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Abstract

We examine the least-squares estimator of change point for nonstationary $I(d)$ data with $0.5 < d < 1.5$. We show that this estimator fails to locate the change point consistently when a change occurs and that it would suggest a spurious change even when there is none.

Keywords: least-squares estimator, change point, nonstationary $I(d)$ process, spurious change

JEL classification: C13, C22

1 Introduction

Consistent estimation of change point(s) is an important issue in the studies of structural changes. It is well known that the least-squares estimator (LSE) of a change point is consistent for $I(0)$ series with a mean change, yet it converges to the end-points of the sample when a change is absent; see, e.g., Bai (1994) and Nunes et al. (1995). When there is a change, Kuan and Hsu (1998) showed that the consistency result carries over to stationary $I(d)$ data (i.e., $-0.5 < d < 0.5$). On the other hand, Nunes et al. (1995) and Kuan and Hsu (1998) demonstrated that, for data that are $I(1)$ or $I(d)$ with $0 < d < 0.5$, the LSE would spuriously suggest a change point when there is none. This paper extends the analysis of Kuan and Hsu (1998) to nonstationary $I(d)$ processes with $0.5 < d < 1.5$. It is shown that the LSE fails to locate the change point consistently when a change occurs and that the LSE leads to a spurious change even when there is none.

This paper proceeds as follows. In Section 2, we review some existing properties of the LSE of change point. The main results are presented in Section 3; the simulation results are reported in Section 4. Section 5 concludes this paper.

2 The Change-Point Estimator for $I(d)$ Processes

We consider the following data generating process:

$$y_t = \begin{cases} \mu_1 + \eta_t, & t = 1, 2, \ldots, [T\tau_o], \\ \mu_2 + \eta_t, & t = [T\tau_0] + 1, [T\tau_o] + 2, \ldots, T, \end{cases} \tag{1}$$

where $0 \leq [T\tau_o] \leq T$ with $[c]$ the integer part of the number c. When $\mu_1 \neq \mu_2$, y_t has a mean change with the change point τ_o; when $\mu_1 = \mu_2$, there is no change and $\tau_o = 0$ or 1. Let $\underline{\tau}$ and $\overline{\tau}$ be two constants such that $0 < \underline{\tau} < \overline{\tau} < 1$. For each hypothetical change point k, $[T\underline{\tau}] \leq k \leq [T\overline{\tau}]$, the pre- and post-change sample means are $\hat{\mu}_1(k) = \sum_{i=1}^{k} y_t/k$ and $\hat{\mu}_2(k) = \sum_{i=k+1}^{T} y_t/(T-k)$. The LSE of τ_o is defined as

$$\hat{\tau} = \inf\{\tau : \tau = \operatorname{argmin}_{\tau \in [\underline{\tau}, \overline{\tau}]} \text{RSS}([T\tau])\}, \tag{2}$$

where $\text{RSS}(k) = \sum_{t=1}^{k} \{y_t - \hat{\mu}_1(k)\}^2 + \sum_{t=k+1}^{T} \{y_t - \hat{\mu}_2(k)\}^2$.

As in Granger and Joyeux (1980) and Hosking (1981, 1984), $\{\eta_t\}$ is an $I(d)$ (fractionally integrated) process if $(1-B)^d \eta_t$ is a stationary and invertible ARMA process, where B is the back-shift operator, and $d > -1$ may be a non-integer such that

$$(1-B)^d = \sum_{j=0}^{\infty} \left(\prod_{k=1}^{j} \frac{k-1-d}{k} \right) B^j.$$

It is well known that An $I(d)$ process is stationary and invertiale when $-0.5 < d < 0.5$, but it is nonstationary when $d > 0.5$.

Bai (1994) showed that $\hat{\tau}$ converges in probability to τ_o when $\{\eta_t\}$ is an $I(0)$ process with a mean change. This consistency result is extended to stationary $I(d)$ data by Kuan and Hsu (1998). When there is no change, $\hat{\tau}$ behaves quite differently for different data. Nunes et al. (1995) demonstrated that, for $I(0)$ data with no change, $\hat{\tau}$ would converge in probability to the set $\{0, 1\}$ when $[\underline{\tau}, \overline{\tau}]$ is expanded to $[0, 1]$. Yet, Nunes et al. (1995) and Kuan and Hsu (1998) showed that, for $I(1)$ data or $I(d)$ data with $0 < d < 0.5$, $\hat{\tau}$ would suggest a sprious change when there is none.

