, as long as the NKPC model provides a good characterization of the data for every j consecutive periods, the k cross-equation restrictions in $\mathbf{c}_j^*(\mathbf{A}, \boldsymbol{\psi})$ provide more information about the model dynamics than the k restrictions in $F^D(\mathbf{A}, \boldsymbol{\psi})$ derived from the DE problem alone. This additional information is the source of the gains in precision from estimating the CF specification versus the DE specification when $\mathbf{A}$ is unknown and needs to be estimated. The discussion so far also indicates that whenever the closed form is too complex to solve or too difficult to approximate reasonably well, it is possible to improve on the DE estimates by imposing additional restrictions on expectations. These take the form of the DE relationship being iterated forward for $j \geq 1$ periods, possibly a much easier task than computing the closed form. We have shown that this is equivalent to imposing the k cross-equation restrictions $\mathbf{c}_j^*(\mathbf{A}, \boldsymbol{\psi}) = \underline{\mathbf{0}}'$.

In sum, the results in this section are related to the literature that illustrates the gains in estimation efficiency from imposing additional restrictions (see Gouriéroux, Monfort, and Trognon, 1985, and Kodde, Palm, and Pfann, 1990). In our context, we show that this gain in efficiency can be obtained by imposing additional model-consistent restrictions on expectations. This is especially desirable in the estimation of forward-looking models, as their dynamics depend crucially on how expectations are formed. What is special about our setup is that all of the expectational constraints can be collapsed into k cross-equations restrictions, no matter how many these constraints are. Indeed, we have shown that the closed-form version of the NKPC also exploits k cross-equation restrictions that characterize model-consistent behavior over the entire expected path of inflation, ad infinitum.

2.3 Monte Carlo Simulations

We now turn to analyze the properties of the estimated vectors of parameters $\widehat{\boldsymbol{\psi}}^D$ and $\widehat{\boldsymbol{\psi}}^C$ obtained in (10) and (14), respectively, in the context of a Monte Carlo exercise. We are interested in ascertaining whether the use of the closed form instead of the difference equation version of the NKPC yields estimates of ψ that, in small samples, are noticeably different in terms of precision. We also consider the estimated vectors $\widehat{\boldsymbol{\psi}}^{D(j)}$ for small values of $j \geq 1$, in order to observe the impact of sequentially adding model discipline on expectations. In particular, we are interested to assess how these estimates relate to $\widehat{\boldsymbol{\psi}}^D$ and $\widehat{\boldsymbol{\psi}}^C$.

We consider first a case in which the estimated NKPC is the true data generating process, so that there are no misspecification issues. We then consider one case of misspecification that we deem relevant in actual data. Specifically, we generate data from an NKPC where lagged inflation enters as a weighted average of inflation at time $t-1$ and $t-2$ through the indexation mechanism, but then estimate an NKPC specification that only allows for lagged inflation at time $t-1$.[22]

2.3.1 No misspecification

The artificial data for inflation in the Monte Carlo exercise are generated according to the NKPC (1). For the marginal cost process, we use a simple univariate AR(2) law of motion. For each of the Monte Carlo repetitions, we estimate a reduced-form VAR with two lags in inflation and marginal costs. The VAR estimation is carried out equation-by-equation via OLS. This provides us with an estimated matrix $\widehat{\mathbf{A}}_{(n)}$, where n denotes the n-th repetition of the Monte Carlo experiment. With this reduced-form coefficients matrix, we can then estimate $\widehat{\boldsymbol{\psi}}^D_{(n)}$ using (10), and $\widehat{\boldsymbol{\psi}}^C_{(n)}$ using (14).[23]

Several considerations about this Monte Carlo exercise are in order. First, note that the NKPC we are estimating, regardless of the chosen representation, is the true data generating process for inflation. In other words, there are no misspecification issues in this exercise. Second, the reduced-form process for real marginal costs, a univariate AR(2), is stylized but not overly counterfactual. The improvement in adjusted R^2 is only 1 percent when we move from the univariate AR(2) representation of marginal costs to a multivariate reduced-form representation that, in addition to two lags of real marginal costs, also includes two lags of inflation, the federal funds rate, and GDP growth.[24] The difficulty in working with this larger information set is that, in order to generate simulated inflation data, the NKPC in (1) needs to be solved first using standard rational expectations solution methods. The solution entails a constrained reduced-form representation of inflation that depends on ψ and on the parameters describing the unconstrained reduced-form

[22] A richer indexation mechanism that depends on inflation at time $t-1$ and $t-2$ is not without economic content. Such a scheme may smooth transitory movements and be more apt to capture more persistent components of inflation.

[23] In this exercise, OLS estimates $\widehat{\mathbf{A}}_{(n)}$ are consistent, as there are no misspecification issues.

[24] The univariate AR(2) representation of real marginal costs we use, which is given by equation (27) in the text (where a constant has been omitted), has an adjusted R^2 of 0.835 over the period 1961:Q1 to 2003:Q4. Granted, if real marginal costs are the relevant driving process for inflation, then changes to the stance of monetary policy should affect real marginal costs. In other words, one would expect the federal funds rate to be a relevant component of the dynamics of real marginal costs. We consider in the next subsection a Monte Carlo exercise with a larger information set that also includes the federal funds rate.

dynamics of the other variables. When using an augmented information set which, in addition to inflation and marginal costs, also includes the federal funds rate and GDP growth, it is not possible to obtain a unique or stable rational expectations solution for a range of relevant values of ψ.[25] For this to be feasible, it is necessary to constrain the dynamics of some of the variables in the information set. Therefore, we first report results from Monte Carlo simulations where the data-generating process for real marginal costs is a simple univariate AR(2). In this case, it is possible to find a stable and unique solution for the NKPC model under a wide range of values for the vector ψ. We later show that our results do not change when the reduced-form dynamics of real marginal costs rely on a larger set of variables.

The AR(2) specification we use to generate the artificial data for marginal costs is the following

$$mc_t^A = 0.98\ mc_{t-1}^A - 0.05\ mc_{t-2}^A + u_{mc,t}^A, \tag{27}$$

where variables have a superscript A to denote that these are artificially generated data. The AR(2) coefficients are obtained from estimating the process on actual U.S. data. With the AR(2) representation for real marginal costs (27), it is easy to derive the constrained, reduced-form solution from which the artificial data for inflation are generated, which is given by

$$\pi_t^A = \rho\pi_{t-1}^A + \frac{0.98 - 0.05\beta}{1 - 0.98\beta + 0.05\beta^2}\zeta mc_{t-1}^A - \frac{0.05}{1 - 0.98\beta + 0.05\beta^2}\zeta mc_{t-2}^A + \zeta u_{mc,t}^A + u_{\pi,t}^A. \tag{28}$$

This expression is a function of the vector ψ of structural parameters in the NKPC.

The artificial data are generated by drawing shocks from a multivariate normal distribution, where the variance-covariance structure of the shocks is estimated on actual data, given the law of motions described in (27) and (28).[26] For each Monte Carlo repetition we discard the first 500 artificially generated observations.[27] We then estimate a reduced-form VAR of order 2 on the artificial data.[28] The sample length for the VAR estimation is set at $S = 176$, which corresponds

[25] Inflation, real marginal costs, the federal funds rate, and GDP growth are the four variables that enter the VAR considered in Cogley and Sbordone (2008).

[26] The estimation period is 1961:Q1 to 2003:Q4. We estimate the errors in (27) and (28), with $\rho = 0.5$ and β and ζ, as described later in the text. For the purpose of estimating the shocks and obtaining their variance-covariance matrix, we include a constant in (27) and in (28). The qualitative features of the Monte Carlo exercises are not affected by reasonable changes in the variance-covariance structure of the shocks used when generating the data.

[27] When generating the artificial data, we take zeros as initial conditions for inflation and marginal costs. This is equivalent to assuming that inflation and marginal costs are at their average levels, since we are not including constants in (27) and (28).

[28] The model-based reduced-form coefficients matrix has zeros for the first and second lags of inflation in the marginal costs equation (27), and a zero for the second lag of inflation in the inflation equation (28). However, when estimating the reduced-form VAR on the simulated data, we are not imposing those zero restrictions on the coefficients matrix. In other words, we are assuming that the econometrician knows that the system is fully characterized by inflation and marginal costs, but the econometrician does not know that marginal costs follow a univariate AR(2) process.

to 44 years of data at a quarterly frequency (roughly the sample size used to estimate the NKPC on actual data).

Once we have estimated the VAR coefficients matrix $\widehat{\mathbf{A}}_{(n)}$, we proceed to estimate $\widehat{\boldsymbol{\psi}}^{D}_{(n)}$ using (10), and $\widehat{\boldsymbol{\psi}}^{C}_{(n)}$ using (14). To keep matters simple, we set β equal to 0.99 and assume that this parameter value is known and thus not estimated. The two parameters in ψ that are left to estimate are, therefore, ρ and ζ. As shown earlier, ζ is a function of the parameters $(\alpha, \theta, \omega, \beta)$ and in this setup only one of the three parameters $(\alpha,\ \theta, \omega)$ can be estimated independently. We set θ equal to 9.8 and ω equal to 0.43. We thus estimate the degree of price indexation ρ and the probability α that a firm will not be able to reset prices optimally in a given period. In generating the data, we set α equal to 0.588,[29] and consider different values for ρ — specifically, $\rho = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$ — to assess whether the degree of indexation affects the properties of the estimated $\widehat{\boldsymbol{\psi}}^{D}_{(n)}$ and $\widehat{\boldsymbol{\psi}}^{C}_{(n)}$.

Results of the Monte Carlo exercises are depicted in Figures 1 and 2, which compare the distributions of $(\widehat{\alpha}^{D}_{(n)}, \widehat{\alpha}^{C}_{(n)})$ and $(\widehat{\rho}^{D}_{(n)}, \widehat{\rho}^{C}_{(n)})$, respectively, for different parametrizations of ρ. Each Monte Carlo exercise consists of 500 repetitions. It is apparent from the pictures that the CF estimates of the NKPC are better centered. This is especially clear when the true ρ in the inflation data-generating process gets closer to unity. Then, the small-sample bias in estimating α from the DE formulation of the NKPC becomes noticeable, with an extremely large mass of estimates at $\alpha = 1$. An estimated value of α equal to unity implies an estimate of ζ equal to zero. In other words, the DE formulation largely fails to detect that marginal costs are the driving process for inflation when the degree of indexation becomes relatively high. Instead, the corresponding distributions for the estimated α under the CF specification of the NKPC do not display mass at unity.

More importantly, the figures also show that the CF estimation of the NKPC produces estimates that are much more efficient. The range between the 5th and the 95th percentiles in the distribution of $\widehat{\alpha}^{C}$ can be three times smaller than the same interquantile range in the distribution of $\widehat{\alpha}^{D}$. Comparing $\widehat{\rho}^{C}$ with $\widehat{\rho}^{D}$, the gain in efficiency is somewhat less pronounced but still evident, especially so when the true ρ in the inflation data-generating process approaches unity. For example, when the true ρ is set equal to 0.7, the range between the 5th and the 95th percentiles in the distribution of $\widehat{\rho}^{C}$ is less than half the same range in the distribution of $\widehat{\rho}^{D}$. It is worth recalling that the estimated coefficients matrix $\widehat{\mathbf{A}}_{(n)}$ from the reduced-form VAR that is used for the esti-

[29] The values for β, ω are common in the literature. The values of θ and α match the median estimates in Cogley and Sbordone (2008).

mation of (α, ρ) is the same in the two minimization problems (10) and (14). The difference in the precision of the estimates is thus only the result of the CF specification imposing model-consistent expectations about future inflation, as discussed previously.

We stressed in the previous section that the efficiency of the DE estimates can be improved by imposing $j \geq 1$ additional restrictions on inflation expectations without resorting to the closed form. In particular, we showed that the set of cross-equation restrictions in this case is

$$\mathbf{c}_j^* (\mathbf{A}, \boldsymbol{\psi}) \equiv \mathbf{F}^D (\mathbf{A}, \boldsymbol{\psi}) \cdot \left(\mathbf{I} + ... + (\beta \mathbf{A})^j\right) = \underline{0}'.$$

Figure 3 illustrates that a small j is sufficient to approach the efficiency of the CF estimates, which corresponds to the limiting case when j goes to infinity. Setting $j = 4$ already generates a substantial improvement in efficiency compared to the DE estimates (which correspond to $j = 0$). The figure also shows that in some instances the gains in efficiency from just having $j = 1$ are quite large. Note that, at quarterly frequency, $j = 4$ means that we are imposing model-consistent constraints on the evolution of expected future inflation for only one year. We find this requirement rather conservative for a model of inflation dynamics.

2.3.2 Robustness

We now check that the large gains in efficiency from estimating the closed form of the NKPC in our baseline Monte Carlo exercise are still present when considering alternative specifications of the reduced-form dynamics for marginal costs. We illustrate two cases that we deem especially important. In the first case, the information set is still restricted to inflation and marginal costs, but we allow feedback from lagged inflation in the evolution of marginal costs over time. This is a particularly relevant case because the NKPC, as shown in (4), implies that current inflation, after controlling for the impact of lagged inflation, is a predictor of the present discounted value of current and future marginal costs. The econometrician may not observe all of the variables useful to forecast marginal costs, but knowing inflation is enough because inflation reveals to the econometrician the forecast of the present discounted value of current and future marginal costs. Therefore, an implication of the NKPC is that inflation should Granger-cause marginal costs when firms use information for forecasting marginal costs beyond the history of that variable.[30] To capture such a feature of the NKPC model, we now assume that, instead of following an AR(2)

[30] This observation, in the context of a consumption problem, was formulated first by Campbell (1987).

process, the reduced-form equation for marginal costs is given by

$$mc_t^A = 0.17\ \pi_{t-1}^A + 0.14\ \pi_{t-2}^A - 0.31\ \pi_{t-3}^A + 0.90\ mc_{t-1}^A + 0.14\ mc_{t-2}^A - 0.13\ mc_{t-3}^A + u_{mc,t}^A. \quad (29)$$

Equation (29) constrains the sum of the coefficients on lagged inflation to sum to zero. This is done to ensure uniqueness and stability of the solution for inflation, given plausible parametrizations of ψ. The process in (29) is data consistent once the zero-sum restriction on the coefficients for lagged inflation is imposed.[31] The Monte Carlo procedure follows the same steps as before, with the modification that the estimated reduced-form VAR in each replication is now of order 3. The parameterization of ψ is the same as in our baseline exercise. The results for $\rho \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$ are shown in Figures 4 and 5. The distributions of $\widehat{\rho}^C$ and $\widehat{\alpha}^C$ are tighter and better centered than the corresponding distributions for $\widehat{\rho}^D$ and $\widehat{\alpha}^D$. In all, these findings are very similar to the baseline case in which there is no feedback from lagged inflation in the dynamics of marginal costs.

Using a larger information set, which in addition to inflation and marginal costs, also includes the federal funds rate and GDP growth, does not change the conclusion that estimates of α and ρ obtained from the CF specification of the NKPC are better centered and more efficient than the corresponding estimates obtained from the DE specification. This is shown in Figures 6 and 7 for $\rho \in \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$. In this exercise, real marginal costs depend only on lagged marginal costs and lagged GDP growth:

$$mc_t^A = -0.07\ g_{t-1}^{y,A} + 0.18\ g_{t-2}^{y,A} + 0.16\ g_{t-3}^{y,A} + 0.80\ mc_{t-1}^A + 0.37\ mc_{t-2}^A - 0.21\ mc_{t-3}^A + u_{mc,t}^A,$$

where g^y denotes GDP growth. This process is consistent with actual data once we impose the restrictions that lagged inflation and the lagged federal funds rate do not enter the reduced-form process for real marginal costs. Again, we place these restrictions to ensure uniqueness and stability of the rational expectations solution for inflation, given plausible parametrizations of ψ. The reduced-form data-generating processes for GDP growth and the federal funds rate include three lags of each variable in the information set and are consistent with actual data. In the Monte Carlo procedure, we use an estimated reduced-form VAR of order 3 to retrieve $\widehat{\mathbf{A}}_{(n)}$ in each replication. The estimation results, overall, are very similar to the results obtained in the baseline exercise.

[31] The process in (29) with the zero restriction on the sum of coefficients for lagged inflation was estimated over the period 1961:Q1 to 2003:Q4. There is some evidence in the data, though not overwhelming, in favor of such a specification: In an unconstrained regression of real marginal costs on three lags of inflation and three lags of marginal costs, the test of the hypothesis that the sum of coefficients on lagged inflation is different from zero has a p-value of 0.126.

2.3.3 Misspecification of the indexation mechanism

We consider here a particular form of misspecification in the estimation of the NKPC. The true NKPC is now given by the following expression

$$\pi_t = \rho(\tau\pi_{t-1} + (1-\tau)\pi_{t-2}) + \beta(E_t\pi_{t+1} - \rho(\tau\pi_t + (1-\tau)\pi_{t-1})) + \zeta mc_t + u_t. \tag{30}$$

In this case, firms that do not reset their prices optimally in a given period follow an indexation mechanism that is not based on last period's inflation only, but on a weighted average of inflation over the past two periods, where $0 \leq \tau \leq 1$ denotes the weight placed on last period's inflation.

The reason for considering such a specification is that estimating the inflation process as a function of two lags of inflation and two lags of real marginal costs over the period 1961:Q1 to 2003:Q4 yields the following OLS estimates

$$\begin{aligned} \pi_t \quad = \quad & \underset{(0.071)}{0.51}\ \pi_{t-1} + \underset{(0.070)}{0.36}\ \pi_{t-2} + \underset{(0.033)}{0.096}\ mc_{t-1} - \underset{(0.034)}{0.078}\ mc_{t-2} + \varepsilon_t, \end{aligned} \tag{31}$$

where standard errors are in parenthesis and ε_t is a reduced-form error.[32] The second lag of inflation is highly significant and, while not as large as the first lag, economically relevant. This result, together with the fact that lags of inflation are not especially important in an estimated reduced-form equation for real marginal costs over the same period, raises the possibility that an NKPC specification as in (30) provides a better characterization of the data than the specification in (1), which constrains τ to unity.

We investigate the misspecification bias that arises when the data-generating process for inflation is an NKPC with $\tau < 1$ as in (30), but the econometrician estimates an NKPC with τ constrained to equal unity. For this purpose, we set up a Monte Carlo exercise that is very similar to our baseline exercise in section 2.3.1. Inflation and marginal costs are the only two variables in the information set. Real marginal costs follow the same reduced-form AR(2) process as in (27). In generating the data, we set τ equal to 0.6. The other parameters (α,β,θ,ω) and the range of values for ρ are set as before. The VAR used to retrieve the matrix $\widehat{\mathbf{A}}_{(n)}$ in each Monte Carlo replication is of order 2. The misspecification in this exercise arises from the fact that the estimated NKPC, regardless of the specification used, constrains τ to unity.

[32] Augmenting the information set to include lagged GDP growth and lagged policy rates does not materially alter the estimates of the coefficients on lagged inflation in (31). The median estimates of the coefficients on the two lags of inflation in the inflation equation from the reduced-form VAR used by Cogley and Sbordone (2008), which is estimated with Bayesian methods, are 0.40 and 0.32, respectively.

Estimation results for α and ρ, with $\rho = \{0.1, 0.3, 0.5, 0.7, 0.9\}$, are reported in Figures 8 and 9, which compare the distributions of $(\widehat{\alpha}^D_{(n)}, \widehat{\alpha}^C_{(n)})$ and $(\widehat{\rho}^D_{(n)}, \widehat{\rho}^C_{(n)})$ for the different parametrizations of ρ.[33] It is apparent that when the NKPC is estimated in the DE form, the estimates $(\widehat{\alpha}^D, \widehat{\rho}^D)$ are biased. The estimate $\widehat{\alpha}^D$ is upward biased, with the estimation procedure often failing to identify real marginal costs as the driving process for inflation. The estimate $\widehat{\rho}^D$ is downward biased. In other words, the estimates point to less indexation to past inflation than is actually present in the true data-generating process. When the NKPC is estimated in closed form via the minimum-distance problem (14), estimates for α are well centered. There is some downward bias, instead, when estimating ρ. However, the bias is not as large as with the DE specification, as the figure clearly shows.

In all, the results in this section highlight the importance of correctly specifying the indexation rule in the NKPC. This is true both in the DE and in the CF specifications of the NKPC, though it is apparent that the CF version is less prone to suffer from this misspecification bias than the DE counterpart.

3 Estimates of the NKPC with Time-Varying Trend Inflation

Given our Monte Carlo findings, we now turn to estimating an NKPC with time-varying trend inflation on actual data. We first provide a brief description of the DE and the CF representations (and the cases in between) of the NKPC in this framework, and highlight the important differences with the zero-trend inflation setup considered in the previous section. We then discuss the estimation method and the empirical findings.

3.1 Model Setup

We adapt the framework in Cogley and Sbordone (2008) to allow for two lags of inflation in the indexation mechanism. This modification, while technically minor, is important from an empirical standpoint as it can reduce the effect of misspecification bias on the estimates. The indexation mechanism takes the form

$$P_t(i) = (\Pi^{\tau}_{t-1}\Pi^{1-\tau}_{t-2})^{\rho} P_{t-1}(i),$$

[33] For brevity, we discuss in the text only the two polar specifications. Estimates for the intermediate cases follow the same pattern (relative to the DE and CF counterparts) as in Section 2.3.1. Results are available from the authors upon request.

where $P_t(i)$ is the price set by firm i when it cannot reoptimize at time t, and $\Pi_t = P_t/P_{t-1}$ is the period t gross rate of inflation. As in section 2, the parameter $\rho \in [0,1]$ measures the degree of indexation, while $\tau \in [0,1]$ represents the weight given to $t-1$ aggregate inflation relative to $t-2$ aggregate inflation. This indexation mechanism nests the one-lag inflation indexation case when $\tau = 1$. The setup with time-varying trend inflation differs from the simple NKPC equation of the previous sections. Previously, the NKPC was derived by log-linearizing the first-order conditions of the Calvo pricing model around a zero-inflation steady state. The log-linearization is now taken around a steady state where trend inflation is an exogenous process that evolves as a random walk. The distinguishing feature of this type of setup, compared with the more standard setup with zero trend inflation, is that the coefficients in the NKPC are a function of trend inflation and, as a result, are time-varying.

We leave the full details of the derivation of the NKPC to Appendix A. In the rest of this section, we provide the equilibrium relationships of the model with time-varying trend inflation that we use at the estimation stage. The first of these relationships is the restriction between trend inflation and steady-state real marginal costs, which takes the form:[34]

$$\left(1-\alpha\overline{\Pi}_t^{(1-\rho)(\theta-1)}\right)^{(1+\theta\omega)/(1-\theta)}\left[\frac{1-\alpha\overline{q}\overline{g}^y\overline{\Pi}_t^{\theta(1+\omega)(1-\rho)}}{1-\alpha\overline{q}\overline{g}^y\overline{\Pi}_t^{(\theta-1)(1-\rho)}}\right]=(1-\alpha)^{(1+\theta\omega)/(1-\theta)}\frac{\theta}{\theta-1}\overline{mc}_t, \qquad (32)$$

where $\overline{q}$ is the steady-state real discount factor, $\overline{g}^y$ is the steady-state growth rate of output, $\overline{\Pi}_t$ is time t gross trend inflation, $\overline{mc}_t$ denotes time t trend real marginal costs, and the other parameters are defined in the previous sections. Denoting by a hat the log-deviation of a variable from its steady-state value, we can write the NKPC as follows

$$\begin{aligned}\widehat{\pi}_t \;=\;& \rho\tau(\widehat{\pi}_{t-1}-\widehat{g}_t^{\pi})+\rho(1-\tau)(\widehat{\pi}_{t-2}-\widehat{g}_{t-1}^{\pi}-\widehat{g}_t^{\pi})+\lambda_t E_t(\widehat{\pi}_{t+1}-\rho\tau\widehat{\pi}_t-\rho(1-\tau)(\widehat{\pi}_{t-1}-\widehat{g}_t^{\pi}))\\ &+\zeta_t\widehat{mc}_t+\gamma_t\widehat{D}_t+u_{\pi,t}, \qquad (33)\end{aligned}$$

where $\widehat{g}_t^{\pi} = \ln(\overline{\Pi}_t/\overline{\Pi}_{t-1})$ is the growth rate of trend inflation, $u_{\pi,t}$ is a structural shock, and $\widehat{D}$ is defined recursively as

$$\widehat{D}_t = \varphi_{1,t}E_t(\widehat{q}_{t,t+1}+\widehat{g}_{t+1}^{y})+\varphi_{1,t}(\theta-1)E_t\left\{\widehat{\pi}_{t+1}-\rho\tau\widehat{\pi}_t-\rho(1-\tau)(\widehat{\pi}_{t-1}-\widehat{g}_t^{\pi})\right\}+\varphi_{1,t}E_t\widehat{D}_{t+1}. \qquad (34)$$

[34] See equation (7) in Cogley and Sbordone (2008). For ease of comparison and reference, we preserve their notation whenever possible.

Some coefficients in (33) and (34) are time-varying. Compared to the standard NKPC specification with zero trend inflation, this specification involves, in addition to the growth rate in trend inflation, other terms summarized by $\widehat{D}$.

It is possible to obtain an exact closed form (ECF) that expresses inflation as a function of its past and the driving process. In contrast with the fixed-coefficients NKPC, this driving process is a linear combination of real marginal costs (current and expected) and expected discounted real output growth:

$$\begin{aligned}\widehat{\pi}_t &= \rho\tau\left(\widehat{\pi}_{t-1}-\widehat{g}_t^{\pi}\right)+\rho\left(1-\tau\right)\left(\widehat{\pi}_{t-2}-\widehat{g}_{t-1}^{\pi}+\widehat{g}_t^{\pi}\right)\\ &\quad+\sum_{j=0}^{\infty}\xi_{1,t}^{j}\sum_{i=0}^{\infty}\xi_{2,t}^{i}E_t\left\{\widehat{\iota}_{t+i+j}\right\}+u_{\pi,t}, \qquad (35)\\ \widehat{\iota}_{t+i+j} &\equiv \zeta_t\left(\widehat{mc}_{t+i+j}-\varphi_{1,t}\widehat{mc}_{t+1+i+j}\right)+\gamma_t\varphi_{1,t}\left(\widehat{q}_{t+i+j,t+1+i+j}+\widehat{g}_{t+1+i+j}^{y}\right).\end{aligned}$$

The derivation of (35), and the definition of $\left\{\xi_{1,t},\ \xi_{2,t}\right\}$ are left to Appendix C.[35] This appendix also discusses how this expression can be interpreted as a limit that imposes all model consistency restrictions on expectations, just as with the constant coefficients case from section 2.

We provide estimates of this exact closed form in section 3.5. These estimates, like the estimates of the DE specification, show that γ_t is typically small and unimportant from an economic standpoint. The main channel through which expected future conditions affect inflation is mainly real marginal costs (and their expected path). It is then possible to derive a simpler expression that imposes model-consistent restrictions only on terms that are unaffected by γ_t. In particular, we can iterate forward equation (33) to obtain an infinite-horizon specification that takes the form

$$\begin{aligned}\widehat{\pi}_t &= \rho\tau(\widehat{\pi}_{t-1}-\widehat{g}_t^{\pi})+\rho(1-\tau)(\widehat{\pi}_{t-2}-\widehat{g}_{t-1}^{\pi}-\widehat{g}_t^{\pi})\\ &\quad+\zeta_t\sum_{i=0}^{\infty}\lambda_t^{i}E_t\widehat{mc}_{t+i}+\gamma_t\sum_{i=0}^{\infty}\lambda_t^{i}E_t\widehat{D}_{t+i}+u_{\pi,t}, \qquad (36)\end{aligned}$$

which we refer to as the CF version of the NKPC. This relationship imposes restrictions at each point in time on the expected dynamics of non-predetermined inflation, in the same spirit of the exact closed form. In particular, equation (36) explicitly isolates the part of inflation that is

[35] In deriving (35) and (36) below, and as in Cogley and Sbordone (2008), expectations are formed using an "anticipated utility" framework (see Kreps, 1998, and Cogley and Sargent, 2008), where at each point in time agents expect all model coefficients to stay constant at their current values going forward. This implies, for instance, that

$$E_t\left\{\xi_{t+i}^{i}\xi_{t+j}^{j}\widehat{\iota}_{t+i+j}\right\}=\xi_t^{i}\xi_t^{j}E_t\widehat{\iota}_{t+i+j}$$

for any integers i, $j>0$.

forward-looking, which is the essential component of this intertemporal model. To see this, note that the non-predetermined part of inflation is[36]

$$\widehat{B}_t \equiv \widehat{\pi}_t - \rho\left[\tau(\widehat{\pi}_{t-1} - \widehat{g}^{\pi}_t) + (1-\tau)(\widehat{\pi}_{t-2} - \widehat{g}^{\pi}_{t-1} - \widehat{g}^{\pi}_t)\right],$$

and that the DE specification can be written as

$$\widehat{B}_t = E_t\left\{\lambda_t \widehat{B}_{t+1} + \zeta_t \widehat{mc}_t + \gamma_t \widehat{D}_t\right\} + u_{\pi,t}.$$

Equation (36) imposes model restrictions on the expected evolution of B_{t+j} as follows[37]

$$E_t\left\{\widehat{B}_{t+j}\right\} = E_t\left\{\lambda_{t+j}\widehat{B}_{t+j+1} + \zeta_{t+j}\widehat{mc}_{t+j} + \gamma_{t+j}\widehat{D}_{t+j}\right\}, \qquad \text{for any } j \geq 1. \tag{37}$$

The set of equations in (37) provides one possible way of consistently expressing the restrictions that the model places on expectations. It is possible to be more exhaustive in requiring model discipline by also recognizing that $\widehat{D}_{t+j}$ depends on $\widehat{B}_{t+j+1}$. This would fully restrict the evolution of expectations, as in the exact closed-form solution. But as long as the model is correct, imposing model consistency through (37) adds useful model information without the extreme demands of an exact closed-form solution. This information is not explicitly taken into account in the DE version of the NKPC, and conceptually it is introduced in the same way as in the standard constant coefficients setup with zero trend-inflation. Moreover, the quasi-closed form in (36) becomes the exact closed form when $\gamma_t = 0$ for all t. We later show that γ_t is estimated to be statistically and economically insignificant, and that setting it equal to zero yields the same deep parameter estimates for the NKPC.[38] Also, the estimates from the exact closed form are remarkably close to the estimates from the CF specification even when γ_t is not forced to be zero. This indicates that the additional restrictions that the exact closed form imposes over (36) are not critical for estimating the NKPC structural parameters.

Our discussion so far has assumed that the present discounted values in (36) are well defined. We address the issue concerning convergence of these geometric series in detail in section 3.4.

[36] The term $\widehat{g}^{\pi}_t$ is exogenous, as trend inflation is modeled as an exogenous process.

[37] This restriction is completely explicit on the impact that $\widehat{mc}_{t+j}$ has on non-predetermined inflation, $\widehat{B}_{t+j}$, since $\widehat{D}_{t+j}$ does not depend on $\widehat{B}_{t+j}$.

[38] Given our estimates of trend inflation and the median estimates of the NKPC deep parameters, $|\gamma_t|$ is typically below 0.001. For instance, 99.9 percent of the exact closed-form estimates satisfy $|\gamma_t| < 0.001$.

3.2 Estimation Approach

We compare estimates of the deep structural parameters $\psi=(\alpha,\rho,\theta,\tau)$ obtained from four types of specifications: The DE form (33), the CF representation (36), the cases with only a finite number of model restrictions on expectations that fall in between the DE and the CF specifications, and the ECF solution. The two-step estimation procedure has already been discussed in section 2.1. In the present context in which trend inflation is time-varying, the reduced-form VAR has drifting coefficients (see Cogley and Sbordone, 2008). The VAR in first-order form can then be written as

$$\mathbf{z}_t = \boldsymbol{\mu}_t + \mathbf{A}_t \mathbf{z}_{t-1} + \boldsymbol{\varepsilon}_{z,t}, \tag{38}$$

where $\boldsymbol{\varepsilon}_{z,t}$ is a possibly heteroskedastic but serially uncorrelated error vector and the coefficients in $\boldsymbol{\mu}_t$ and $\mathbf{A}_t$ are assumed to evolve as a random walk. The evolution of the coefficients in $\mathbf{A}_t$ is constrained so that the roots of $\mathbf{A}_t$ at each point in time lie inside the unit circle. Using the VAR in (38), the forecasting rule (6) is modified as follows

$$E_t \widehat{y}_{t+k} = \mathbf{e}_y' \mathbf{A}_t^k \widehat{\mathbf{z}}_t, \tag{39}$$

where $\widehat{\mathbf{z}}_t$ is the vector of variables expressed in deviations from the time-varying trends

$$\widehat{\mathbf{z}}_t \equiv \mathbf{z}_t - (\mathbf{I} - \mathbf{A}_t)^{-1} \boldsymbol{\mu}_t,$$

and $\mathbf{e}_y'$ is the selection vector for variable y_t in $\mathbf{z}_t$. Given the forecasting rule (39) and equations (33) and (34), we obtain the conditional expectation of inflation in the DE form, based on information at $t-2$, as follows

$$\begin{aligned}
\mathbf{e}_\pi' \mathbf{A}_{t-2}^2 \widehat{\mathbf{z}}_{t-2} \;=\;& \widetilde{\rho}_{1,t-2}^D \mathbf{e}_\pi' \mathbf{A}_{t-2} \widehat{\mathbf{z}}_{t-2} + (1-\tau) \widetilde{\rho}_{2,t-2}^D \mathbf{e}_\pi' \widehat{\mathbf{z}}_{t-2} + \widetilde{\zeta}_{t-2}^D \mathbf{e}_{mc}' \mathbf{A}_{t-2}^2 \widehat{\mathbf{z}}_{t-2} \\
& + d_{1,t-2}^D \mathbf{e}_\pi' \mathbf{A}_{t-2}^3 \widehat{\mathbf{z}}_{t-2} + d_{2,t-2}^D \varphi_{1,t-2} \mathbf{e}_\pi' \mathbf{J}_{t-2} \mathbf{A}_{t-2}^4 \widehat{\mathbf{z}}_{t-2} \\
& + d_{3,t-2}^D (\mathbf{e}_Q' \mathbf{J}_{t-2} \mathbf{A}_{t-2}^2 \widehat{\mathbf{z}}_{t-2} + \mathbf{e}_{g^y}' \mathbf{J}_{t-2} \mathbf{A}_{t-2}^3 \widehat{\mathbf{z}}_{t-2}),
\end{aligned} \tag{40}$$

where

$$\mathbf{J}_t \equiv (\mathbf{I} - \varphi_{1,t} \mathbf{A}_t)^{-1}. \tag{41}$$

We take expectations as of $t-2$ because the indexation mechanism is based on two lags of inflation, and this allows us to ignore the terms involving the growth rate of trend inflation.[39] The full

[39] As already mentioned, trend inflation evolves as a (driftless) random walk, and therefore the expected future growth rate of trend inflation is zero.

derivation of (40) and the definition of coefficients is left to Appendix B. The same conditions ensuring that the steady-state relationship (32) is well defined imply that $\left|\varphi_{1,t}\right| < 1.$[40] This, together with the fact that the roots of $\mathbf{A}_t$ are constrained to lie inside the unit circle, guarantees that the series $\mathbf{I} + \varphi_{1,t}\mathbf{A}_t + \varphi_{1,t}^2\mathbf{A}_t^2 + ...$, is convergent and can be compactly written as in equation (41). The vector of cross-equation restrictions implied by the conditional expectation in (40) is then

$$\begin{aligned}
\mathbf{e}'_\pi\mathbf{A}_{t-2}^2 &= \widetilde{\rho}_{1,t-2}^D\mathbf{e}'_\pi\mathbf{A}_{t-2} + (1-\tau)\widetilde{\rho}_{2,t-2}^D\mathbf{e}'_\pi\mathbf{I} + \widetilde{\zeta}_{t-2}^D\mathbf{e}'_{mc}\mathbf{A}_{t-2}^2 \\
&\quad + d_{1,t-2}^D\mathbf{e}'_\pi\mathbf{A}_{t-2}^3 + d_{2,t-2}^D\varphi_{1,t-2}\mathbf{e}'_\pi\mathbf{J}_{t-2}\mathbf{A}_{t-2}^4 \\
&\quad + d_{3,t-2}^D(\mathbf{e}'_Q\mathbf{J}_{t-2}\mathbf{A}_{t-2}^2 + \mathbf{e}'_{g^y}\mathbf{J}_{t-2}\mathbf{A}_{t-2}^3) \\
&\equiv \mathbf{g}^D(\boldsymbol{\mu}_{t-2}, \mathbf{A}_{t-2}, \boldsymbol{\psi}), \qquad (42)
\end{aligned}$$

which we can write as

$$\mathbf{F}_1^D(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}) \equiv \mathbf{e}'_\pi\mathbf{A}_t^2 - \mathbf{g}^D(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}) = \underline{0}', \forall t.$$

In contrast, for the CF specification of the NKPC in (36) the conditional expectation of inflation based on information at $t-2$ is now

$$\begin{aligned}
\mathbf{e}'_\pi\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} &= \widetilde{\rho}_{1,t-2}^C\mathbf{e}'_\pi\mathbf{A}_{t-2}\widehat{\mathbf{z}}_{t-2} + (1-\tau)\,\widetilde{\rho}_{2,t-2}^C\mathbf{e}'_\pi\widehat{\mathbf{z}}_{t-2} + \widetilde{\zeta}_{t-2}^C\mathbf{e}'_{mc}\mathbf{K}_{t-2}\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} \\
&\quad + d_{0,t-2}^C\mathbf{e}'_\pi\mathbf{K}_{t-2}\mathbf{A}_{t-2}\widehat{\mathbf{z}}_{t-2} + d_{1,t-2}^C\mathbf{e}'_\pi\mathbf{K}_{t-2}\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} \\
&\quad + d_{2,t-2}^C\mathbf{e}'_\pi\mathbf{K}_{t-2}\mathbf{A}_{t-2}^3\widehat{\mathbf{z}}_{t-2} + d_{2,t-2}^C\varphi_{1,t-2}\mathbf{e}'_\pi\mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}_{t-2}^4\widehat{\mathbf{z}}_{t-2} \\
&\quad + d_{3,t-2}^C(\mathbf{e}'_Q\mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} + \mathbf{e}'_{g^y}\mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}_{t-2}^3\widehat{\mathbf{z}}_{t-2}), \qquad (43)
\end{aligned}$$

where

$$\mathbf{K}_t \equiv (\mathbf{I} - \lambda_t\mathbf{A}_t)^{-1}. \qquad (44)$$

We leave the derivation of (43) and the definition of coefficients to Appendix B. It is important to stress that for the series $\mathbf{I} + \lambda_t\mathbf{A}_t + \lambda_t^2\mathbf{A}_t^2 + ...$, to have a representation as in (44), the roots of $\lambda_t\mathbf{A}_t$ need to lie inside the unit circle, that is $\|\lambda_t\mathbf{A}_t\| < 1$. This is not guaranteed by model conditions alone, and it is therefore an empirical issue. Still, if this condition does not hold, the NKPC is incompatible with any structural model with a reduced-form VAR representation as in (38), unless one is willing to make additional assumptions on other structural equations besides the NKPC. As we discuss in Appendix D, the reduced-form VAR would instead involve an infinite number of lags of

[40] These conditions are given by the inequalities (39) and (40) in Cogley and Sbordone (2008), which we also impose at the estimation stage.

the endogenous variables, or equivalently the error term would have a moving average component. Neither of these two representation would be consistent with the law of motion in (38). Then, trying to match the cross-equation restrictions of the NKPC to a reduced-form VAR representation (38) that the model cannot possibly admit would invalidate the estimation procedure. In essence, the estimation procedure adopted here (and in Cogley and Sbordone, 2008) assumes a reduced form as in (38), and hence implies that $\|\lambda_t \mathbf{A}_t\| < 1$. This condition is only necessary for the VAR in (38) to be valid (as we show in section 3.5), but empirical violations of this condition would render meaningless the estimates for all the specifications we consider.[41] In the discussion of the empirical results we show that the CF estimates typically satisfy this condition, while the DE estimates do not. We also provide the necessary and sufficient condition for the validity of the first stage (the estimation of (38)) when discussing the ECF estimates, and reach the same conclusion.

The vector of cross-equation restrictions implied by the conditional expectation in (36) is given by

$$\begin{aligned}
\mathbf{e}'_\pi \mathbf{A}^2_{t-2} &= \widetilde{\rho}^C_{1,t-2}\mathbf{e}'_\pi \mathbf{A}_{t-2} + (1-\tau)\,\widetilde{\rho}^C_{2,t-2}\mathbf{e}'_\pi \mathbf{I} + \widetilde{\zeta}^C_{t-2}\mathbf{e}'_{mc}\mathbf{K}_{t-2}\mathbf{A}^2_{t-2} \\
&\quad + d^C_{0,t-2}\mathbf{e}'_\pi \mathbf{K}_{t-2}\mathbf{A}_{t-2} + d^C_{1,t-2}\mathbf{e}'_\pi \mathbf{K}_{t-2}\mathbf{A}^2_{t-2} \\
&\quad + d^C_{2,t-2}\mathbf{e}'_\pi \mathbf{K}_{t-2}\mathbf{A}^3_{t-2} + d^C_{2,t-2}\varphi_{1,t-2}\mathbf{e}'_\pi \mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}^4_{t-2} \\
&\quad + d^C_{3,t-2}(\mathbf{e}'_Q \mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}^2_{t-2} + \mathbf{e}'_{g^y}\mathbf{K}_{t-2}\mathbf{J}_{t-2}\mathbf{A}^3_{t-2}) \\
&\equiv \mathbf{g}^C(\boldsymbol{\mu}_{t-2}, \mathbf{A}_{t-2}, \boldsymbol{\psi}), \qquad (45)
\end{aligned}$$

and the relevant distance for the estimation is

$$\mathbf{F}^C_1(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}) \equiv \mathbf{e}'_\pi \mathbf{A}^2_t - \mathbf{g}^C(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}) = \underline{0}'.$$

As we already discussed, the function $\mathbf{g}^C(\boldsymbol{\mu}_t, \mathbf{A}_t, \psi)$ imposes model-consistent restrictions that are not captured in $\mathbf{g}^D(\boldsymbol{\mu}_t, \mathbf{A}_t, \psi)$.

When trend inflation varies over time, the long-run relationship given by equation (32) provides an additional condition that is minimized at the estimation stage for all specifications. Correspondingly, we define

$$\mathbf{F}_2(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}) \equiv \left(1 - \alpha \overline{\Pi}_t^{(1-\rho)(\theta-1)}\right)^{(1+\theta\omega)/(1-\theta)} \left[\frac{1 - \alpha \overline{q}\overline{g}^y \overline{\Pi}_t^{\theta(1+\omega)(1-\rho)}}{1 - \alpha \overline{q}\overline{g}^y \overline{\Pi}_t^{(\theta-1)(1-\rho)}}\right] - (1-\alpha)^{(1+\theta\omega)/(1-\theta)} \frac{\theta}{\theta-1} \overline{mc}_t = \underline{0}'.$$

[41] This is an issue that arises in limited information estimation, when estimates for $\mathbf{A}_t$ and λ_t are obtained at different stages.