3 Main results

Given $\{y_t\}$ in (1), let $\lambda = \mu_1 - \mu_2$ and $\eta_t^* = \eta_t + \lambda \mathbf{1}_{\{t > [T\tau_o]\}}$, where $\mathbf{1}_A$ be the indicator function of the event A. We can write $\text{RSS}([T\tau]) = \sum_{t=1}^{T} (\eta_t^*)^2 - J_T([T\tau])$, with

$$J_T([T\tau]) = \frac{1}{[T\tau]} \left(\sum_{t=1}^{[T\tau]} \eta_t^* \right)^2 + \frac{1}{T - [T\tau]} \left(\sum_{t=[T\tau]+1}^{T} \eta_t^* \right)^2.$$

It follows that $\hat{\tau} = \inf\{\tau : \tau = \mathrm{argmax}_{\tau \in [\underline{\tau}, \overline{\tau}]}\ T^{-\alpha} J_T([T\tau])\}$, for any real number α.

We focus on the case that $\{\eta_t\}$ is a nonstationary $I(d)$ process with $0.5 < d < 1.5$. Then, $(1-B)\eta_t = z_t$ is a stationary $I(d-1)$ process with $-0.5 < d < 0.5$. We require z_t obeying a functional central limit theorem, as stated in the assumption below; more specific regularity conditions ensuring this assumption can be found in, e.g., Sowell (1990). Let $\Rightarrow$ denotes weak convergence (of associated probability measures) and B_d denote the fractional Brownian motion with

$$B_d(\tau) \equiv \frac{1}{\Gamma(1+d)} \left[\int_0^\tau (\tau - s)^d dB_0(s) + \int_{-\infty}^0 (\tau - s)^d - (-s)^d dB_0(s) \right],$$

and $\Gamma(\cdot)$ is the Gamma function.

Assumption A For $0.5 < d < 1.5$, $T^{-(d-0.5)} \sum_{t=1}^{[T\tau]} z_t \Rightarrow \kappa B_{d-1}(\tau)$ on $[0,1]$, where κ is a positive real number.

The result below shows that the weak limit of $T^{-2d} J_T([T\tau])$ is not affected by the presence of a mean change.

Theorem 3.1 *Given that* $\{\eta_t\}$ *is a nonstationary* $I(d)$ *process with* $0.5 < d < 1.5$ *and* $(1-B)\eta_t$ *satisfies Assumption A, then* $T^{-2d} J_T([T\tau]) \Rightarrow \Psi_d(\tau)$ *on* $[\underline{\tau}, \overline{\tau}] \subset (0,1)$, *where*

$$\Psi_d(\tau) = \frac{\kappa^2 [\int_0^\tau B_{d-1}(x) dx]^2}{\tau} + \frac{\kappa^2 [\int_\tau^1 B_{d-1}(x) dx]^2}{1-\tau}.$$

Thus, for $[\underline{\tau}, \overline{\tau}] \subset (0,1)$, $\hat{\tau} \Rightarrow \arg\max_{\tau \in [\underline{\tau}, \overline{\tau}]} \Psi_d(\tau)$.

This result indicates that $\hat{\tau}$ loses consistency for nonstationary $I(d)$ data with a mean change because its limit does not depend on τ_o, in contrast with the consistency results for stationary $I(0)$ and $I(d)$ data; cf. Bai (1994) and Kuan and Hsu (1998). When a change is absent, this result is analogous to Theorem 3.3 of Kuan and Hsu (1998) for stationary $I(d)$ data.

To extend Theorem 3.1 when $[\underline{\tau}, \overline{\tau}]$ is expanded to $[0,1]$, we first show that $\Psi_d(\tau)$ would be a continuous function on $[0,1]$ if we define $\Psi_d(0) = \Psi_d(1) = \kappa^2 [\int_0^1 B_{d-1}(x) dx]^2$.

Lemma 3.2 $\Psi_d(\tau)$ *converges almost surely to* $\Psi_d(0)$ *and* $\Psi_d(1)$ *as* $\tau \to 0$ *and* $\tau \to 1$, *respectively.*

With Lemma 3.2, Theorem 3.1 carries over when $[\underline{\tau}, \overline{\tau}] = [0,1]$.