The complete set of cross-equation restrictions that need to be satisfied is then given by

$$\begin{aligned}\mathcal{F}^{\,h}(\mathbf{\Theta}) &\equiv \left[\mathcal{F}_1^{\,h}\,,...,\mathcal{F}_T^{\,h}\right] \quad \text{for } h \,\epsilon\, \{D,\, C\}\,, \\ \mathcal{F}_t^{\,h} &\equiv \left[\mathbf{F}_1^{\,h}\left(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}\right),\ \mathbf{F}_2\left(\boldsymbol{\mu}_t, \mathbf{A}_t, \boldsymbol{\psi}\right)\right],\end{aligned}$$

where

$$\mathbf{\Theta} \equiv \{\boldsymbol{\mu}_t, \mathbf{A}_t\}_{t=1}^{T}\,.$$

The first step of the estimation procedure uses the Bayesian method in Cogley and Sargent (2005) to characterize the posterior distribution of $\mathbf{\Theta}$ from a set of M estimates $\{\widehat{\mathbf{\Theta}}_i\}_{i=1}^{M}$. In the second step we obtain the structural parameter estimates

$$\widehat{\boldsymbol{\psi}}_i^{\,h} \equiv \arg\min \mathcal{F}^{\,h}(\widehat{\mathbf{\Theta}}_i) \cdot \mathcal{F}^{\,h}(\widehat{\mathbf{\Theta}}_i)' \quad \text{for } i = 1,..,M,$$

where h indicates the different type of NKPC specifications being considered. These specifications are given by the DE form ($h = D$), the CF formulation ($h = C$), and the exact closed form ($h = ECF$). The cross-equation restrictions for the exact closed-form case are discussed in Appendix C, as their derivation offers little additional insight to the discussion here.[42] In addition to these specifications, we estimate $\widehat{\boldsymbol{\psi}}_i^{D(j)}$, which impose model discipline on expectations of future inflation for (a finite number of) $j \geq 1$ consecutive periods. The cross-equation restrictions involved in this case can be obtained by postmultiplying the DE restrictions in (42) by $(\mathbf{I} + \lambda_t \mathbf{A}_t + ... + \lambda_t^j \mathbf{A}_t^j)$. These formulations are therefore in between the DE and the CF ($j \to \infty$) specifications, which correspond to $j = 0$ and $j \to \infty$, respectively. They are important because they gauge the impact of imposing model discipline on expectations without resorting to the extreme requirements of the closed form. In addition, they do not impose the assumption that $\|\lambda_t \mathbf{A}_t\| < 1$, which is used to derive the CF estimates. In this dimension, their underlying assumptions are the same as those of the DE estimates, and as long as j is small, the roots of $\lambda_t \mathbf{A}_t$ are not a source of concern. Likewise, it remains true that a violation of the condition $\|\lambda_t \mathbf{A}_t\| < 1$ would make the first-stage estimation incompatible with the second stage.

We use the same data as in Cogley and Sbordone (2008).[43] The technical aspects of the first-stage estimation and the data are described in their paper. We provide results for their same sample period, but as a robustness check we also report sub-sample (post-1983) findings. Here, we just

[42] Appendix C also links explicitly the analysis of the model with time-varying trends and the fixed-coefficients model from section 2.

[43] These data and Cogley and Sbordone's code, upon which our code builds, are available at the AEA website.

mention that the quarterly reduced-form time-varying VAR is of order 2 and has four variables: inflation, real marginal costs (as proxied by the labor share), output growth, and a nominal discount factor, based on the federal funds rate. Because the first-stage Bayesian estimation yields an entire posterior distribution $\{\widehat{\boldsymbol{\Theta}}_i\}_{i=1}^M$, the second stage also provides a distribution of estimates $\{\widehat{\boldsymbol{\psi}}_i^h\}_{i=1}^M$, where h denotes the type of specification we consider. The number of ensembles M is set to 5,000, and the same ensembles are used to compute the distributions of all four types of second-stage estimates.

3.3 Estimation Results

Tables 1 and 2 display our main full-sample (from 1960:Q1 to 2003:Q4) estimation results.[44] In Table 1, we compare the DE and CF estimates, and in Table 2 we focus on the important cases that lie in between the DE and CF specifications. Specifically, Table 2 reports the estimates $\{\widehat{\boldsymbol{\psi}}_i^{D(j)}\}_{i=1}^M$ when $j = 2, 4, 6$, and 8 quarters. The exact closed-form (ECF) estimates are discussed in section 3.5. Throughout our empirical analysis, the parameters β and ω are set as in section 2.[45] In Table 1, we start by replicating Cogley and Sbordone (2008) baseline estimates, which are reported in the first row of the table (denoted as *DE_ con*). These correspond to the DE estimates $\widehat{\boldsymbol{\psi}}^D$ with τ constrained to equal 1. The indexation parameter ρ is estimated at zero, and the estimated α implies that the median time-span to re-optimization is between one and two quarters.

We then consider the DE specification without constraining τ to equal unity. This is a small but potentially important change in the specification, as the reduced-form inflation equation from the estimated VAR places a significant weight on the second lag of inflation. Our Monte Carlo results in section 2.3 illustrate that the misspecification bias from incorrectly constraining τ to unity can be very large, especially so when the NKPC is estimated in the DE form. Estimation results (denoted as *DE_ uncon* in the tables) show that this simple modification produces a very different estimate for ρ, suggesting misspecification bias when estimating the NKPC with τ set to 1. The median estimated value for ρ jumps from zero in the previous case to 0.64. While imprecisely estimated, the 90 percent trust region does not include zero. Moreover, τ is estimated at 0.56, and its 90 percent trust region is bounded well away from one. This implies that the first and the second lags of inflation in the indexation mechanism receive approximately the same weight, suggesting that the average inflation over half a year is more relevant for indexation than just the most recent

[44] We report as a point estimate the median of the distribution of estimates for each parameter.
[45] The parameters are set to 0.99 and 0.43, respectively, for ease of comparison with Cogley and Sbordone (2008).

quarter.

Next we consider the estimates obtained from the CF specification of the NKPC. To highlight the differences in results when this specification is used instead of the DE form, we report constrained ($\tau = 1$) and unconstrained estimates. For the constrained case (CF_con in the tables) the median estimate of ρ is 0.59, with the 90 percent trust region that does not encompass zero. Note also that the median estimate of α increases to 0.84, implying a median time between optimal resets of approximately 5 quarters, almost three times the estimate obtained from the DE specification (DE_con). Moreover, the 90 percent trust regions for this parameter in the DE and CF specifications do not overlap.

Estimating the CF specification that does not fix τ (CF_uncon in the tables) yields a median estimate for ρ of 0.89. This is an even larger estimate than the one obtained from the DE counterpart (DE_uncon), and it is estimated much more precisely. The median estimate of α, at 0.88, implies that re-optimization now occurs every 5 quarters on average. It is important to stress that this refers to the frequency at which firms optimally reset prices. In the presence of indexation, prices are changed more frequently than they are being reoptimized. This relative infrequency of optimal resets is consistent with evidence at the micro level suggesting that the information necessary to set the optimal markup is costly to obtain (see Zbaracki et al., 2004). The estimated indexation mechanism in the unconstrained CF specification places a larger weight on the most recent quarter of inflation than in the DE counterpart; the median estimate is now 0.69. Still, the 90 percent trust region for τ does not contain unity. The median estimate of θ is 12.3, which implies a steady-state markup of 8.8 percent. This estimate for the steady-state markup is somewhat lower than the steady-state markup of 11 percent estimated from the corresponding DE specification. In addition, the 90 percent trust region for θ tends to be larger using the CF specification. In all, the CF estimates for ρ, α, and θ suggest that when all of the model restrictions on expectations are taken into account, the link from marginal costs to inflation becomes weaker.

Table 2 illustrates how the estimates change when moving from the DE to the CF specification by progressively imposing additional model consistent restrictions on inflation expectations. We mentioned in section 2.2 that the DE and the CF specifications can be interpreted as two extremes, with the CF specification imposing, relative to the DE form, additional model-consistent expectations on *all* future realizations of inflation. But we have emphasized that there is an important middle ground where model discipline is explicitly imposed on expectations for just a finite number of periods. Such middle ground can be an effective way of trading off the relatively unstructured handling of expectations in the DE form, and the CF specification's high internal consistency requirements. For this reason, these estimates are no less relevant than the two extremes examined in Table 1.

In this second table, we focus on the specification of the NKPC that does not restrict τ to unity. To facilitate the discussion, the table also reports the corresponding DE and CF estimates already shown in Table 1. The results mirror our Monte Carlo findings regarding the quick convergence of this type of estimates to the CF limiting case. Setting $j = 4$ (*DE_uncon(4)* in the table), which corresponds to four quarters of model-consistent expectations, results in estimates for ρ and α that are noticeably different from their DE counterparts. The *DE_uncon(4)* estimates have already bridged about half of the gap between the DE and CF estimates. When considering $j = 8$ (*DE_uncon(8)* in the table), which corresponds to two years of model consistent expectations, the median estimates for ρ and α become even closer to the corresponding CF estimates. As we already mentioned, these estimates are important because for the small values of j we consider here, the assumption that $\|\lambda_t \mathbf{A}_t\| < 1$ is not imposed at the estimation stage. Therefore, they provide additional support to the CF results, as they illustrate that the CF estimates are not crucially affected by such an assumption.

3.4 A Necessary Condition for the Validity of Parameter Estimates

Although the condition $\|\lambda_t \mathbf{A}_t\| < 1$ is assumed when deriving the CF specification, a violation of this condition would invalidate all of the estimates reported in Tables 1 and 2, as we mentioned in section 3.2. In particular, the estimates in Tables 1 and 2 rely on this implicit assumption as a necessary condition for the first and second estimation stages to be compatible with each other.[46]

Panel A in Figure 10 shows the distribution of the largest estimated root of $\hat{\lambda}_t \hat{\mathbf{A}}_t$ in absolute value for the *CF_uncon* specification. Violations of this assumption are infrequent. The 99th percentile of the distribution stays most of the time well below one. It is only during the period 1973 to 1979 that the 99th percentile of the distribution rises above unity, but the 95th percentile is far below one throughout the entire sample. Still, a possible concern is that assuming that

[46] When the condition $\|\lambda_t \mathbf{A}_t\| < 1$ is violated, the NKPC solution is indeterminate. In this case, discussed in Appendix C, inflation dynamics are affected by a predictable error unless the structural shock to inflation u_π is assumed to be zero for all t. The presence of this predictable error would imply that expectations formed with the VAR in (5) are biased, unless one imposes additional structural assumptions on the behavior of other equations besides the NKPC.

$\|\lambda_t \mathbf{A}_t\| < 1$ in order to derive the closed form of the NKPC could bias the estimates towards satisfying this condition. We have shown, however, that the estimates $\widehat{\boldsymbol{\psi}}^{D(4)}$ are already close to the CF estimates. Not surprisingly then, the distribution (not shown) of $\left\|\hat{\lambda}_t \hat{\mathbf{A}}_t\right\|$ for $\widehat{\boldsymbol{\psi}}^{D(4)}$ is still largely below unity. Moreover, increasing the number of model consistent restrictions on expectations shifts this distribution towards the one based on the CF estimates.[47]

The corresponding findings for the *DE_ con* specification are sharply different, indicating an incompatibility between first- and second-stage estimates. In this case, reported in panel B of Figure 10, the distribution of $\left\|\hat{\lambda}_t \hat{\mathbf{A}}_t\right\|$ has the 95th percentile above unity almost always. Between 1970 and 1985 the fraction of estimates violating the stability condition ranges from 10 to 33 percent. Worse yet, the fraction of ensembles for which the *DE_ con* estimates never violate this stability condition over the estimated sample period is a mere 11 percent. In contrast, using this more restrictive criterion the corresponding fraction for the *CF_ uncon* estimates is 79 percent.

In all, these findings point to an inconsistency between the assumption of a reduced-form VAR as in (38) and the reduced form implied by the DE estimates when $\|\lambda_t \mathbf{A}_t\| \geq 1$. As highlighted in section 3.2 and in Appendix D, when $\|\lambda_t \mathbf{A}_t\| \geq 1$ the finite-order reduced-form VAR would involve a moving average error. Failure to take this into account yields to regressors in the VAR being contemporaneously correlated with the error term. The stark contrast between the distributions in the two panels of Figure 10 indicates that this is much less of an issue for the CF estimates. For this reason, the estimation strategy seems better suited for the CF specification.

Our discussion so far raises the issue of whether the estimates change when we consider only the ensembles that do not violate the necessary condition $\|\lambda_t \mathbf{A}_t\| < 1$. For the set of specifications that do not fix τ to unity in the estimation, which are more relevant given the fraction of surviving ensembles, the results are reported in Table 3. If anything, this evidence corroborates our previous findings that imposing additional model-consistent restrictions has a substantial impact on the estimates. Too, for the DE form (*DE_ uncon*) the median estimated ρ is already 0.78. Convergence of ρ and α to the CF estimates, as illustrated by $\widehat{\boldsymbol{\psi}}^{D(4)}$, is rapid.

[47] For $\widehat{\boldsymbol{\psi}}^{D(4)}$, the 95th percentile of the distribution of $\left\|\hat{\lambda}_t \hat{\mathbf{A}}_t\right\|$ is above unity from 1972 to 1979, but the 90th percentile is always below unity. Results for $\widehat{\boldsymbol{\psi}}^{D(j)}$, where $j = \{2, 4, 6, 8\}$, are available upon request.

3.5 Exact Closed-Form Estimates

We now discuss the exact closed-form estimates and their relationship with the CF estimates reported in Tables 1 and 2. The relevant exact closed form is given in (35), and the cross-equation restrictions used in the estimation are detailed in Appendix C. These estimates are especially relevant because they provide a check on whether the expectational restrictions that we did not impose on our CF estimates play a role in our results. The necessary and sufficient conditions for the validity (and uniqueness) of the ECF solution are particularly transparent, and as a result they facilitate assessing whether the first-stage estimation is consistent with the second-stage estimation. These conditions are given by[48]

$$\left\|\xi_{i,t} \cdot A_t\right\| < 1 \text{ for } i = 1, 2.$$

Without loss of generality, for $\xi_{1,t} > \xi_{2,t}$ we have

$$\begin{aligned} \xi_{1,t} &\geq \lambda_t \\ \xi_{2,t} &\leq \varphi_{1,t}, \end{aligned}$$

with equality for $\gamma_t = 0$. It is clear then that the condition $\|\lambda_t A_t\| < 1$ discussed in the context of our CF estimates is just a necessary condition for determinacy and hence, for the existence of the exact closed form (35).

Table 4 provides the estimates for this specification without restricting τ. We present four sets of estimates. In the top row (ECF) we report median parameter estimates for all the optimized ensembles. Using these parameter estimates, Figure 11 examines the necessary and sufficient conditions for determinacy. The 95th percentile of $\left\|\hat{\xi}_{i,t} \cdot \hat{A}_t\right\|$, for $i = 1, 2$, is always below one with the exception of a few small violations during the mid-1970s. The other rows in Table 4 report medians for ensembles that do not violate the necessary and sufficient conditions for the validity of the ECF specification more than 100-, 95-, or 90-percent of the time, respectively. The most restrictive criterion (100 percent) allows one to preserve only those ensembles for which no violations ever occur over the 174 quarters. This criterion is satisfied by 60 percent of the optimized ensembles. The 95 and 90 percent criteria allow one to preserve 90 and 98 percent of the ensembles, respectively.

It is apparent from the table that regardless of the criterion chosen, the estimates are remarkably similar. What is most relevant for our analysis is that the distribution of estimates is virtually the

[48] The condition $\left\|\varphi_{1,t} \cdot A_t\right\| < 1$ is required in all specifications.

same as for our *CF_ uncon* specification. The medians differ by less than 1 percent. This implies that neglecting the additional restrictions imposed by the exact closed form relative to the CF specification is inconsequential for our findings.

Next, we consider a particular case in which the CF specification is the exact closed form. This requires omitting terms involving the discount factor, output growth, and terms involving higher-order leads of inflation. These terms do not appear to be empirically relevant for our estimates. Since this amounts to setting $\gamma_t = 0$ for all t, it is apparent from equation (36) that the CF specification provides the exact closed form. For this set of estimates, we also shut down time variation in the NKPC coefficients.[49] Note that in this case λ_t is simply equal to β, and as a result, the stability requirement that $\|\beta \mathbf{A}_t\| < 1$ is always satisfied. In practical terms, this means that the second-stage estimates cannot produce evidence contrary to the implicit assumption in the first stage of model determinacy.[50] Estimation results are reported in Table 5 for the same specifications considered in Table 1. It is evident that the findings are largely unaltered from a qualitative standpoint. Quantitatively, the changes in the estimates are very small across all specifications. It is then apparent that differences between the DE and CF estimates (constrained or unconstrained), do not depend on whether the stability requirement is met, and are instead the consequence of the CF specification exploiting additional model-consistent restrictions.

3.6 Subsample Analysis

We discuss here a robustness check that we deem especially interesting, which concerns the estimates of the NKPC over the post-1983 sample. The 20 years (1984:Q1 to 2003:Q4) we examine were a period of mostly stable and low inflation. It is relevant to ask whether this low inflation (and low inflation volatility) environment was accompanied by noticeable changes in the values taken by some of the deep parameters of the NKPC. For this exercise, we consider the very same specifications of the NKPC with time-varying coefficients examined in Table 1, with the only difference consisting in the chosen sample period. We use the shorter sample period in both the first and the second

[49] Cogley and Sbordone (2008) also report DE estimates for this simpler specification in Table C.3. The estimation procedure still uses the time-varying VAR in (38) to form expectations, and the long-run restriction (32).

[50] Even with a standard NKPC with constant coefficients it is possible to have indeterminate equilibria — for instance if policy does not satisfy the Taylor principle (see Lubik and Schorfheide, 2003). In our exercise this potential outcome is ruled out by assumption, since the VAR is assumed to properly characterize the unique stationary dynamics of the model, and because $\beta < 1$ (the real discount factor) is not estimated.

stage of estimation.[51] Estimation results for this exercise are reported in Table 6. The pattern of findings is largely unchanged. The role of the second lag in inflation indexation is now somewhat larger and close to the weight given to the first lag. Hence, backward dependence seems to be more closely related to semiannual inflation than to higher (quarterly) inflation frequencies. Again, only the *DE_ con* case yields estimates of ρ equal to zero, this time even more precisely than in the full sample. The *DE_ uncon* case and the CF specifications in the table yield a much larger and significant estimate of ρ. Estimates (not reported) for the exact closed form (35), as for the full sample, are very close to the *CF_ uncon* specification estimates.[52] Overall, the subsample estimates do not point to a shift in parameters towards a purely forward-looking NKPC. Similar conclusions hold when the sample is further restricted to the period 1989 to 2003.[53]

3.7 Discussion of Results And Relationship to Previous Literature

Overall, the estimation findings from specifications that impose model discipline on expectations are noticeably different from the unstructured DE specification. This is true even after estimating a specification that entails a more plausible (at least empirically) indexation mechanism. In addition to the CF and ECF results, we would like to emphasize that the estimates that discipline expectations for only a few periods already provide significant evidence on the importance of using these restrictions on expectations. After all, these restrictions are the defining characteristic of forward-looking rational expectations models and should not be neglected at the estimation stage. One can argue whether explicitly imposing rationality ad infinitum is an overly stringent empirical test for a model. For this reason, we have shown that it is not only possible but also relevant to constrain expectations to be model consistent for a short period of time. In our application, this turns out to be a particularly effective way of validating (and also intuitively explaining) the closed-form estimation results.

When interpreting the results, note that having ρ equal to one implies that the NKPC (expressed in the DE form) would assign about the same weights to past inflation and to expected future inflation.[54] Other studies have shown that the autocorrelation properties of detrended inflation are

[51] For the Bayesian estimation of the time-varying coefficients VAR in the first stage, the pre-estimation training sample goes from 1963:Q4 to 1983:Q4. We checked that the second-stage estimates of the NKPC are not overly sensitive to the choice of training sample in the first stage.

[52] Estimation results for this specification are available upon request.

[53] We looked at this case in order to consider a training sample for the first-stage estimates that is restricted to the Great Moderation period. Estimation results are available upon request.

[54] This case would correspond to the NKPC assumed in Christiano, Eichenbaum, and Evans (2005), though their

better captured by a high value of ρ. For instance, Fuhrer (2009) notes that after removing the time-varying trend in this model, the first autocorrelation of inflation in the data over the period 1960:Q3 to 2003:Q4 is equal to 0.81. With ρ equal to zero, the first autocorrelation implied by the NKPC is very low, and a value of ρ close to one is necessary to match this crucial feature of the data.

Our closed-form estimates of ρ are similar to some previous estimates in the literature, although we are using a different estimation method and we are explicitly taking into account time-varying trend inflation. In particular, an almost even split between past and future inflation when characterizing inflation dynamics in the NKPC appears to be common to those estimation procedures which, like ours, take explicitly into account the constraints placed by the NKPC on all future expectations of inflation (Fuhrer and Moore, 1995; Fuhrer and Olivei, 2005; Lindé, 2005).[55] Still, it is important to stress that in our setup expectations are formed using the same reduced-form estimated VAR in all the specifications of the NKPC that we consider. In this very specific sense, the closed form does not use more information at the estimation stage than the DE form. But more generally the closed form does exploit additional information: It requires that the DE relationship hold at all future points in time. In this way, the closed form provides additional model structure to the expected evolution of future inflation that the DE form does not provide. The same holds true for the specifications that impose model consistency for only a finite number of future periods.

The CF and ECF estimates in our context are not the result of a full information estimation method, because the expectations obtained in the first stage come from an unconstrained reduced form. As such, the reduced form does not take into account any structural cross-equation restrictions. Still, imposing model discipline on expectations in the manner illustrated here allows a partial information method to include more model restrictions that are especially relevant for testing forward-looking models. We have applied this method in the context of a minimum-distance estimation, but the requirement that the cross-equation restrictions be satisfied in expected value in future periods could be exploited in other partial information methods such as GMM.[56]

specification does not include trend inflation.

[55] The finding of a ρ significantly different from zero is also consistent with work by Kozicki and Tinsley (2002) that explicitly considers time-varying trend inflation.

[56] In this respect, our work contributes to previous literature (Fuhrer, Moore, and Schuh, 1995, and Fuhrer and Olivei, 2005) that compares the properties of DE and CF estimates, albeit in different settings and using different estimation methods. In those papers, the CF relationship is estimated in a single stage by means of full-information methods, so that the forecasting rule differs from the reduced-form forecasting rule used to estimate the DE specification.

4 Conclusions

In this paper we illustrate why estimation of forward-looking models via limited information methods can differ substantially when the model is expressed in Euler equation (DE) form rather than in closed form (CF). The reason for the difference in estimates is that the closed form imposes model-consistent restrictions on expectations that are not explicitly imposed in the DE form. These restrictions contribute model information that results in much more precise estimates in small samples.

We also show that estimating the DE and CF specifications of a forward-looking model can be thought of as two polar cases: The CF specification imposes model discipline on expectations ad infinitum while the DE form does not use model structure to characterize expectations. There is also an important middle ground where model discipline on expectations is imposed on only a finite number of consecutive periods. This relaxes the strong model-consistency requirements of the closed form, and provides a useful tool to assess how much model discipline is needed to explain differences between the DE and CF estimates.

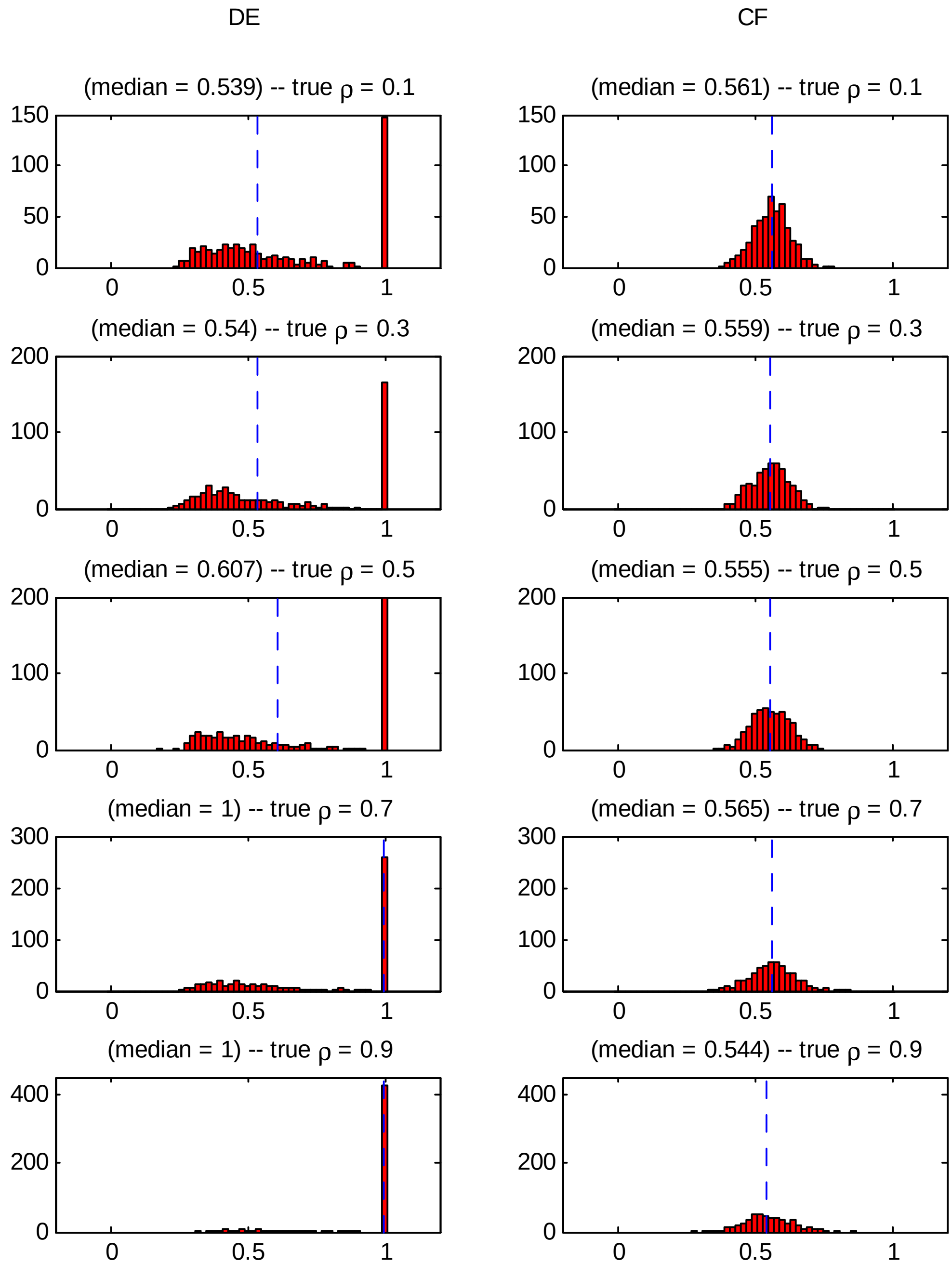
A desirable property of our framework is that imposing additional model-consistent restrictions on the DE form does not increase the scale of the estimation problem. While these restrictions can be interpreted as additional moment conditions, we show that the number of cross-equation restrictions is always the same, regardless of how much model discipline the econometrician is willing to impose explicitly. This is especially useful when estimating forward-looking Euler equations in a limited-information framework, as model-consistent expectations are an essential element of these models (Hansen and Sargent 1982).

Our application focuses on the estimation of a New Keynesian Phillips curve (NKPC). In Monte Carlo simulations, we show how the addition of model-consistent constraints (which do not necessarily hold in the DE form) yields more efficient and less biased estimates. On actual data, moving from the DE specification (Cogley and Sbordone, 2008) to the CF version of the NKPC leads to substantially different results concerning the importance of lagged inflation in the NKPC. The estimated role for backward-looking indexation goes from zero in some DE specifications to almost one using the CF specification. This implies that the NKPC assigns similar weights to lagged inflation and to expected future inflation. The estimation of the closed form suggests that accounting for time-varying trend inflation in the NKPC does not explain away inflation inertia. In addition, the CF specification implies that prices are re-optimized much less frequently than what is suggested by the DE form (approximately 12 months versus 4 months).

Our empirical findings hold for both the 1960–2003 and the post-1983 samples. Moreover, the results show that imposing model discipline on expectations for four quarters is already enough to yield estimates that are much more precise than their DE counterparts. These estimates are similar in distribution to the CF estimates.

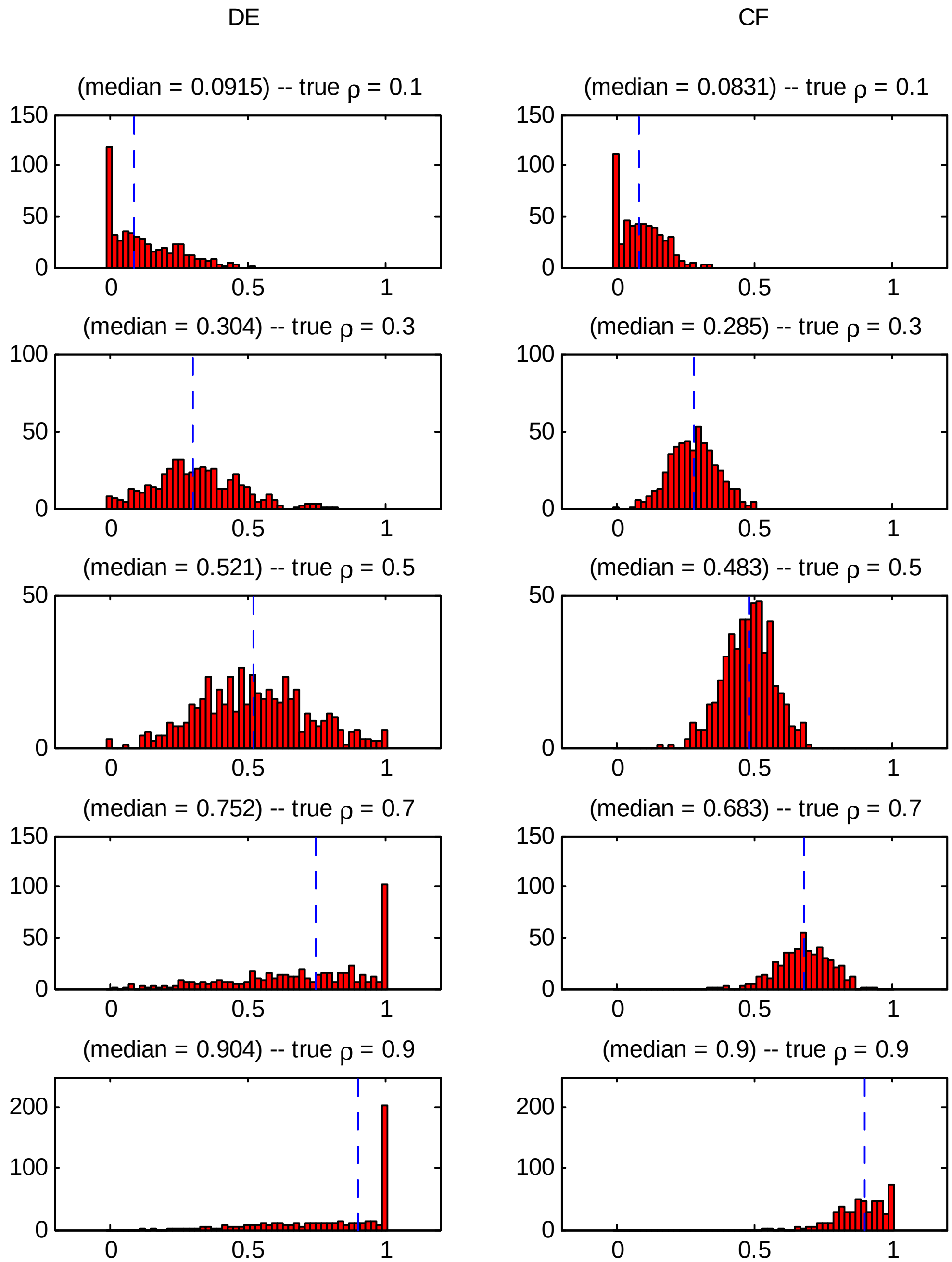
Whether inflation exhibits autonomous inertia has important implications for the dynamics of inflation. It is therefore important to approach the estimation of the NKPC using methods that take the forward-looking nature of the model as seriously as possible. The inflation model estimated here is used simply as a tool to illustrate the importance of acknowledging model discipline on expectations in estimation settings that do not necessarily require it. This widely used model assumes autonomous inflation inertia rather than deriving it from microfoundations. This lack of microfoundations does not mean that we can a priori reject inertia as a feature of the data. Moreover, "indexation" might just help to capture capture more complex structural behavior such as the impact of learning on pricing (Gumbau-Brisa, 2005). For the period we consider, our findings indicate that autonomous inertia is a relevant feature of the inflation process, and continue to highlight the need for better microfoundations. In this respect it should also be noted that our estimates question the relevance of marginal costs (as proxied by the labor share) as the driving process for inflation. The current economic environment may provide an important test of the model as a whole, and of the relevance of autonomous inertia when inflation is very low.

Figure 1: Monte Carlo simulations, estimates of α
Difference equation (DE) vs. closed form (CF)



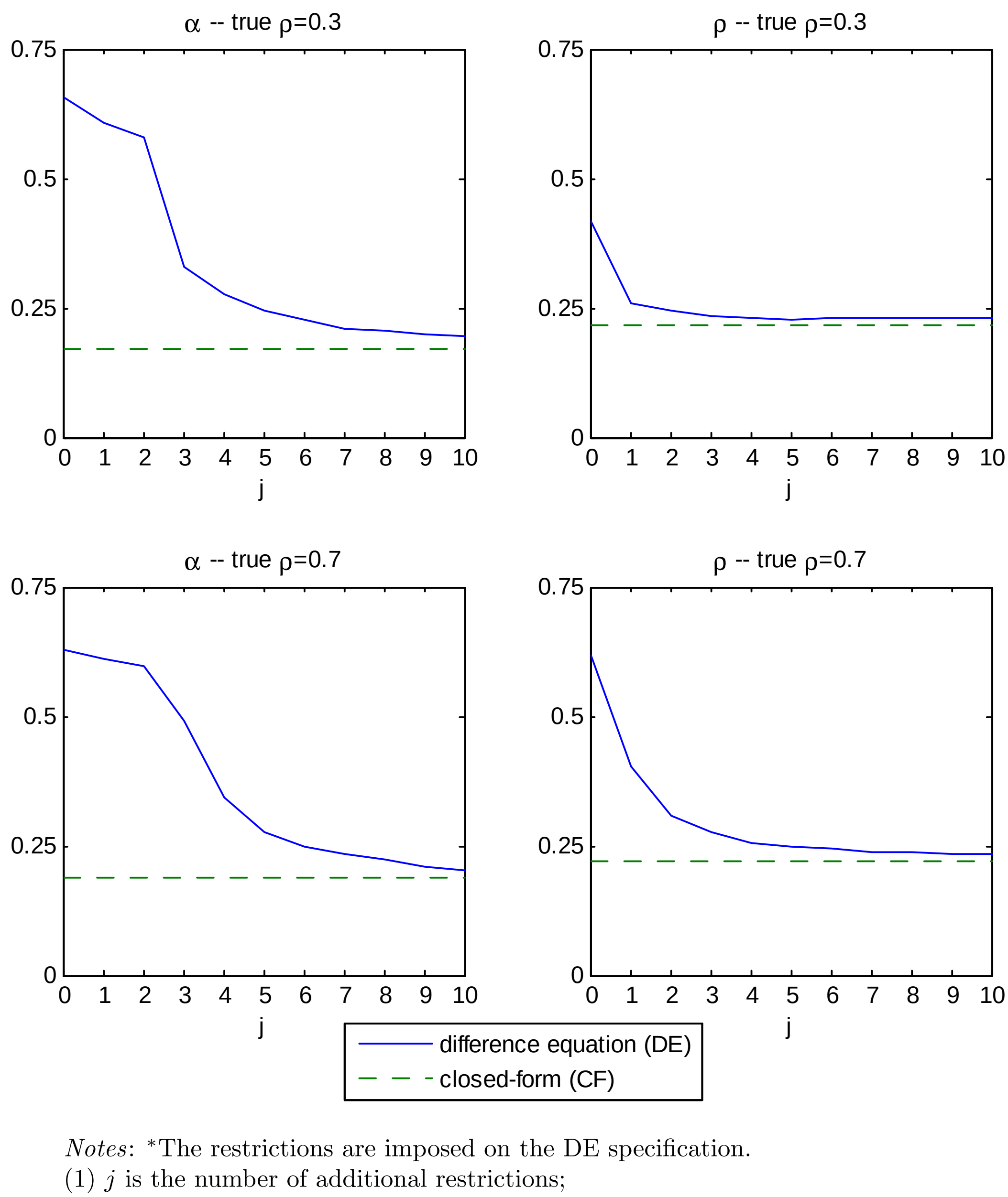
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 2: Monte Carlo simulations, estimates of ρ
Difference equation (DE) vs. closed form (CF)



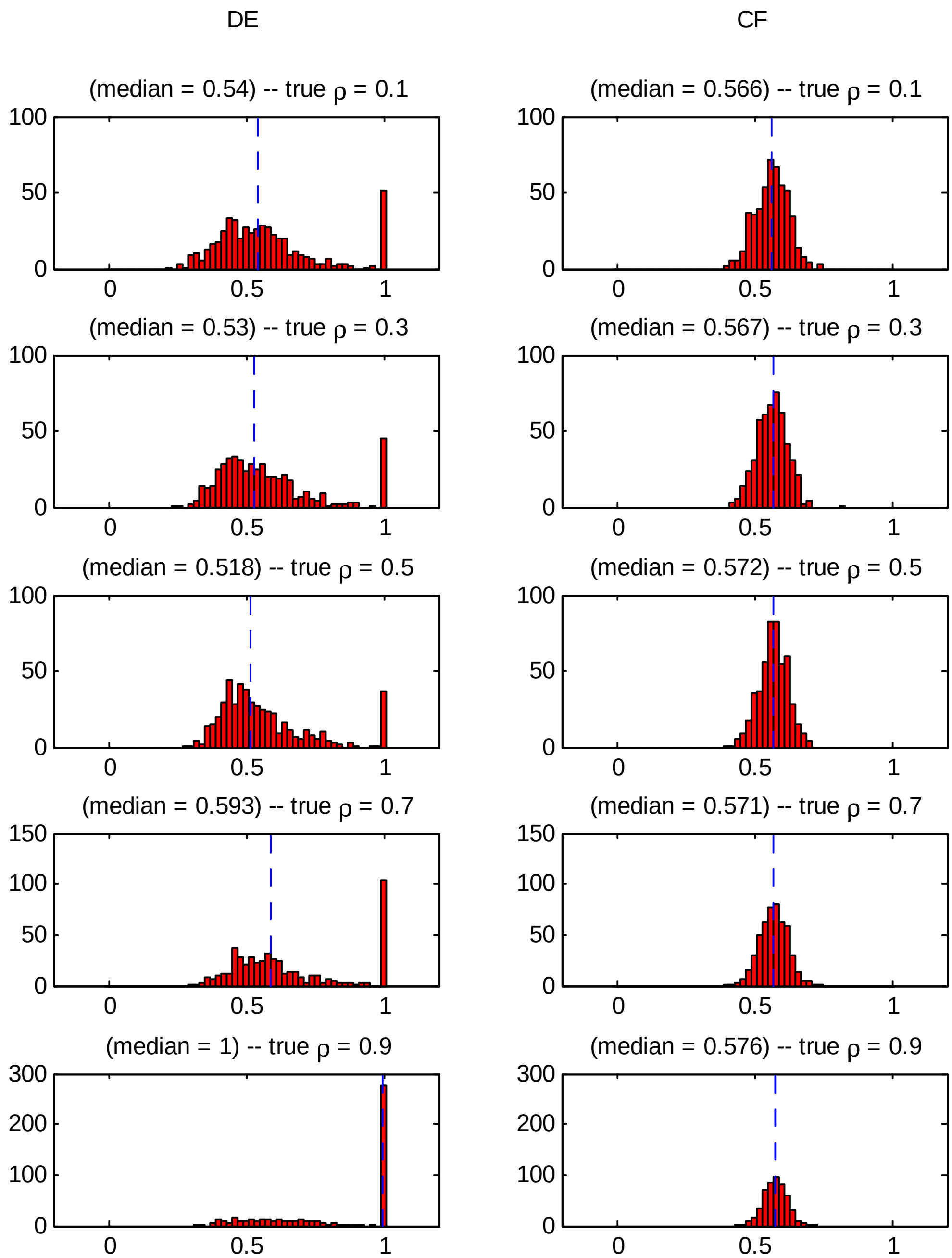
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 3: 90% confidence interval width after imposing
j additional expectational restrictions*
(Monte Carlo results)



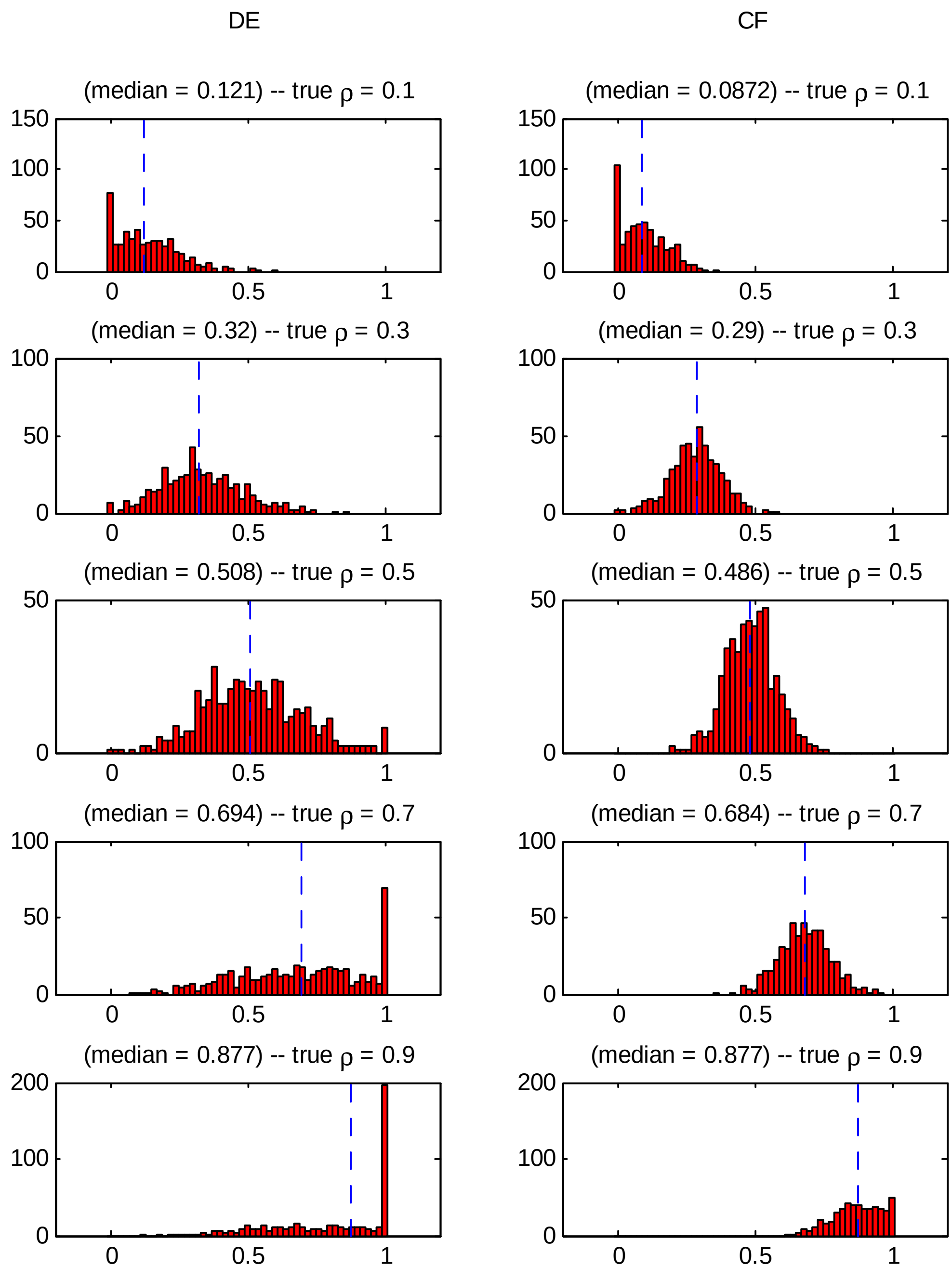
Notes: *The restrictions are imposed on the DE specification.
(1) j is the number of additional restrictions;
(2) $j = 0$ and $j \to \infty$ correspond to the DE and CF specifications, respectively;
(3) real marginal cost is generated from an AR(2) process.