Theorem 3.3 *Given that* $\{\eta_t\}$ *is a nonstationary* $I(d)$ *process with* $0.5 < d < 1.5$ *and* $(1-B)\eta_t$ *satisfies Assumption A,* $\hat{\tau} \Rightarrow \arg\max_{\tau \in [0,1]} \Psi_d(\tau)$, *such that with probability one,* $\Psi_d(0) = \Psi_d(1) < \Psi_d(\tau)$ *for all* $0 < \tau < 1$.

Theorem 3.3 shows that Ψ_d can not attain the maximum at 0 or 1, so that the support of the limiting distribution of $\hat{\tau}$ is $(0, 1)$. Thus, $\hat{\tau}$ must suggest a spurious change in the sample when there is none. This problem is analogous to that discussed in Nunes et al. (1995), Bai (1998), and Kuan and Hsu (1998).

4 Simulations

In our simulations, the data y_t are generated according to (1). We set $\mu_1 = \mu_2 = 0$ when there is no change, and $\mu_1 = 0$, $\mu_2 = 1$ and $\tau_o = 0.5$ when there is a change. To generate a nonstationary $I(d)$ process with $0.5 < d < 1.5$, we first generate a stationary $I(d-1)$ process $\boldsymbol{Z}$ following McLeod and Hipel (1978) and Hosking (1984), and compute the $I(d)$ process as $\eta_t = \sum_{i=1}^{t} z_i$. We consider the cases $d = 0.7, 1.0, 1.3$. The sample size is $T = 200$; the number of replication is 50000. The empirical distributions of $\hat{\tau}$ are plotted in Figure 1, where the graphs on the left are the distributions when there is no change and those on the right are the distributions when there is a change.

We find that without a change, the values of $\hat{\tau}$ do not cluster around 0 or 1, but have a distribution on $(0, 1)$. This verifies the spurious change problem. Given a mean change at $\tau_o = 0.5$, the distributions of $\hat{\tau}$ do not concentrate on 0.5, showing inconsistency of $\hat{\tau}$. Even for larger samples, the distributions of $\hat{\tau}$ are qualitatively similar (the results are not plotted to save space). Also, for a given d, the distributions of $\hat{\tau}$ are virtually the same, regardless of the presence of a change, as shown in Theorems 3.1 and 3.3.

5 Conclusions

This paper shows that the LSE of change point in nonstationary $I(d)$ data is not practically useful. First, it is inconsistent when there is a change, in contrast with the case of stationary $I(d)$ data. Second, it suffers from the spurious change problem, as in $I(1)$ and stationary $I(d)$ data.

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Appendix

Proof of Theorem 3.1: By Assumption A,

$$\frac{1}{T^{d+0.5}}\sum_{t=1}^{[T\tau]}\eta_t=\frac{1}{T}\sum_{t=1}^{[T\tau]}\left(\frac{1}{T^{d-0.5}}\sum_{i=1}^{t}z_i\right)\Rightarrow\kappa\int_0^\tau B_{d-1}(x)dx,$$

$$\frac{1}{T^{d+0.5}}\sum_{t=[T\tau]+1}^{T}\eta_t=\frac{1}{T}\sum_{t=[T\tau]+1}^{T}\left(\frac{1}{T^{d-0.5}}\sum_{i=1}^{t}z_i\right)\Rightarrow\kappa\int_\tau^1 B_{d-1}(x)dx,$$

as $T\to\infty$. It follows from the continuous mapping theorem that

$$\frac{T^{-2d}}{[T\tau]}\left(\sum_{t=1}^{[T\tau]}\eta_t\right)^2=\frac{T}{[T\tau]}\left(\frac{1}{T^{d+0.5}}\sum_{t=1}^{[T\tau]}\eta_t\right)^2\Rightarrow\frac{\kappa^2\left(\int_0^\tau B_{d-1}(x)dx\right)^2}{\tau},$$

and

$$\frac{T^{-2d}}{T-[T\tau]}\left(\sum_{t=[T\tau]+1}^{T}\eta_t\right)^2=\frac{T}{T-[T\tau]}\left(\frac{1}{T^{d+0.5}}\sum_{t=[T\tau]+1}^{T}\eta_t\right)^2\Rightarrow\frac{\kappa^2(\int_\tau^1 B_{d-1}(x)dx)^2}{1-\tau}.$$

When there is no change, $\eta_t^*=\eta_t$, so that $T^{-2d}J_T([T\tau])\Rightarrow\Psi_d(\tau)$ on $[\underline{\tau},\overline{\tau}]$.