Figure 4: Monte Carlo simulations, estimates of α
Difference equation (DE) vs. closed form (CF)
(Allowing for inflation feedback in the marginal cost equation)



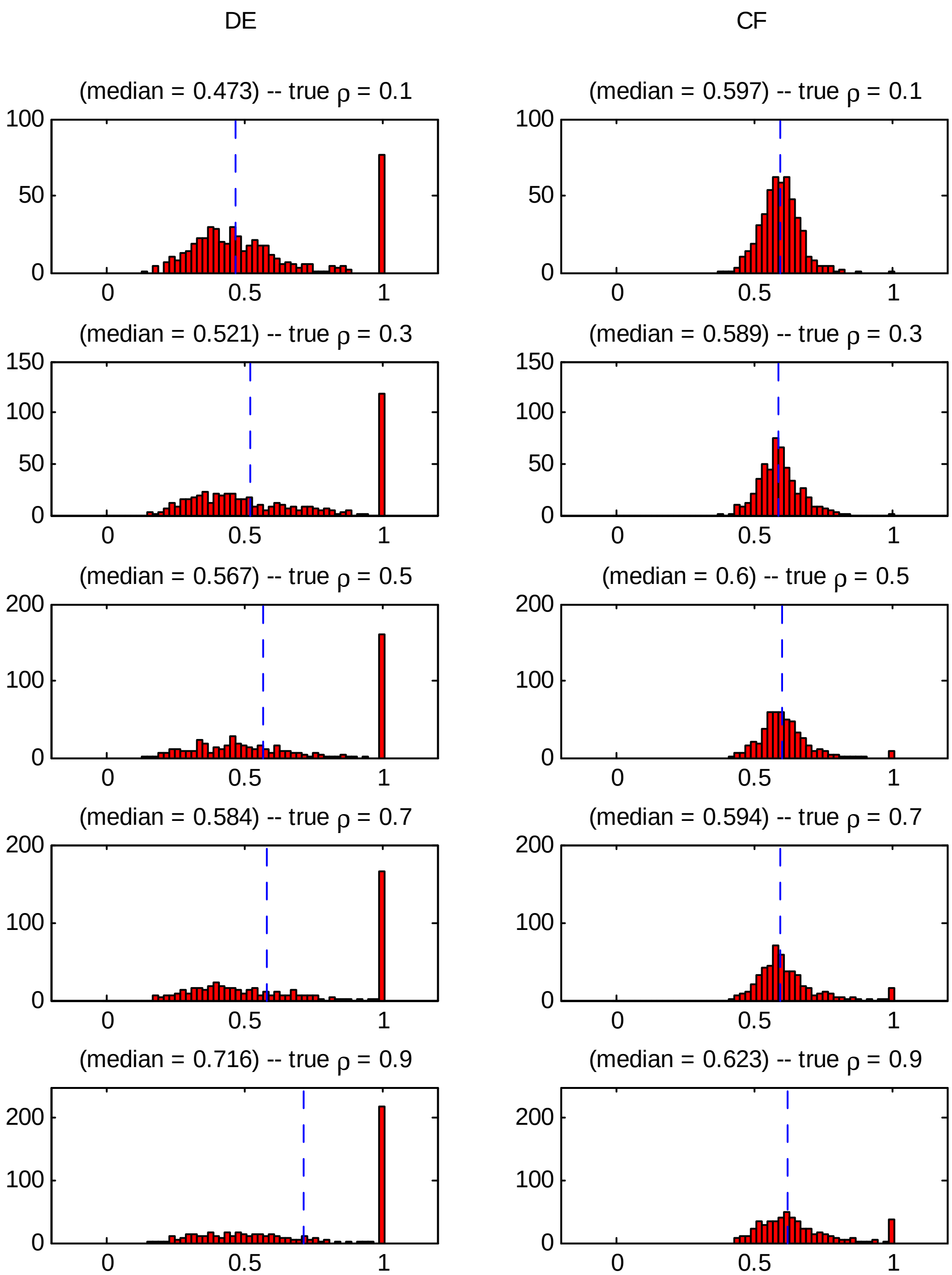
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 5: Monte Carlo simulations, estimates of ρ
Difference equation (DE) vs. closed form (CF)
(Allowing for inflation feedback in the marginal cost equation)



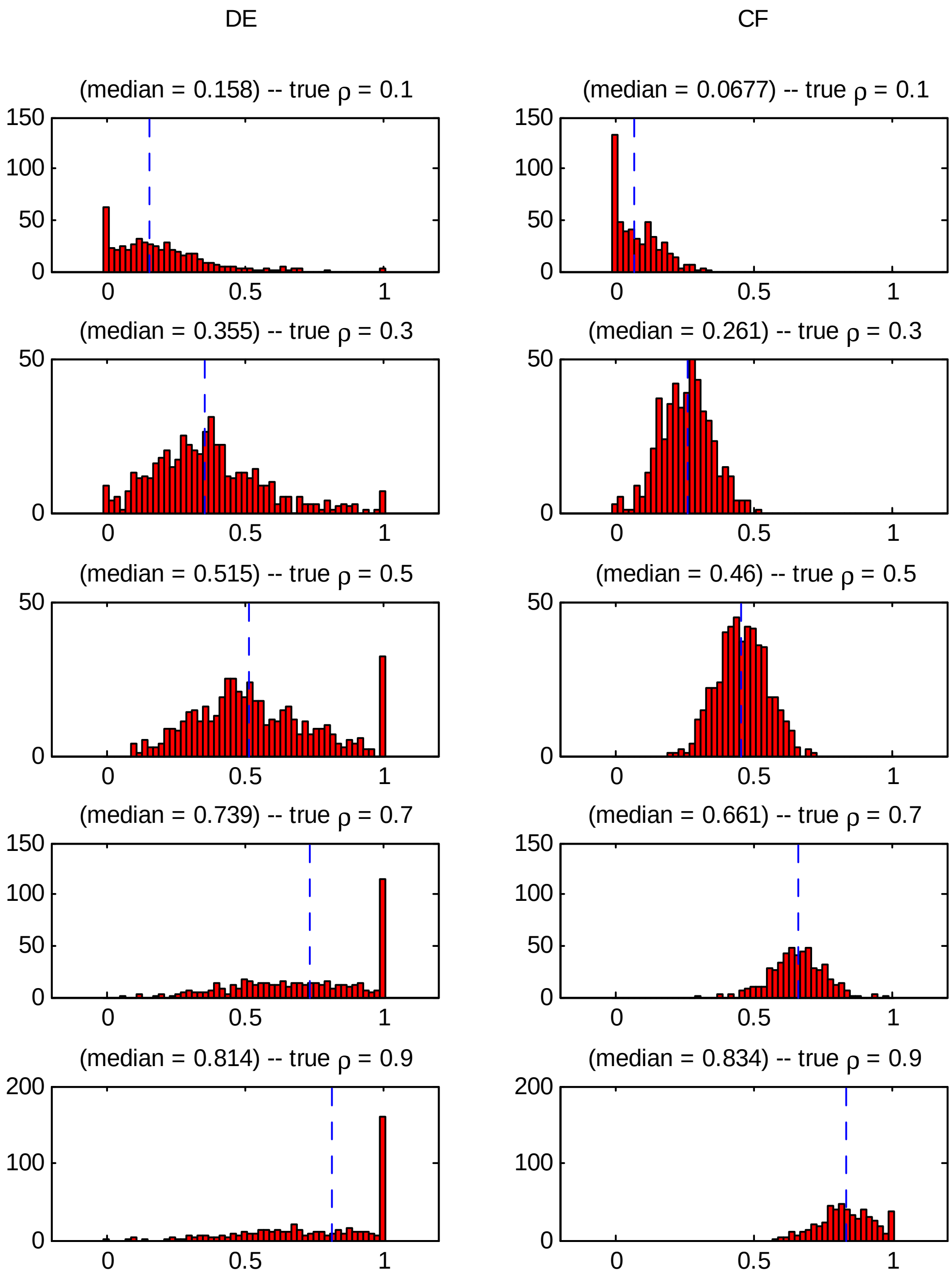
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 6: Monte Carlo simulations, estimates of α
Difference equation (DE) vs. closed form (CF)
(Larger information set)



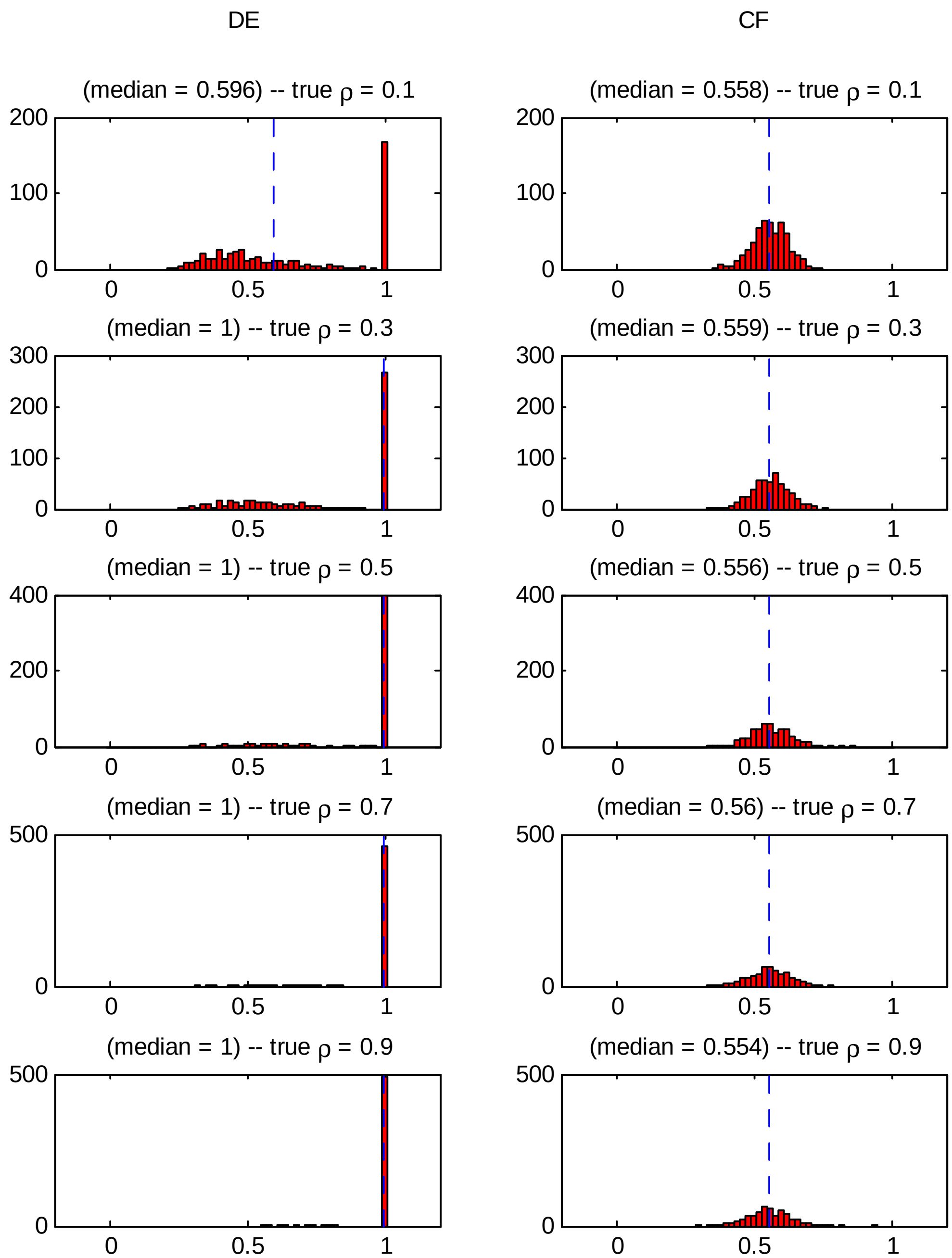
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 7: Monte Carlo simulations, estimates of ρ
Difference equation (DE) vs. closed form (CF)
(Larger information set)



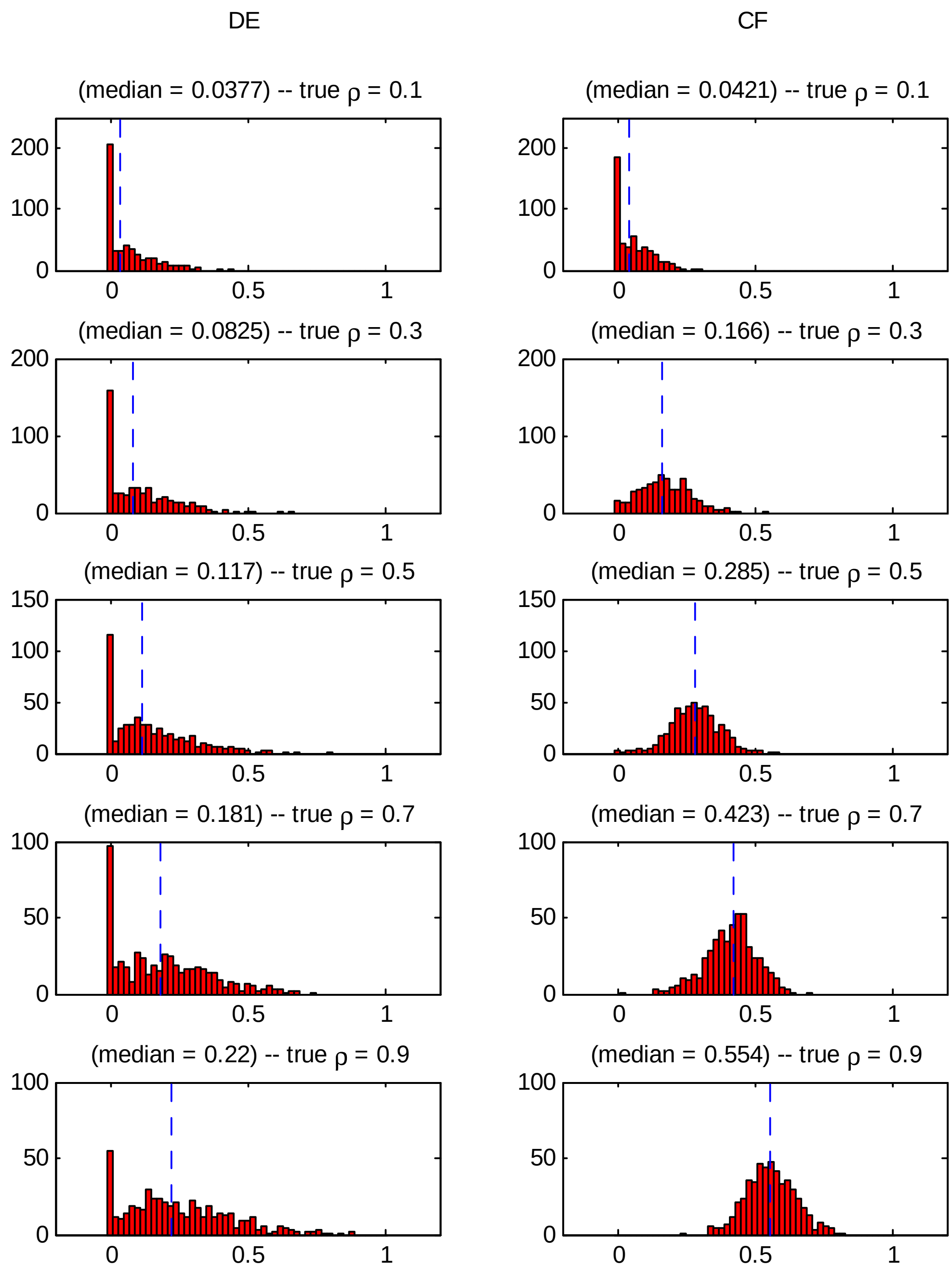
Notes:
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 8: Monte Carlo simulations, α parameter
Difference equation (DE) vs. closed form (CF), with misspecification*



Notes: * The data are generated with $\tau = 0.6$, but estimations are done with $\tau = 1$.
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

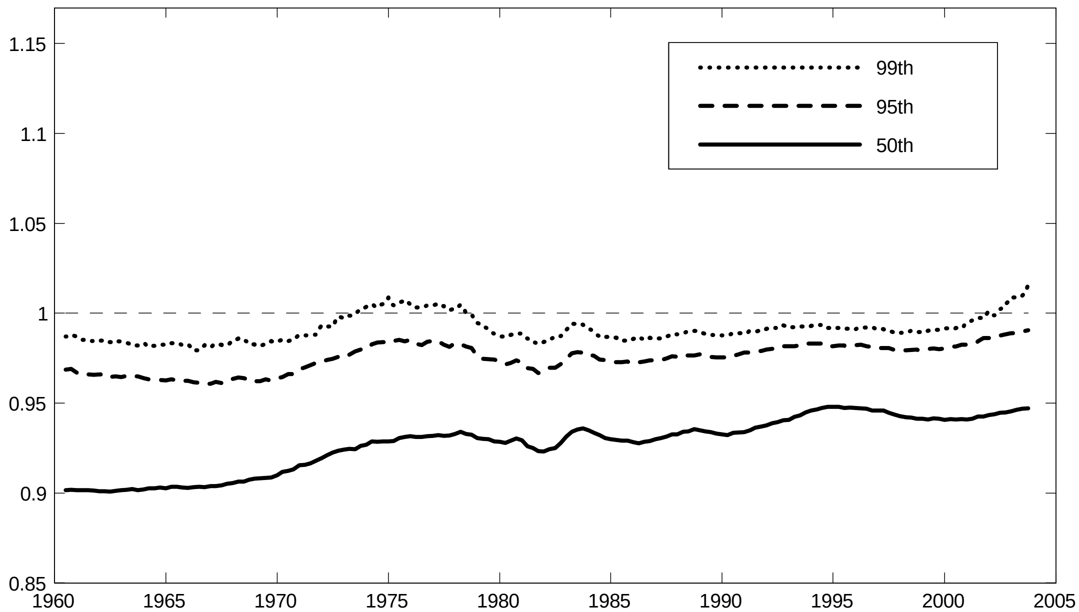
Figure 9: Monte Carlo simulations, ρ parameter
Difference equation (DE) vs. closed form (CF), with misspecification*



Notes: * The data are generated with $\tau = 0.6$, but estimations are done with $\tau = 1$.
(1) left and right panels correspond to DE and CF specifications, respectively;
(2) the vertical axis is the number of repetitions. Total number of repetitions is 500.

Figure 10: Distribution of largest root of $\hat{\lambda}_t \cdot \hat{\mathbf{A}}_t$ in absolute value
Median, 95th, and 99th percentiles.
(Same $\hat{\mathbf{A}}_t$ estimates in both panels)

Panel A: Closed Form, distribution of $\left\|\hat{\lambda}_t^{CF}\hat{\mathbf{A}}_t\right\|$.



Panel B: Difference-Equation Form, distribution of $\left\|\hat{\lambda}_t^{DE}\hat{\mathbf{A}}_t\right\|$

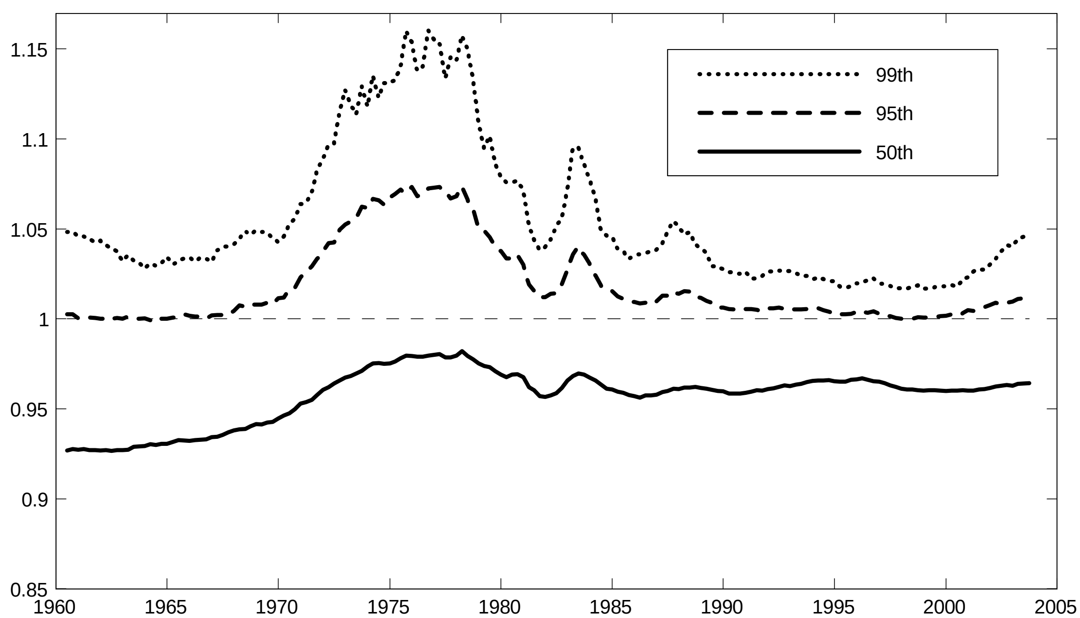


Figure 11: Distribution of largest root of $\left\{\widehat{\xi}_{i,t} \cdot \hat{\mathbf{A}}_t\right\}_{i=1,2}$ in absolute value. Median, 95th, and 99th percentiles.

Exact Closed Form

Table 1: Structural parameter estimates (median and 90% trust region)
Sample period: 1960:Q1–2003:Q4

	ρ	α	θ	τ
DE_ con	0 (0,0.17)	0.582 (0.45,0.67)	9.76 (7.66,12.46)	1 —
DE_ uncon	0.64 (0.29,1)	0.597 (0.51,0.72)	11.87 (10.25,14.80)	0.56 (0.23,0.80)
CF_ con	0.59 (0.29,0.91)	0.836 (0.73,0.91)	10.01 (7.90,14.73)	1 —
CF_ uncon	0.89 (0.76,0.99)	0.877 (0.79,0.93)	12.28 (10.77,20.19)	0.69 (0.49,0.90)

Notes: (1) numbers in parentheses are 90% trust regions; (2) *DE_ con* and *CF_ con* correspond to difference equation (DE) and closed-form (CF) specifications with $\tau = 1$; (3) *DE_ uncon* and *CF_ uncon* correspond to DE and CF cases with unconstrained τ, respectively; (4) *DE_ con* is the benchmark case in Cogley and Sbordone (2008).

Table 2: Structural parameter estimates (median and 90% trust region)
Sample period: 1960.Q1–2003.Q4

	ρ	α	θ	τ
DE_ uncon	0.64 (0.29,1)	0.597 (0.51,0.72)	11.87 (10.25,14.80)	0.56 (0.23,0.80)
DE_ uncon(2)	0.81 (0.50,1)	0.67 (0.59,0.77)	11.78 (10.30,15.00)	0.68 (0.45,0.93)
DE_ uncon(4)	0.86 (0.59,1)	0.722 (0.64,0.81)	11.75 (10.33,14.89)	0.71 (0.50,0.96)
DE_ uncon(6)	0.87 (0.64,1)	0.76 (0.67,0.84)	11.78 (10.36,14.91)	0.71 (0.50,0.96)
DE_ uncon(8)	0.87 (0.67,1)	0.781 (0.70,0.86)	11.82 (10.40,14.86)	0.71 (0.51,0.95)
CF_ uncon	0.89 (0.76,0.99)	0.877 (0.79,0.93)	12.28 (10.77,20.19)	0.69 (0.49,0.90)

Notes: (1) numbers in parentheses are 90% trust regions; (2) *DE_ uncon* and *CF_ uncon* correspond to DE and CF cases with unconstrained τ, respectively; (3) *DE_ uncon(j)* corresponds to the DE case with j consecutive quarters of model-consistent restrictions on expectations.

Table 3: Structural parameter estimates (median and 90% trust region)
Sample period: 1960.Q1–2003.Q4

	ρ	α	θ	τ
DE_uncon	0.78 (0.34,1)	0.612 (0.52,0.74)	11.70 (10.23,13.75)	0.59 (0.31,0.81)
DE_uncon(2)	0.90 (0.57,1)	0.684 (0.60,0.78)	11.63 (10.30,13.57)	0.69 (0.47,0.92)
DE_uncon(4)	0.92 (0.66,1)	0.732 (0.65,0.82)	11.64 (10.33,13.60)	0.71 (0.50,0.95)
DE_uncon(6)	0.90 (0.68,1)	0.76 (0.69,0.84)	11.67 (10.38,13.69)	0.71 (0.51,0.95)
DE_uncon(8)	0.90 (0.70,1)	0.788 (0.71,0.86)	11.71 (10.42,13.79)	0.71 (0.51,0.94)
CF_uncon	0.89 (0.76,0.99)	0.875 (0.80,0.93)	12.13 (10.74,15.16)	0.68 (0.49,0.90)

Notes: (1) numbers in parentheses are 90% trust regions; (2) *DE_uncon* and *CF_uncon* correspond to DE and CF cases with unconstrained τ, respectively; (3) 45.39% (out of 4,902) ensembles remain in *DE_uncon*; (4) 63.47% (out of 4,588) ensembles remain in *DE_uncon(2);* (5) 68.91% (out of 4,587) ensembles remain in *DE_uncon*(4); (6) 71.45% (out of 4,588) ensembles remain in *DE_uncon(6)*; (7) 72.31% (out of 4586) ensembles remain in *DE_uncon(8)*; (8) 79.24% (out of 4,543) ensembles remain in *CF_uncon*.

Table 4: Structural parameter estimates (median and 90% trust region)
Sample period: 1960.Q1–2003.Q4

	ρ	α	θ	τ
ECF	0.90	0.884	12.38	0.69
	(0.80,0.98)	(0.82,0.93)	(10.77,20.79)	(0.50,0.90)
ECF (100% criterion)	0.90	0.883	12.01	0.68
	(0.79,0.99)	(0.82,0.93)	(10.70,14.26)	(0.50,0.89)
ECF (95% criterion)	0.89	0.885	12.25	0.69
	(0.80,0.98)	(0.82,0.93)	(10.75,17.14)	(0.50,0.90)
ECF (90% criterion)	0.90	0.885	12.34	0.69
	(0.80,0.98)	(0.82,0.93)	(10.77,18.87)	(0.50,0.90)

Notes: (1) numbers in parentheses are 90% trust regions;
(2) The x% criterion preserves ensembles that satisfy the determinacy condition for x% of the 174 quarters. (3) The number of optimized ensembles is 4402. (4) The 100% criterion retains 60.2% of the ensembles, the 95% retains 90.4%, and the 90% retains 97.5%.

Table 5: Structural parameter estimates (median and 90% trust region)
Sample period: 1960.Q1–2003.Q4; Training sample: 1954.Q1–1959.Q4
(constant coefficients and removing higher-order leads)

	ρ	α	θ	τ
DE_ con	0	0.562	12.07	1
	(0,0.11)	(0.44,0.66)	(8.12,15.22)	—
DE_ uncon	0.70	0.612	11.72	0.58
	(0.37,1)	(0.52,0.73)	(10.21,13.95)	(0.32,0.81)
CF_ con	0.42	0.777	11.61	1
	(0.17,0.71)	(0.68,0.85)	(8.46,15.10)	—
CF_ uncon	0.87	0.864	12.29	0.68
	(0.72,0.99)	(0.78,0.92)	(10.77,16.11)	(0.48,0.89)

Notes: (1) numbers in parentheses are 90% trust regions;
(2) *DE_ con* and *CF_ con* correspond to difference equation (DE) and closed-form (CF) specifications with $\tau = 1$; (3) *DE_ uncon* and *CF_ uncon* correspond to DE and CF cases with unconstrained τ, respectively.

Table 6: Structural parameter estimates (median and 90% trust region)
Sample period: 1984.Q1–2003.Q4; Training sample: 1963.Q4–1983.Q4

	ρ	α	θ	τ
DE_ con	0	0.622	9.19	1
	(0,0.09)	(0.43,0.74)	(7.51,11.88)	—
DE_ uncon	0.83	0.643	10.51	0.55
	(0.42,1)	(0.53,0.76)	(8.46,15.23)	(0.31,0.77)
CF_ con	0.49	0.838	9.51	1
	(0.10,0.88)	(0.73,0.92)	(7.51,14.13)	—
CF_ uncon	0.86	0.869	10.98	0.57
	(0.66,0.99)	(0.77,0.93)	(9.04,31.74)	(0.35,0.81)

Notes: (1) numbers in parentheses are 90% trust regions; (2) *DE_ con* and *CF_ con* correspond to difference equation (DE) and closed-form (CF) specifications with $\tau = 1$; (3) *DE_ uncon* and *CF_ uncon* correspond to DE and CF cases with unconstrained τ, respectively.

Appendix A: Derivation of the NKPC in difference equation (DE) form

In this appendix, we derive the NKPC in difference-equation (DE) form as described in (33) and (34). We also show how to combine them into a final form used in the estimation procedure described in section 3.2. and show the cross-equation restrictions implied by conditional expectation based on information at $t-2$. The NKPC derivation closely follows that in Cogley and Sbordone (2008).[57]

First, we derive the log-linear approximation of the evolution of aggregate prices. Let X_t be the optimal nominal price at time t chosen by firms that are allowed to adjust their prices, which happens with constant per-period probability $(1-\alpha)$. Based on our indexation mechanism, the price of an individual firm i that is not allowed to adjust (with probability α) evolves according to

$$P_t(i) = (\Pi_{t-1}^{\tau}\Pi_{t-2}^{1-\tau})^{\rho} P_{t-1}(i) \ .$$

Hence, the aggregate price based on the CES aggregator is given by

$$P_t = \left[(1-\alpha)X_t^{1-\theta} + \alpha\left\{(\Pi_{t-1}^{\tau}\Pi_{t-2}^{1-\tau})^{\rho}P_{t-1}\right\}^{1-\theta}\right]^{\frac{1}{1-\theta}} \ .$$

Dividing by the price level P_t, we have

$$1 = (1-\alpha)x_t^{1-\theta} + \alpha\left\{(\Pi_{t-1}^{\tau}\Pi_{t-2}^{1-\tau})^{\rho}\Pi_t^{-1}\right\}^{1-\theta} \ , \tag{A1}$$

where x_t is the optimal relative price at time t. Next define stationary variables $\widetilde{\Pi}_t = \Pi_t/\overline{\Pi}_t$, $g_t^{\bar{\pi}} = \overline{\Pi}_t/\overline{\Pi}_{t-1}$, $g_t^{y} = Y_t/Y_{t-1}$, and $\widetilde{x}_t = x_t/\overline{x}_t$. Here, for any variable k_t, $\overline{k}_t$ is its time-varying trend. Equation (A1) can then be transformed in terms of these stationary variables to yield (after some algebra):

$$\begin{aligned} 1 \quad = \quad & (1-\alpha)\widetilde{x}_t^{1-\theta}\overline{x}_t^{1-\theta} \\ & +\alpha\left[\begin{array}{c} \widetilde{\Pi}_{t-2}^{\rho(1-\tau)(1-\theta)}\widetilde{\Pi}_{t-1}^{\rho\tau(1-\theta)}\widetilde{\Pi}_t^{-(1-\theta)}\overline{\Pi}_t^{(1-\rho)(\theta-1)}\cdot \\ (g_{t-1}^{\bar{\pi}})^{-\rho(1-\tau)(1-\theta)}(g_t^{\bar{\pi}})^{-\rho(1-\tau)(1-\theta)}(g_t^{\bar{\pi}})^{-\rho\tau(1-\theta)} \end{array}\right] \ . \end{aligned} \tag{A2}$$

In the steady state where $\widetilde{x}_t = \widetilde{\Pi}_t = g_t^{\bar{\pi}} = 1$, (A2) can be solved for $\overline{x}_t$ as a function of $\overline{\Pi}_t$:

$$\overline{x}_t = \left[\frac{1-\alpha\overline{\Pi}_t^{(1-\rho)(\theta-1)}}{1-\alpha}\right]^{\frac{1}{1-\theta}} \ . \tag{A3}$$

Defining $\widehat{\pi}_t \equiv \ln\widetilde{\Pi}_t \equiv \ln(\Pi_t/\overline{\Pi}_t)$ and $\widehat{x}_t \equiv \ln\widetilde{x}_t$, imposing (A3), and rearranging, the log-linear approximation of (A2) around the steady state can be expressed as

$$\begin{aligned} \widehat{x}_t \quad = \quad & -\frac{1}{\varphi_{0,t}}\rho(1-\tau)\left(\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\bar{\pi}} - \widehat{g}_t^{\bar{\pi}}\right) \\ & -\frac{1}{\varphi_{0,t}}\rho\tau\left(\widehat{\pi}_{t-1} - \widehat{g}_t^{\bar{\pi}}\right) \\ & +\frac{1}{\varphi_{0,t}}\widehat{\pi}_t \ , \end{aligned} \tag{A3}$$

[57] Note that in Cogley and Sbordone (2008), the indexation is constrained to the first lag of inflation, which corresponds to $\tau = 1$.

where $\varphi_{0,t} = \frac{1-\alpha\overline{\Pi}_t^{(1-\rho)(\theta-1)}}{\alpha\overline{\Pi}_t^{(1-\rho)(\theta-1)}}$.

Next, we take the log-linear approximation to the first-order condition (FOC) of firms' pricing problem. Identically to the one-lag indexation case in Cogley and Sbordone (2008), the firms' FOC can be written as

$$E_t \sum_{j=0}^{\infty} \alpha^j Q_{t,t+j} Y_{t+j} P_{t+j} \Psi_{tj}^{1-\theta} \left(X_t^{(1+\theta\omega)} - \frac{\theta}{\theta-1} MC_{t+j} \Psi_{tj}^{-(1+\theta\omega)} P_{t+j}^{\theta\omega} \right) = 0 \ , \tag{A4}$$

where $Q_{t,t+j}$ and MC_{t+j} are the nominal discount factor and average marginal cost at $t+j$, respectively. The variable Ψ_{tj} enters in the CES demand function for any good i, $Y_{t+j}(i) = Y_{t+j} \left(\frac{P_{t+j}(i)\Psi_{tj}}{P_{t+j}} \right)$, with

$$\Psi_{tj} = \begin{cases} 1 & j = 0 \\ \prod_{k=0}^{j-1} \left(\Pi_{t+k}^{\tau} \Pi_{t+k-1}^{1-\tau} \right)^{\rho} & j \geq 1 \end{cases} \tag{A5}$$

The second line of (A5) makes clear that prices are indexed to a weighted average of the first two lags of inflation if they are not set optimally. Combining (A4) and (A5) and rearranging leads to

$$X_t^{1+\theta\omega} = \frac{C_t}{D_t} \ ,$$

where C_t and D_t are recursively defined by

$$\begin{aligned} C_t &= \frac{\theta}{\theta-1} Y_t P_t^{\theta(1+\omega)-1} MC_t \\ &\quad + E_t \left[\alpha q_{t,t+1} \Pi_t^{-\rho\tau\theta(1+\omega)} \Pi_{t-1}^{-\rho(1-\tau)\theta(1+\omega)} C_{t+1} \right] \end{aligned} \tag{A6}$$

$$\begin{aligned} D_t &= Y_t P_t^{\theta-1} \\ &\quad + E_t \left[\alpha q_{t,t+1} \Pi_t^{\rho\tau(1-\theta)} \Pi_{t-1}^{\rho(1-\tau)(1-\theta)} D_{t+1} \right] \ , \end{aligned} \tag{A7}$$

where $q_{t,t+1}$ now is the real discount factor. Defining the stationary variables $\widetilde{C}_t = \frac{C_t}{Y_t P_t^{\theta(1+\omega)}}$ and $\widetilde{D}_t = \frac{D_t}{Y_t P_t^{\theta-1}}$, we have based on (A6) and (A7):

$$\begin{aligned} \widetilde{C}_t &= \frac{\theta}{\theta-1} MC_t \\ &\quad + E_t \left[\alpha q_{t,t+1} g_{t+1}^{y} \Pi_{t+1}^{\theta(1+\omega)} \Pi_t^{-\rho\tau\theta(1+\omega)} \Pi_{t-1}^{-\rho(1-\tau)\theta(1+\omega)} \widetilde{C}_{t+1} \right] \end{aligned} \tag{A8}$$

$$\widetilde{D}_t = 1 + E_t \left[\alpha q_{t,t+1} g_{t+1}^{y} \Pi_{t+1}^{(\theta-1)} \Pi_t^{\rho\tau(1-\theta)} \Pi_{t-1}^{\rho(1-\tau)(1-\theta)} \widetilde{D}_{t+1} \right] \ . \tag{A9}$$

Also note that

$$\frac{\widetilde{C}_t}{\widetilde{D}_t} = \frac{C_t}{D_t} \frac{1}{P_t^{(1+\theta\omega)}} = x_t^{1+\theta\omega} \ , \tag{A10}$$

where $x_t \equiv X_t / P_t$. Evaluating (A8) and (A9) at the steady state leads to

$$\overline{C}_t = \frac{\frac{\theta}{\theta-1} \overline{mc}_t}{1 - \alpha \overline{q} \overline{g}^{y} \overline{\Pi}_t^{\theta(1+\omega)(1-\rho)}}$$

$$\overline{D}_t = \frac{1}{1 - \alpha \overline{q} \overline{g}^y \overline{\Pi}_t^{(\theta-1)(1-\rho)}}.$$

Combining the two expressions above with (A3) and using (A10) leads to the steady-state restriction (32). This restriction does not depend on τ and hence is identical to the case in Cogley and Sbordone (2008) with $\tau = 1$. Next, define $\widehat{C}_t = \ln \frac{\widetilde{C}_t}{\overline{C}_t}$, $\widehat{D}_t = \ln \frac{\widetilde{D}_t}{\overline{D}_t}$, and $\widehat{mc}_t = \ln \frac{mc_t}{\overline{mc}_t}$. Log-linearizing (A10) yields

$$(1 + \theta\omega)\widehat{x}_t = (\widehat{C}_t - \widehat{D}_t) \ . \tag{A11}$$

Combining (A11) with (A3) and rearranging leads to an intermediate expression for $\widehat{\pi}_t$:

$$\begin{aligned} \widehat{\pi}_t \;&=\; \rho\tau \left[\widehat{\pi}_{t-1} - \widehat{g}_t^{\overline{\pi}}\right] \\ &\quad + \rho(1-\tau) \left[\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\overline{\pi}} - \widehat{g}_t^{\overline{\pi}}\right] \\ &\quad + \frac{\varphi_{0,t}}{(1+\theta\omega)}(\widehat{C}_t - \widehat{D}_t) \ . \end{aligned} \tag{A12}$$

We can obtain the expressions for $\widehat{C}_t$ and $\widehat{D}_t$ by log-linearizing (A8) and (A9). Combining the resulting expressions with (A11) leads to equations (33) and (34) in the main text:

$$\begin{aligned} \widehat{\pi}_t \;&=\; \rho\tau(\widehat{\pi}_{t-1} - \widehat{g}_t^{\overline{\pi}}) + \rho(1-\tau)(\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\overline{\pi}} - \widehat{g}_t^{\overline{\pi}}) \\ &\quad + \lambda_t E_t(\widehat{\pi}_{t+1} - \rho\tau\widehat{\pi}_t - \rho(1-\tau)(\widehat{\pi}_{t-1} - \widehat{g}_t^{\overline{\pi}})) + \zeta_t \widehat{mc}_t + \gamma_t \widehat{D}_t + u_{\pi,t} \end{aligned} \tag{A13}$$

$$\begin{aligned} \widehat{D}_t \;&=\; \varphi_{1,t} E_t(\widehat{q}_{t,t+1} + \widehat{g}_{t+1}^y) \\ &\quad + \varphi_{1,t}(\theta - 1) E_t \left\{\widehat{\pi}_{t+1} - \rho\tau\widehat{\pi}_t - \rho(1-\tau)(\widehat{\pi}_{t-1} - \widehat{g}_t^{\overline{\pi}})\right\} + \varphi_{1,t} E_t \widehat{D}_{t+1} \ , \end{aligned} \tag{A14}$$

with the time-varying coefficients given by

$$\begin{aligned} \zeta_t &= \chi_t \varphi_{3,t} \\ \lambda_t &= \varphi_{2,t}(1 + \varphi_{0,t}) \\ \gamma_t &= \frac{\chi_t(\varphi_{2,t} - \varphi_{1,t})}{\varphi_{1,t}} \\ \chi_t &= \frac{\varphi_{0,t}}{1 + \theta\omega} \\ \varphi_{1,t} &= \alpha \overline{q} \overline{g}^y \overline{\Pi}_t^{(\theta-1)(1-\rho)} \\ \varphi_{2,t} &= \alpha \overline{q} \overline{g}^y \overline{\Pi}_t^{\theta(1+\omega)(1-\rho)} \\ \varphi_{3,t} &= 1 - \varphi_{2,t}. \end{aligned}$$

Finally, iterating $\widehat{D}_t$ in (A14) forward, substituting the resulting expression for $\widehat{D}_t$ in (A13), converting real discount factors $\widehat{q}_{t+j,t+j+1}$ into nominal discount factors $\widetilde{Q}_{t+j,t+j+1}$ for ease of comparison with Cogley and Sbordone (2008), and rearranging terms yields the NKPC in DE form:

$$\begin{aligned}
\widehat{\pi}_t &= \widetilde{\rho}_{1,t}^D \left(\widehat{\pi}_{t-1} - \widehat{g}_t^{\overline{\pi}}\right) + (1-\tau)\widetilde{\rho}_{2,t}^D \left(\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\overline{\pi}} - \widehat{g}_t^{\overline{\pi}}\right) \\
&\quad + \widetilde{\zeta}_t^D \widehat{mc}_t \\
&\quad + d_{1,t}^D E_t \widehat{\pi}_{t+1} \\
&\quad + d_{2,t}^D E_t \sum_{j=2}^{\infty} \varphi_{1,t}^{j-1} \widehat{\pi}_{t+j} \\
&\quad + d_{3,t}^D E_t \sum_{j=0}^{\infty} \varphi_{1,t}^{j} \left[\widehat{Q}_{t+j,t+j+1} + \widehat{g}_{t+j+1}^y\right] + \widetilde{u}_{\pi,t},
\end{aligned} \tag{PC-DE}$$

where the coefficients are defined by

$$\begin{aligned}
\widetilde{\rho}_{1,t}^D &= \left[\rho\tau - \lambda_t\rho(1-\tau) - \gamma_t(\theta-1)\rho(1-\tau)\varphi_{1,t}\right]/\Delta_t \\
\widetilde{\rho}_{2,t}^D &= \rho/\Delta_t \\
d_{1,t}^D &= \widetilde{d}_{1,t}^D + d_{3,t}^D \\
d_{2,t}^D &= \widetilde{d}_{2,t}^D + d_{3,t}^D \\
d_{3,t}^D &= [\gamma_t\varphi_{1,t}]/\Delta_t \\
\widetilde{\zeta}_t^D &= \zeta_t/\Delta_t \\
\Delta_t &= 1 + \rho\tau\lambda_t + \gamma_t(\theta-1)\rho\varphi_{1,t}\left\{\tau + (1-\tau)\varphi_{1,t}\right\} \\
\widetilde{d}_{1,t}^D &= \left[\lambda_t + \gamma_t(\theta-1)\varphi_{1,t}\left\{1 - \rho\tau\varphi_{1,t} - \rho(1-\tau)\varphi_{1,t}^2\right\}\right]/\Delta_t \\
\widetilde{d}_{2,t}^D &= \left[\gamma_t(\theta-1)\varphi_{1,t}\left\{1 - \rho\tau\varphi_{1,t} - \rho(1-\tau)\varphi_{1,t}^2\right\}\right]/\Delta_t.
\end{aligned}$$

Note that as in Cogley and Sbordone (2008), we use the "anticipated utility" assumption (Kreps, 1998) in deriving the NKPC in (PC-DE) so that $E_t \prod_{k=0}^{i} \varphi_{1,t+k}\widehat{h}_{t+i} = \varphi_{1,t}^{i+1} E_t \widehat{h}_{t+i}$ for any variable $\widehat{h}_{t+i}$.