When $\mu_1\neq\mu_2$, $\eta_t^*=\eta_t+\lambda\mathbf{1}_{\{t>[T\tau_o]\}}$. Then for $\tau\leq\tau_o$,

$$\frac{1}{T^{d+0.5}}\sum_{t=1}^{[T\tau]}\eta_t^*=\frac{1}{T^{d+0.5}}\sum_{t=1}^{[T\tau]}\eta_t\Rightarrow\kappa\int_0^\tau B_{d-1}(x)dx,$$

$$\frac{1}{T^{d+0.5}}\sum_{t=[T\tau]+1}^{T}\eta_t^*=\frac{1}{T^{d+0.5}}\sum_{t=[T\tau]+1}^{T}\eta_t+\frac{(T-[T\tau_o])\lambda}{T^{d+0.5}}\Rightarrow\kappa\int_\tau^1 B_{d-1}(x)dx,$$

because $T^{-(d+0.5)}(T-[T\tau_o])=0$. It is easily seen that these two limits also hold when $\tau>\tau_o$. Thus, $T^{-2d}J_T([T\tau])\Rightarrow\Psi_d(\tau)$ on $[\underline{\tau},\overline{\tau}]$, as in the case of no change. Hence,

$$\sup_{[\underline{\tau},\overline{\tau}]}T^{-2d}J_T([T\tau])\Rightarrow\sup_{[\underline{\tau},\overline{\tau}]}\Psi_d(\tau).$$

The assertion on $\hat{\tau}$ now follows because $\hat{\tau}$ maximizes $T^{-2d}J_T([T\tau])$.

Proof of Lemma 3.2: By Theorem A1 in Taqqu (1977), Corollary 3.2 in Mori and Oodaira (1986) and Eqn. (4) in Kuan and Hsu (1998), we have

$$\limsup_{t\to 0}\frac{B_{d-1}(t)}{\{ct^{2d-1}\log\log(1/t)\}^{\frac{1}{2}}}=1, \tag{3}$$

almost surely (a.s.). Let A denote the ω-set that (3) holds; then $P(A) = 1$. Given that $B_{d-1}(t)$ is symmetric relative to 0, then for each $\omega \in A$ and for all $\varepsilon > 0$, we can find a small $\delta > 0$ such that $\left|B_{d-1}(t;\omega)\right| \leq (1+\varepsilon)\{ct^{2d-1}\log\log(1/t)\}^{\frac{1}{2}}$ when $t < \delta$. And for all $x > 0$, when $t \to 0$, $t^x\{\log\log(1/t)\}^{\frac{1}{2}} \to 0$. Hence, when t is small, $\left|B_{d-1}(t;\omega)\right| \leq \sqrt{c}(1+\varepsilon)(t^{d-\frac{1}{2}-x})$. By picking a small x such that $\gamma = d - \frac{1}{2} - x > -\frac{1}{2}$, we have

$$\int_0^\tau B_{d-1}(t;\omega)dt \leq \int_0^\tau \left|B_{d-1}(t;\omega)\right|dt \leq \frac{\sqrt{c}(1+\varepsilon)}{1+\gamma}(\tau^{1+\gamma}).$$

Therefore,

$$\lim_{\tau\to 0} \frac{\left(\int_0^\tau B_{d-1}(t;\omega)dt\right)^2}{\tau} \leq \lim_{\tau\to 0} \frac{c(1+\varepsilon)^2}{(1+\gamma)^2}\tau^{1+2\gamma} = 0.$$

In other words,

$$A \subseteq \left\{\omega : \lim_{\tau\to 0} \frac{\left(\int_0^\tau B_{d-1}(t;\omega)dt\right)^2}{\tau} = 0\right\}.$$

Hence, $\lim_{\tau\to 0}\left(\int_0^\tau B_{d-1}(t)dt\right)^2/\tau = 0$ a.s. Also, $\lim_{\tau\to 1}(\int_\tau^1 B_{d-1}(t)dt)^2/(1-\tau) = 0$ a.s. This proves that $\Psi_d(0)$ and $\Psi_d(1)$ so defined are indeed the almost sure limits of $\Psi_d(\tau)$.