Two limiting cases of (PC-DE) are worth mentioning. First, when $\tau = 1$ so that the indexation is constrained to the first lag of inflation, we have the NKPC in Cogley and Sbordone (2008). Second, if the prices of non-adjusting firms are fully indexed to a mixture of past inflation (first and second lags) and current trend inflation, the NKPC collapses to the case with constant coefficients and where there are no extra lead terms beyond $t+1$. Furthermore, in the constant-trends case with $\tau = 1$, one obtains the NKPC as in Christiano, Eichenbaum, and Evans (2005) exhibited in (2).

Cross-equation restrictions. Given the forecasting rule (39) and equation (PC-DE), we obtain the conditional expectation of inflation based on information at $t-2$ in the DE form as follows

$$\begin{aligned}
\mathbf{e}_\pi'\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} &= \widetilde{\rho}_{1,t-2}^D\mathbf{e}_\pi'\mathbf{A}_{t-2}\widehat{\mathbf{z}}_{t-2} + (1-\tau)\widetilde{\rho}_{2,t-2}^D\mathbf{e}_\pi'\widehat{\mathbf{z}}_{t-2} + \widetilde{\zeta}_{t-2}^D\mathbf{e}_{mc}'\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} \\
&\quad + d_{1,t-2}^D\mathbf{e}_\pi'\mathbf{A}_{t-2}^3\widehat{\mathbf{z}}_{t-2} + d_{2,t-2}^D\varphi_{1,t-2}\mathbf{e}_\pi'\mathbf{J}_{t-2}\mathbf{A}_{t-2}^4\widehat{\mathbf{z}}_{t-2} \\
&\quad + d_{3,t-2}^D(\mathbf{e}_Q'\mathbf{J}_{t-2}\mathbf{A}_{t-2}^2\widehat{\mathbf{z}}_{t-2} + \mathbf{e}_{g^y}'\mathbf{J}_{t-2}\mathbf{A}_{t-2}^3\widehat{\mathbf{z}}_{t-2}),
\end{aligned} \tag{A15}$$

where $\mathbf{J}_t \equiv (\mathbf{I} - \varphi_{1,t}\mathbf{A}_t)^{-1}$. Hence, the vector of cross-equation restrictions is given by

$$
\begin{aligned}
\mathbf{e}'_\pi \mathbf{A}^2_{t-2} &= \widetilde{\rho}^D_{1,t-2}\mathbf{e}'_\pi \mathbf{A}_{t-2} + (1-\tau)\widetilde{\rho}^D_{2,t-2}\mathbf{e}'_\pi \mathbf{I} + \widetilde{\zeta}^D_{t-2}\mathbf{e}'_{mc}\mathbf{A}^2_{t-2} \\
&\quad + d^D_{1,t-2}\mathbf{e}'_\pi \mathbf{A}^3_{t-2} + d^D_{2,t-2}\varphi_{1,t-2}\mathbf{e}'_\pi \mathbf{J}_{t-2}\mathbf{A}^4_{t-2} \\
&\quad + d^D_{3,t-2}(\mathbf{e}'_Q \mathbf{J}_{t-2}\mathbf{A}^2_{t-2} + \mathbf{e}'_{g^y}\mathbf{J}_{t-2}\mathbf{A}^3_{t-2}) \\
&\equiv \mathbf{g}^D(\boldsymbol{\mu}_{t-2}, \mathbf{A}_{t-2}, \boldsymbol{\psi}) \ . \qquad \text{(A16)}
\end{aligned}
$$

Appendix B: Derivation of the CF specification

In this appendix we derive the CF representation of the NKPC based on (A13) and (A14). First, define an auxiliary variable

$$\widehat{B}_t = \widehat{\pi}_t - \rho\tau(\widehat{\pi}_{t-1} - \widehat{g}_t^{\bar{\pi}}) - \rho(1-\tau)(\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\bar{\pi}} - \widehat{g}_t^{\bar{\pi}}) \ ,$$

so that

$$E_t\widehat{B}_{t+1} = \widehat{\pi}_{t+1} - \rho\tau\widehat{\pi}_t - \rho(1-\tau)(\widehat{\pi}_{t-1} - \widehat{g}_t^{\bar{\pi}}) \ .$$

Note that the expectation above reflects the fact that $\widehat{g}_t^{\bar{\pi}}$ is an innovation process so that $E_t\widehat{g}_{t+j}^{\bar{\pi}} = 0$ for $j \geq 1$. Using this definition, we can rewrite (A13) as

$$\widehat{B}_t = \lambda_t E_t \widehat{B}_{t+1} + \zeta_t \widehat{mc}_t + \gamma_t \widehat{D}_t + u_{\pi,t}. \tag{B1}$$

Solving forward (B1) yields

$$\widehat{B}_t = \zeta_t E_t \sum_{j=0}^{\infty} \lambda_t^j \widehat{mc}_{t+j} + \gamma_t E_t \sum_{j=0}^{\infty} \lambda_t^j \widehat{D}_{t+j} + u_{\pi,t}. \tag{B2}$$

In deriving (B2) (and (B3) below), the "anticipated utility" assumption is used so that

$$\begin{aligned} E_t\zeta_{t+j} \prod_{k=0}^{j} \lambda_{t+k} \widehat{mc}_{t+j} &= \zeta_t \lambda_t^{j+1} E_t \widehat{mc}_{t+j} \\ E_t\gamma_{t+j} \prod_{k=0}^{j} \lambda_{t+k} \widehat{D}_{t+j} &= \gamma_t \lambda_t^{j+1} E_t \widehat{D}_{t+j} \end{aligned}$$

for any $j > 0$. Next, solving forward (A14), converting real discount factors into nominal ones, and rearranging leads to

$$\begin{aligned} \widehat{D}_t &= \varphi_{1,t} E_t \sum_{j=0}^{\infty} \varphi_{1,t}^j \left[\widehat{Q}_{t+j,t+j+1} + \widehat{g}_{t+j+1}^{y}\right] \\ &\quad -\kappa_{1,t}\left[\widehat{\pi}_{t-1} - \widehat{g}_t^{\bar{\pi}}\right] + \kappa_{2,t}\widehat{\pi}_t + \kappa_{3,t}\widehat{\pi}_{t+1} \\ &\quad +\kappa_{3,t} E_t \sum_{j=2}^{\infty} \varphi_{1,t}^{j-1} \widehat{\pi}_{t+j} \ , \end{aligned} \tag{B3}$$

with the new coefficients defined by

$$\begin{aligned} \kappa_{1,t} &= (\theta-1)\rho(1-\tau)\varphi_{1,t} \\ \kappa_{2,t} &= (\theta-1)\rho\tau\varphi_{1,t} + (\theta-1)\rho(1-\tau)\varphi_{1,t}^2 \\ \kappa_{3,t} &= \theta\varphi_{1,t} - (\theta-1)\rho\tau\varphi_{1,t}^2 - (\theta-1)\rho(1-\tau)\varphi_{1,t}^3. \end{aligned}$$

We next remove the auxiliary variables $\widehat{B}_t$ and $\widehat{D}_t$ and derive the NKPC. Using the definition of $\widehat{B}_t$, we reintroduce inflation into (B2) so that

$$\begin{aligned} \widehat{\pi}_t &= \rho\tau(\widehat{\pi}_{t-1} - \widehat{g}_t^{\bar{\pi}}) + \rho(1-\tau)(\widehat{\pi}_{t-2} - \widehat{g}_{t-1}^{\bar{\pi}} - \widehat{g}_t^{\bar{\pi}}) \\ &\quad +\zeta_t E_t \sum_{j=0}^{\infty} \lambda_t^j \widehat{mc}_{t+j} + \gamma_t E_t \sum_{j=0}^{\infty} \lambda_t^j \widehat{D}_{t+j} \ . \end{aligned} \tag{B4}$$

Finally, we substitute for $\widehat{D}_{t+j}$ terms in (B4) using (B3) and rearrange the resulting expression to obtain the CF representation of NKPC:

$$\begin{aligned}
\widehat{\pi}_t \;=\;& \widetilde{\rho}^C_{1,t}(\widehat{\pi}_{t-1} - \widehat{g}^{\bar{\pi}}_t) + (1-\tau)\widetilde{\rho}^C_{2,t}(\widehat{\pi}_{t-2} - \widehat{g}^{\bar{\pi}}_{t-1} - \widehat{g}^{\bar{\pi}}_t) \\
&+\widetilde{\zeta}^C_t E_t \sum_{j=0}^{\infty} \lambda_t^j \widehat{mc}_{t+j} \\
&+d^C_{0,t} E_t \sum_{k=0}^{\infty} \lambda_t^k \left[\widehat{\pi}_{t+k-1} - \widehat{g}^{\bar{\pi}}_{t+k}\right] \\
&+d^C_{1,t} E_t \sum_{k=0}^{\infty} \lambda_t^k \widehat{\pi}_{t+k} \\
&+d^C_{2,t} E_t \sum_{k=0}^{\infty} \lambda_t^k \widehat{\pi}_{t+k+1} \qquad \text{(PC-CF)} \\
&+d^C_{2,t} E_t \sum_{k=0}^{\infty} \lambda_t^k \sum_{j=2}^{\infty} \varphi_{1,t}^{j-1} \widehat{\pi}_{t+j+k} \\
&+d^C_{3,t} E_t \sum_{k=0}^{\infty} \lambda_t^k \sum_{j=0}^{\infty} \varphi_{1,t}^{j} \left[\widehat{Q}_{t+j+k,t+j+k+1} + \widehat{g}^y_{t+j+k+1}\right] + \; u_{\pi,t},
\end{aligned}$$

with the new coefficients defined as follows

$$\begin{aligned}
\widetilde{\rho}^C_{1,t} &= \rho\tau \\
\widetilde{\rho}^C_{2,t} &= \rho \\
\widetilde{\zeta}^C_t &= \zeta_t \\
d^C_{0,t} &= -\gamma_t \kappa_{1,t} \\
d^C_{1,t} &= -\gamma_t \kappa_{2,t} \\
d^C_{2,t} &= \gamma_t \kappa_{3,t} \\
d^C_{3,t} &= \gamma_t \varphi_{1,t}.
\end{aligned}$$

Cross-equation restrictions.

As before, given the forecasting rule (39), the $t-2$ conditional expectation of (PC-CF) is in the form

$$\begin{aligned}
\mathbf{e}'_\pi \mathbf{A}^2_{t-2} \widehat{\mathbf{z}}_{t-2} \;=\;& \widetilde{\rho}^C_{1,t-2} \mathbf{e}'_\pi \mathbf{A}_{t-2} \widehat{\mathbf{z}}_{t-2} + (1-\tau)\,\widetilde{\rho}^C_{2,t-2} \mathbf{e}'_\pi \widehat{\mathbf{z}}_{t-2} + \widetilde{\zeta}^C_{t-2} \mathbf{e}'_{mc} \mathbf{K}_{t-2} \mathbf{A}^2_{t-2} \widehat{\mathbf{z}}_{t-2} \\
&+d^C_{0,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}_{t-2} \widehat{\mathbf{z}}_{t-2} + d^C_{1,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}^2_{t-2} \widehat{\mathbf{z}}_{t-2} \\
&+d^C_{2,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}^3_{t-2} \widehat{\mathbf{z}}_{t-2} + d^C_{2,t-2} \varphi_{1,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^4_{t-2} \widehat{\mathbf{z}}_{t-2} \\
&+d^C_{3,t-2} (\mathbf{e}'_Q \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^2_{t-2} \widehat{\mathbf{z}}_{t-2} + \mathbf{e}'_{gy} \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^3_{t-2} \widehat{\mathbf{z}}_{t-2}), \qquad \text{(B5)}
\end{aligned}$$

where $\mathbf{K}_t \equiv (\mathbf{I} - \lambda_t \mathbf{A}_t)^{-1}$. Hence, the vector of cross-equation restrictions is given by

$$\begin{aligned}
\mathbf{e}'_\pi \mathbf{A}^2_{t-2} &= \widetilde{\rho}^C_{1,t-2} \mathbf{e}'_\pi \mathbf{A}_{t-2} + (1-\tau)\, \widetilde{\rho}^C_{2,t-2} \mathbf{e}'_\pi \mathbf{I} + \widetilde{\zeta}^C_{t-2} \mathbf{e}'_{mc} \mathbf{K}_{t-2} \mathbf{A}^2_{t-2} \\
&\quad + d^C_{0,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}_{t-2} + d^C_{1,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}^2_{t-2} \\
&\quad + d^C_{2,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{A}^3_{t-2} + d^C_{2,t-2} \varphi_{1,t-2} \mathbf{e}'_\pi \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^4_{t-2} \\
&\quad + d^C_{3,t-2} (\mathbf{e}'_Q \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^2_{t-2} + \mathbf{e}'_{g^y} \mathbf{K}_{t-2} \mathbf{J}_{t-2} \mathbf{A}^3_{t-2}) \\
&\equiv \mathbf{g}^C(\boldsymbol{\mu}_{t-2}, \mathbf{A}_{t-2}, \boldsymbol{\psi}). \qquad \text{(B6)}
\end{aligned}$$

Appendix C: The exact closed form (ECF), derivation and discussion

In this appendix we derive the exact closed form, and the corresponding cross-equation restrictions used for estimation. In order to simplify notation, in this appendix we omit time subscripts where immaterial, and hats on variables. All expectations assume anticipated utility (Kreps, 1998).

As in the main text, we use B_t to denote non-predetermined inflation

$$B_t \equiv \pi_t - \rho\tau\left(\pi_{t-1} - g_t^{\pi}\right) - \rho\left(1-\tau\right)\left(\pi_{t-2} - g_{t-1}^{\pi} + g_t^{\pi}\right).$$

Then, the NKPC is given by the expression

$$\begin{aligned} B_t &= \lambda E_t\left\{B_{t+1}\right\} + \gamma\varphi_1\left(\theta-1\right)\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+1+i}\right\} \\ &\quad +\zeta mc_t + \gamma\varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{q_{t+i,t+1+i} + g_{t+1+i}^{y}\right\} + u_{\pi,t}. \end{aligned}$$

After rearranging the inflation terms as follows

$$\begin{aligned} & B_t - \lambda E_t\left\{B_{t+1}\right\} - \gamma\varphi_1\left(\theta-1\right)\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+1+i}\right\} \\ = & \zeta mc_t + \gamma\varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{q_{t+i,t+1+i} + g_{t+1+i}^{y}\right\} + u_{\pi,t}, \end{aligned}$$

we add and subtract from the left-hand side of this equation the following term:

$$\lambda\varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+2+i}\right\} + \varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+1+i}\right\}.$$

This results in the equation

$$\begin{aligned} & \sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+i}\right\} - \left(\lambda + \varphi_1\left(\gamma\left(\theta-1\right)+1\right)\right)\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+1+i}\right\} \\ & +\lambda\varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{B_{t+2+i}\right\} \\ = & \zeta mc_t + \gamma\varphi_1\sum_{i=0}^{\infty}\varphi_1^i E_t\left\{q_{t+i,t+1+i} + g_{t+1+i}^{y}\right\} + u_{\pi,t}. \end{aligned}$$

From now on we use several factorizations of the form $(1 - root \cdot F)$ in the expectational terms. We discuss later the conditions under which inverting this factor is appropriate. The first factorization extracts $(1-\varphi_1 F)^{-1}$ from the left- and right-hand sides of the last equation to obtain

$$\begin{aligned} & E_t\left\{\left(1-\varphi_1 F\right)^{-1}\cdot\left[B_t - \left(\lambda + \varphi_1\left(\gamma\left(\theta-1\right)+1\right)\right)B_{t+1} + \lambda\varphi_1 B_{t+2}\right]\right\} \\ = & \zeta mc_t + \gamma\varphi_1 E_t\left\{\left(1-\varphi_1 F\right)^{-1}\cdot\left[q_{t,t+1} + g_{t+1}^{y}\right]\right\} + u_{\pi,t}, \end{aligned}$$

which can be rewritten as

$$\begin{aligned}&E_t\left\{B_t-(\lambda+\varphi_1(\gamma(\theta-1)+1))B_{t+1}+\lambda\varphi_1 B_{t+2}\right\}\\=\;&\zeta E_t\left\{(1-\varphi_1 F)\cdot mc_t\right\}+\gamma\varphi_1 E_t\left\{q_{t,t+1}+g_{t+1}^y\right\}+u_{\pi,t}.\end{aligned}$$

Factoring the polynomial in the left-hand side we obtain

$$E_t\left\{(1-\xi_1 F)(1-\xi_2 F)B_t\right\}=\zeta E_t\left\{(1-\varphi_1 F)\cdot mc_t\right\}+\gamma\varphi_1 E_t\left\{q_{t,t+1}+g_{t+1}^y\right\}+u_{\pi,t} \tag{C1}$$

with the roots $\{\xi_1,\xi_2\}$ given by

$$\xi_1+\xi_2 = \lambda+\varphi_1+\varphi_1\gamma(\theta-1), \tag{C2}$$

$$\xi_1\cdot\xi_2 = \lambda\varphi_1. \tag{C3}$$

The unique closed form under determinacy is therefore

$$\begin{aligned}\pi_t=\;&\rho\tau\left(\pi_{t-1}-g_t^\pi\right)+\rho(1-\tau)\left(\pi_{t-2}-g_{t-1}^\pi+g_t^\pi\right)\\&+\zeta E_t\left\{\sum_{j=0}^{\infty}\xi_1^j\sum_{i=0}^{\infty}\xi_2^i\left(mc_{t+i+j}-\varphi_1 mc_{t+1+i+j}\right)\right\}\\&+\gamma\varphi_1 E_t\left\{\sum_{j=0}^{\infty}\xi_1^j\sum_{i=0}^{\infty}\xi_2^i\left(q_{t+i+j,t+1+i+j}+g_{t+1+i+j}^y\right)\right\}+u_{\pi,t}.\end{aligned} \tag{C4}$$

This closed form states that inflation depends on (1) indexation, (2) the expected path of real marginal costs, and (3) the real expected discounted value of future output growth. The expected path of real marginal costs enters as a quasi-difference, indicating that both the expected level and the expected rate of change of real marginal costs have an impact on inflation. While the roots $\{\xi_1,\xi_2\}$ can be complex, equation (C4) facilitates the discussion of the necessary and sufficient conditions for determinacy. At the end of this appendix we show that this exact closed form is in fact a function of real-valued parameters only. For our purposes, what is important is that the cross-equation restrictions obtained from (C4), to which we now turn, involve only real-valued coefficients.

To derive the cross-equation restrictions, we use the forecasting rule (39) to form expectations. Taking expectations of (C4) conditional on information available at time $t-2$, we obtain

$$\begin{aligned}e_\pi'\mathbf{A}_{t-2}^2=\;&\rho\tau e_\pi'\mathbf{A}_{t-2}+\rho(1-\tau)e_\pi'\mathbf{I}\\&+\zeta_{t-2}e_{mc}'\mathbf{A}_{t-2}^2\left(\mathbf{I}-\varphi_{1,t-2}\mathbf{A}_{t-2}\right)\mathbf{W}_{t-2}^{-1}\\&+\gamma_{t-2}\varphi_{1,t-2}(\theta-1)\left[\widetilde{e}_{q,t-2}'\mathbf{I}+e_y'\mathbf{A}_{t-2}\right]\mathbf{A}_{t-2}^2\mathbf{W}_{t-2}^{-1},\end{aligned}$$

where the invertible matrix $\mathbf{W}_t$ is defined as

$$\begin{aligned}\mathbf{W}_t\equiv\;&\left(\mathbf{I}-\xi_{1,t}\mathbf{A}_t\right)\left(\mathbf{I}-\xi_{2,t}\mathbf{A}_t\right)\\=\;&\left[(\mathbf{I}-\lambda_t\mathbf{A}_t)(\mathbf{I}-\varphi_t\mathbf{A}_t)-\gamma_t\varphi_{1,t}(\theta-1)\mathbf{A}_t\right],\end{aligned}$$

with the last equality following immediately from (C1)-(C3) and the forecasting rule, and the vector $\widetilde{e}_{q,t}'$ is given by

$$\widetilde{e}_{q,t}'=e_Q'+e_\pi'\mathbf{A}_t.$$

The vectors e'_x are row vectors with zeros everywhere except for a single one in the position of variable x_t in $\mathbf{z}_t$ (see equation (38)). The vector $\widetilde{e}'_{q,t-2}$ selects the expected real discount factor by adding expected inflation to the nominal discount factor, which is the discount factor used in the estimated VAR. The exact closed form for inflation cannot depend on expected future nominal variables, as these would contain expected future inflation. For this reason, it is necessary to add expected inflation back to the nominal discount factor, in order to obtain the expected real discount factor.

From expression (C4) it is possible to extract the necessary and sufficient conditions for the validity of the exact closed-form estimates (see also Appendix D). These conditions are

$$\begin{aligned} \|\varphi_1 \mathbf{A}\| &< 1 \\ \|\xi_i \mathbf{A}\| &< 1 \text{ for } i = 1, 2. \end{aligned}$$

As stated in section 3 (see also Figure 11), these conditions appear to be satisfied most of the time in the data. A property of $\{\xi_1, \xi_2\}$ worth noting is that as long as $\gamma_t \varphi_{1,t} (\theta - 1) \geq 0$, which occurs in over 99 percent of the estimates, we have that

$$\begin{aligned} \xi_{1,t} &\geq \lambda_t \\ \xi_{2,t} &\leq \varphi_{1,t}. \end{aligned}$$

This can be shown from simple but tedious algebra based on the expression of these roots

$$\{\xi_1, \xi_2\} = \frac{2\lambda\varphi_1}{\lambda + \varphi_1 + \varphi_1\gamma(\theta - 1) \pm \sqrt{(\lambda + \varphi_1 + \varphi_1\gamma(\theta - 1))^2 - 4\lambda\varphi_1}}.$$

We also note that when $\gamma_t \varphi_{1,t} (\theta - 1) = 0$ we obtain

$$\mathbf{W}_t^{-1} = (\mathbf{I} - \varphi_t \mathbf{A}_t)^{-1} (\mathbf{I} - \lambda_t \mathbf{A}_t)^{-1},$$

which shows that in this case $\xi_1 = \lambda_t$, and $\xi_2 = \varphi_{1,t}$. Hence, conditional on $\left\|\varphi_{1,t} \mathbf{A}_t\right\| < 1$, the necessary and sufficient condition would be given by

$$\|\lambda_t \mathbf{A}_t\| < 1,$$

which is the same condition discussed in the main text for the existence of CF estimators (given that $\left\|\varphi_{1,t} \mathbf{A}_t\right\| < 1$). In our exact closed-form (unrestricted) estimation we obtain that $\left|\gamma_t \varphi_{1,t} (\theta - 1)\right|$ has a median of $8.7x10^{-4}$, a mode of $2.8x10^{-6}$, and is below $3x10^{-3}$ for all ensembles.[58] In short, the necessary and sufficient conditions for the existence of the exact closed form are extremely close to the conditions affecting the CF specifications. Also, as discussed in the main text, the conditions for the existence of our CF estimators are always met whenever the exact closed-form solution exists.

To conclude, we show that even when the roots $\{\xi_1,\ \xi_2\}$ are complex, the exact closed form in (C4) depends only on real-valued coefficients.[59] To see this, note that

$$\sum_{j=0}^{\infty} \xi_1^j \sum_{i=0}^{\infty} \xi_2^i E_t \{mc_{t+i+j}\} = \sum_{j=0}^{\infty} \delta(\xi_2 + \xi_1, \xi_2\xi_1, j) E_t \{mc_{t+j}\},$$

[58] Negative values of $\gamma_t \varphi_{1,t} (\theta - 1)$ are typically due to $\gamma_t < 0$, which occurs for 1.17 percent of the 765,948 point estimates. We have only one negative estimate of $\varphi_{1,t}$, and the estimation is restricted to produce estimates of $\theta > 1$.

[59] This occurs for fewer than 0.06 percent of our estimates.

where δ is a real-valued function of real arguments.[60] This function is given by

$$\delta\left(\xi_2+\xi_1,\ \xi_2\xi_1,\ j\right) = \begin{cases} \left(\xi_2+\xi_1\right)^j \quad \text{for } j=0,\ 1. \\ \left(\xi_2+\xi_1\right)^j + \sum_{i=1}^{M(j)} (-1)^j \binom{j-i}{i} \left(\xi_2+\xi_1\right)_1^{j-2i} \left(\xi_2\xi_1\right)^i \quad \text{for } j \geq 2 \end{cases} \tag{C5}$$

$$M(j) = \begin{cases} \frac{j}{2} & \text{for } j \text{ even,} \\ \frac{j-1}{2} & \text{for } j \geq 3 \text{ and } j \text{ odd,} \end{cases}$$

where $\binom{j-i}{i}$ is the binomial coefficient ${}_{j-i}C_i$, and $M(j)$ determines the number of elements in the right-hand side summation. Despite its complicated appearance, this function essentially provides the moving average coefficients from inverting a stationary $AR(2)$ process of the form $y_t\ (1-\xi_2 L)(1-\xi_1 L) = e_t$.

Relationship between cross-equation restrictions of the DE specification and the exact closed-form solution.

Next, we discuss the relationship between the cross-equation restrictions of the DE specification and the exact closed-form solution, along the same lines as our discussion for the constant-coefficients NKPC in sections 2.1 and 2.2 in the main text.

The cross-equation restriction errors for the DE specification with time-varying trends can be written as

$$\begin{aligned} \mathbf{F}^D\left(\mathbf{A}_t,\psi,t\right) \equiv\ & e_\pi'\mathbf{B}_t^*\left(\mathbf{I}-\lambda_t\mathbf{A}_t\right) \\ & -\zeta e_{mc}'\mathbf{A}_t\left(\mathbf{I}-\varphi\mathbf{A}\right) \\ & -\gamma_t\varphi_{1,t}\left(\theta-1\right)e_\pi'\mathbf{B}_t^*\mathbf{A}_t \\ & -\gamma_t\varphi_{1,t}\left[\widetilde{e}_{q,t}'\mathbf{I}+e_y'\mathbf{A}\right], \end{aligned} \tag{C6}$$
$$\text{with } \mathbf{B}_t^* \equiv \left(\mathbf{A}_t^2-\rho\tau\mathbf{A}_t-\rho\left(1-\tau\right)\mathbf{I}\right)\left(\mathbf{I}-\varphi\mathbf{A}\right),$$

although in the main text we provide the expression closest to Cogley and Sbordone (2008) for ease of comparison (this also motivates the particular presentation of the CF cross-equation restrictions in the main text). The differences are only due to the grouping of specific terms in the equation, and for this appendix we adopt the expression (C6) to facilitate our discussion here. This expression can be directly derived from the difference equation that results from quasi-differencing forward the Phillips curve defined by (33) and (34), with factor φ_1.

The cross-equation restriction errors of the ECF specification are then given by

$$\mathbf{F}^{ECF}\left(\mathbf{A}_t,\boldsymbol{\psi},t\right) = \mathbf{F}^D\left(\mathbf{A}_t,\boldsymbol{\psi},t\right)\left(\mathbf{I}+\xi_1\mathbf{A}+\left(\xi_1\mathbf{A}\right)^2+...\right)\left(\mathbf{I}+\xi_2\mathbf{A}+\left(\xi_2\mathbf{A}\right)^2+...\right),$$

which illustrates that $\mathbf{F}^{ECF}\left(\mathbf{A}_t,\boldsymbol{\psi},t\right)$ is a reweighting of the difference equation errors, as in the constant coefficients case, and that they can be obtained as a weighted sum of the restrictions imposed by the model on expectations of future variables. In this sense, the only difference with the constant coefficients case is that this involves a double summation. Nonetheless, since the idea behind imposing model-consistency restrictions on expectations in our application is to explicitly build the link between expected period $t+j$ inflation and expected period $t+j$ marginal costs, imposing this link for m periods is equivalent to

$$\mathbf{F}^m\left(\mathbf{A}_t,\boldsymbol{\psi},t\right) = \mathbf{F}^D\left(\mathbf{A}_t,\boldsymbol{\psi},t\right)\left(\mathbf{I}+\delta_{t,\xi}\left(1\right)\mathbf{A}_t+...+\delta_{t,\xi}\left(m\right)\mathbf{A}_t^m\right),$$

[60] The arguments $(\xi_1+\xi_2)$ and $(\xi_1\xi_2)$ are given in (C2) and (C3) above as a real-valued function of the parameter values, which are always real. The third argument is a non-negative integer.

with $\delta_{t,\xi}(j) \equiv \delta\left(\xi_{1,t} + \xi_{2,t},\ \xi_{1,t}\xi_{2,t},\ j\right)$ defined in (C5), and hence $\mathbf{F}^m \to \mathbf{F}^{ECF}$ when $m \to \infty$.

Likewise, expression (15) in our discussion of the case with constant coefficients can also be generalized for the case with time-varying trends. This expression is now

$$\mathbf{g}^D\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right) = \mathbf{g}^{ECF}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right) + \left(\hat{\xi}_1 + \hat{\xi}_2\right)\mathbf{k}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right)\hat{\mathbf{A}}_t - \left(\hat{\xi}_1\hat{\xi}_2\right)\mathbf{k}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right)\hat{\mathbf{A}}_t^2.$$

Since having both $\hat{\xi}_1 + \hat{\xi}_2 = 0$ and $\hat{\xi}_1\hat{\xi}_2 = 0$ would render the NKPC meaningless (see (C2) and (C3)), this expression implies that $\mathbf{g}^D\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right) = \mathbf{g}^{ECF}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right)$ if and only if[61]

$$\mathbf{k}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right) = 0.$$

It is possible to show, using only equations provided in this appendix, that when $\mathbf{k}(.) = 0$ we have

$$\mathbf{k}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right) = \mathbf{F}^{ECF}\left(\hat{\mathbf{A}}_t, \boldsymbol{\psi}, t\right).$$

As a result, all of our discussion on the role of model-consistency restrictions for the NKPC with constant coefficients naturally extends to the NKPC with time-varying coefficients.

[61] As we have pointed out earlier, $\hat{\mathbf{A}}_t$ is a full rank matrix for all t, and all the eigenvalues of $\hat{\mathbf{A}}_t$ are inside the unit circle. Hence $\left\|\hat{\mathbf{A}}_t^2\right\| < \left\|\hat{\mathbf{A}}_t\right\| < 1$.

Appendix D: Misspecification of the first-stage VAR in the presence of sunspots

In this appendix we show that the VAR representation (5) with a finite number of lags and i.i.d. shocks implies that there is a unique forward solution to the model. In particular, we show that indeterminacy of the model solution yields to a reduced-form representation that involves a moving average error term. In such a setting, a reduced-form VAR representation with i.i.d. shocks entails an infinite number of lags.[62] Therefore, estimation of the finite-order VAR in (5) results in biased and inconsistent coefficient estimates (truncation bias). This discussion is relegated to an appendix because its content draws on a vast literature on solutions to rational expectations models; see for instance the textbook treatment in Pesaran (1989) and the references therein.

For ease of exposition and without loss of generality, we consider a simple setup with only two structural equations.[63] The first equation is a purely forward-looking version of the NKPC with zero (constant) trend inflation:

$$\pi_t = \Lambda E_t \pi_{t+1} + \zeta mc_t + u_{\pi,t}, \tag{D1}$$

where $\zeta \geq 0$ is the elasticity of inflation with respect to marginal costs, $\Lambda > 0$ determines the extent of forward-looking behavior, and $u_{\pi,t}$ denotes the structural i.i.d. shock associated with the NKPC. The presence of this structural shock is critical for modern DSGE models and is at the heart of the discussion in this appendix.[64] The second equation closes the model, and states that marginal costs follow a simple univariate autoregressive process

$$mc_t = \chi mc_{t-1} + u_{mc,t}, \tag{D2}$$

where $1 \geq \chi > 0$ and $u_{mc,t}$ is also a structural i.i.d. shock.

Before analyzing the conditions for determinacy, notice that regardless of the value of $\Lambda > 0$, it is always possible to express (D1) backwards as

$$\pi_t = \frac{1}{\Lambda}\pi_{t-1} - \frac{\zeta}{\Lambda}mc_{t-1} - \frac{1}{\Lambda}u_{\pi,t-1} + \left[\pi_t - E_{t-1}\pi_t\right]. \tag{D3}$$

In a determinate solution the model equations impose just enough conditions to uniquely determine the forecast error $[\pi_t - E_{t-1}\pi_t]$ as a linear combination of new information revealed at time t by the two structural shocks $u_{\pi,t}$, and $u_{mc,t}$. When there are not enough conditions to uniquely pin down the forecast error $[\pi_t - E_{t-1}\pi_t]$, rational expectations dictate only that the error be unpredictable as of time $t-1$, and as a result, this error could be partly driven by a sunspot.[65] Whether the stationary solution for the system in (D1) and (D2) is indeterminate cannot be established just by looking at the parameters of the NKPC. In fact, in this stylized framework, whether a stationary solution involves sunspots or not is uniquely determined by the parameter interaction $\Lambda\chi$.

[62] For a discussion of this non-invertibility problem see Fernandez-Villaverde et al. (2007).

[63] The arguments here carry over with some minor adjustments to the anticipated utility framework with time-varying trends (or parameters). Also, note that our discussion focuses on the issue of determinacy and assumes for convenience that a stationary solution is feasible.

[64] As already mentioned in the main text, this shock can be interpreted as capturing potential misspecifications in the relationship or a markup shock.

[65] See Pesaran (1989, Chapter 5) for a discussion of the different methods to refine the solution set under indeterminacy.

Consider first the case in which $\Lambda\chi < 1$. Then a unique stationary forward-looking solution for the NKPC exists and takes the following form:

$$\begin{aligned} \pi_t &= \zeta \sum_{i=0}^{\infty} \Lambda^i E_t mc_{t+i} + u_{\pi,t} \\ \Leftrightarrow \pi_t &= \zeta mc_t \sum_{i=0}^{\infty} (\Lambda\chi)^i + u_{\pi,t} \\ \Leftrightarrow \pi_t &= \frac{\zeta}{1-\Lambda\chi} mc_t + u_{\pi,t}. \end{aligned} \tag{D4}$$

In constrained reduced form, the last expression can therefore be rewritten as

$$\pi_t = \frac{\zeta\chi}{1-\Lambda\chi} mc_{t-1} + u_{\pi,t} + \frac{\zeta}{1-\Lambda\chi} u_{mc,t}. \tag{D5}$$

Equations (D2) and (D5) form a constrained reduced-form VAR with serially uncorrelated shocks. In order to estimate the deep parameters of the model, it is possible to minimize the distance between the constrained VAR given by (D2) and (D5) and an estimated unconstrained reduced-form VAR as in (5). In the specific case of our application, we try to match the cross-equation restrictions implied by model expectations, with forecasts obtained from the estimated unrestricted VAR. For this exercise to be correct, the estimated unrestricted VAR in (5) and the restricted VAR implied by (D2) and (D5) need to share two crucial characteristics: (1) a finite number of lags, and (2) serially uncorrelated (unpredictable) errors.

Using (D4) it is possible to eliminate $u_{\pi,t-1}$ in (D3), to obtain

$$\pi_t = \frac{\zeta\chi}{1-\Lambda\chi} mc_{t-1} + (\pi_t - E_{t-1}\pi_t). \tag{D6}$$

Next, equations (D5) and (D6) can be used to solve for the forecast error as a linear combination of the structural shocks:

$$\pi_t - E_{t-1}\pi_t = u_{\pi,t} + \frac{\zeta}{1-\Lambda\chi} u_{mc,t}.$$

Hence, in the determinate solution the time t values of inflation, marginal costs, and expected inflation are all uniquely determined by the two structural shocks in the model. At most two of these variables can be linearly independent. This property is what allows one to properly form inflation expectations using a VAR where the only right-hand-side variables are inflation and marginal costs. As we discuss next, this property does not hold in the presence of sunspots.

The indeterminate (sunspot) solution arises when $\Lambda\chi \geq 1$. In this case, the stationary forward-looking solution we just discussed is no longer feasible because equations (D4), (D5), and (D6) do not hold. This implies that the forecast errors cannot be uniquely determined by the new time t information provided by $u_{\pi,t}$ and $u_{mc,t}$. Following Lubik and Schorfheide (2003), the forecast error in this case can be written as

$$\pi_t - E_{t-1}\pi_t = M_1 u_{\pi,t} + M_2 u_{mc,t} + h_t, \tag{D7}$$

where h_t is an arbitrary martingale difference (the sunspot shock), and M_1 and M_2 are parameters that are not pinned down by the structural parameters of the model. As a result, the parameters

M_1 and M_2 cannot be guaranteed to take any specific value, although in principle they can be estimated (see Lubik and Schorfheide 2004).[66]

Replacing the forecast error in (D3) by (D7) yields the following constrained reduced-form representation of inflation

$$\pi_t = \frac{1}{\Lambda}\pi_{t-1} - \frac{\zeta}{\Lambda}mc_{t-1} + M_2 u_{mc,t} + \xi_t \tag{D8}$$

$$\xi_t = \left[M_1 u_{\pi,t} - \frac{1}{\Lambda}u_{\pi,t-1}\right] + h_t. \tag{D9}$$

Under determinacy, equation (D4) implies that the error $u_{\pi,t-1}$ in (D9) can be replaced by a linear combination of π_{t-1} and mc_{t-1}, and as a result inflation does not contain a moving average error term. But in the indeterminate case equation (D4) does not hold, and the moving average error in (D9) becomes an important problem for estimation. More specifically, the equilibrium under indeterminacy is spanned by three independent shocks ($u_{\pi,t}$, $u_{mc,t}$, and ξ_t), and as a result marginal costs, inflation, and expected inflation are all linearly independent.[67] To avoid misspecification, a finite-order VAR would need to include all three variables: inflation, marginal costs, and most importantly inflation expectations themselves.[68]

It follows from this discussion that the dynamics of marginal costs and inflation given by (D2) and (D8) cannot have an unrestricted VAR representation such as (5), because (5) does not allow for a moving average error component. In other words, the problem with the unrestricted VAR in (5) is the omission of a variable that belongs in the model. In the presence of sunspots the estimated VAR parameters are biased and inconsistent, since the lagged endogenous variables at $t-1$ are correlated with the moving average error term. Moreover, even if consistent estimates for the VAR in (5) part of the process were available, it would still be incorrect to use (5) to proxy for one-period-ahead expectations.[69] The error term $u_{\pi,t-1}$ would contribute valuable information to form time $t-1$ inflation expectations beyond what would be contributed by inflation and marginal costs.

Overall, this simple example shows that assuming the VAR representation (5) as an unconstrained reduced form for the NKPC model amounts to imposing the existence of a stable forward-looking solution, that is, $\Lambda\chi < 1$. If $\Lambda\chi \geq 1$, the reduced-form VAR representation contains a moving average error term as we have already discussed. In such a case, the VAR errors would only be uncorrelated if the VAR contains an infinite number of lags. Otherwise estimating (5) with a finite number of lags results in truncation bias in the estimated coefficients. This bias would undermine the validity of the second stage, whereby estimates of the parameters of the NKPC are obtained by imposing model-consistent, cross-equation restrictions on the estimated VAR in equation (5). Moreover, because the estimation of (5) in the first stage imposes the assumption that no sunspots are present in the equilibrium, it is important that the second-stage estimates be supportive of this assumption. Evidence to the contrary would invalidate the estimation because of an incongruence between the two estimation stages. In our application, determinacy of the system

[66] For estimation purposes, they cannot be assumed to be zero, or to equal their determinacy values $M_1 = 1$ and $M_2 = \zeta/(1-\Lambda\chi)$.

[67] Note that as long as (D4) does not hold, inflation is affected by the moving average ξ_t even if we assume $h_t = 0$ for all t. More generally, the problems posed by ξ_t for the invertibility of the DSGE model are present even if $M_1 = 0$.

[68] This is, in essence, the non-invertibility problem that arises whenever the VAR has to incluede variables that are unobservable to the econometrician (see Fernandez-Villaverde et al., 2007).

[69] In order to obtain these consistent estimates the empirical model could be a $VARMA(p,q)$ model. In models that involve additional leads and lags of inflation or the driving process(es), the problem is how to determine the correct values of p and q.

hinges on the largest eigenvalues of $\widehat{\lambda}_t \widehat{\mathbf{A}}_t$. While Figure 10 illustrates that this is not a problem for our CF estimates, Figure 11 points at an important problem for the DE estimates (which replicate those in Cogley and Sbordone, 2008).

Two additional issues are worth noting here. First, truncation bias would imply that the estimated errors from the VAR follow an infinite-order moving average (see Fernandez-Villaverde et al., 2007). Second, even if a finite-order VAR might provide a relatively good approximation to the dynamics of the true infinite order VAR for specific shocks, this is ultimately irrelevant for our estimation because we exploit cross-equation restrictions.[70] In particular, the dynamics of the variables are determined by linear combinations of the columns of the VAR, while the cross-equation restrictions select very specific elements of the estimated VAR matrix to determine the structural parameters. Even if the linear combinations of the inconsistently estimated VAR matrix may provide relatively good impulse responses for specific shocks, the cross-equation restrictions hinge much more critically on each point estimate involved in a cross-equation restriction. In this sense, for the purpose of minimum-distance estimation as carried out in our paper and in Cogley and Sbordone (2008), the warning in Fernandez-Villaverde et al. (2007) against simply assuming that a finite-order VAR appropriately replicates model dynamics is particularly relevant.

[70] See Sims and Zha (2006) and Sims (2009) for applications where the non-invertibility of a DSGE model does not pose a particularly serious issue for estimation of a finite-order VAR.

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