Proof of Theorem 3.3: By continuity of Ψ_d on $[0,1]$, it is easy to see that Theorem 3.1 carries over for $[\underline{\tau}, \overline{\tau}] = [0,1]$. To prove the second assertion, note that

$$\Psi_d(0) - \Psi_d(\tau) = -\kappa^2 \frac{[\tau\int_0^1 B_{d-1}(x)dx - \int_0^\tau B_{d-1}(x)dx]^2}{\tau(1-\tau)} < 0,$$

with probability one. Similarly, we have $\Psi_d(1) - \Psi_d(\tau) < 0$ with probability one.

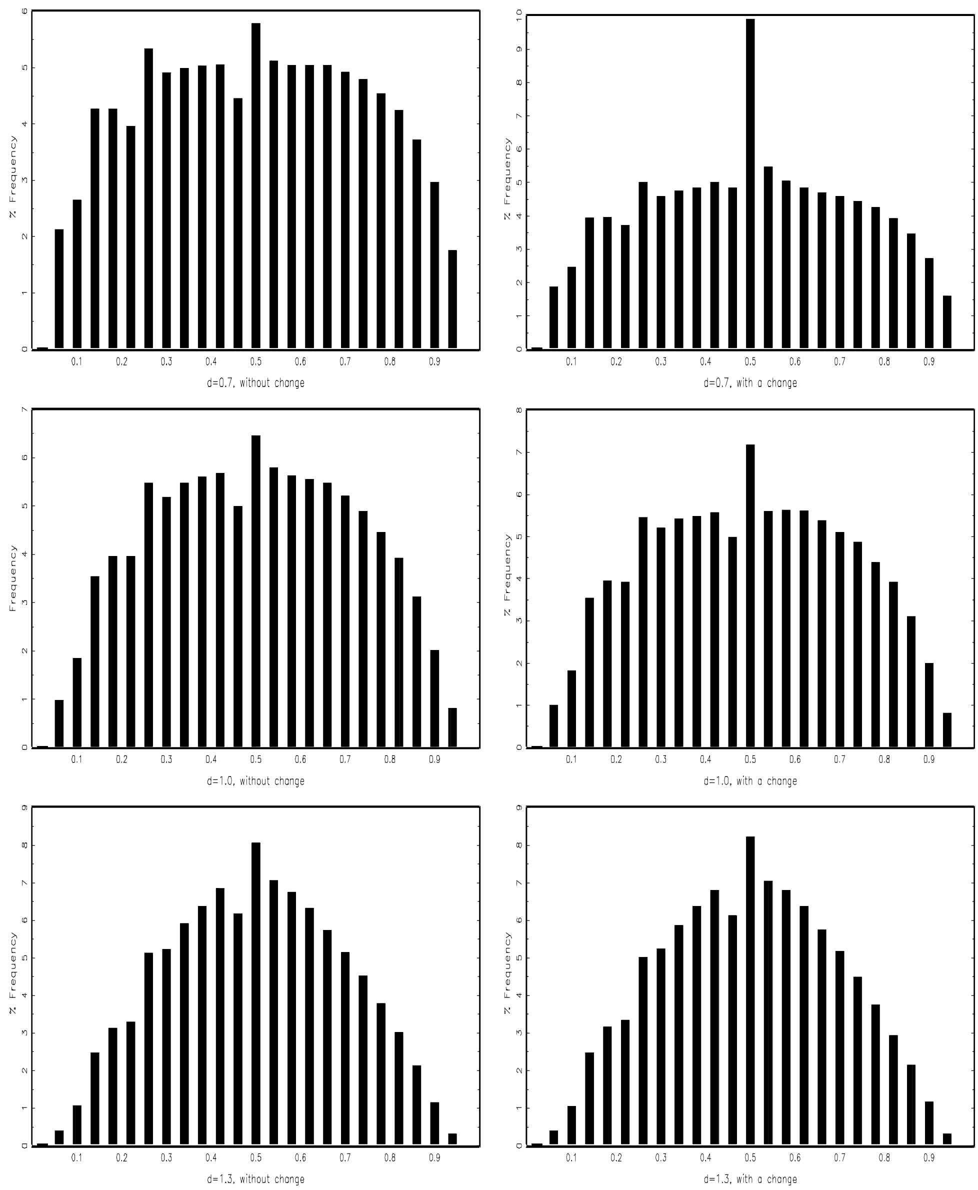

Figure 1: Empirical distributions of $\hat{\tau}$.

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