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« Var for Quadratic Portfolio's with Generalized Laplace Distributed Returns »

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VAR FOR QUADRATIC PORTFOLIO'S WITH GENERALIZED LAPLACE DISTRIBUTED RETURNS

R.BRUMMELHUIS AND J.SADEFO-KAMDEM

Abstract. This paper is concerned with the efficient analytical computation of Value-at-Risk (VaR) for portfolios of assets depending quadratically on a large number of joint risk factors that follows a multivariate Generalized Laplace Distribution. Our approach is designed to supplement the usual Monte-Carlo techniques, by providing an asymptotic formula for the quadratic portfolio's cumulative distribution function, together with explicit error-estimates. The application of these methods is demonstrated using some financial applications examples.

1. Introduction

This paper is concerned with the efficient numerical computation of static Value-at-Risk (VaR) for portfolios of assets depending quadratically on a large number of risk factors $\mathbb{X}_t = (X_{1,t}, \cdots, X_{n+1,t})$ (t representing time), under the assumption that $\mathbb{X}_t$ follows a Generalized Laplace Distribution or GLD. Our approach is designed to supplement the usual Monte-Carlo techniques, by providing an asymptotic formula for the quadratic portfolio's cumulative distribution function, together with explicit error-estimates. The basic philosophy is the same as in Brummelhuis, Cordoba, Quintanilla and Seco [1], where such an asymptotic formula was derived in the case of normally distributed risk factors. Here the result of [1] will be extended to a class of non-Gaussian $\mathbb{X}_t$'s, and even slightly improved upon in the normal case). More importantly, the asymptotic formula will be supplemented with estimates for the error-term, which were lacking in [1]. This will allow us to establish a rigorous interval in which the true quadratic VaR will lie, rather than just give an approximation which is asymptotically exact when the VaR confidence parameter tends to 1.

The typical way in which quadratic portfolios arise in practice are as a $\Gamma - \Delta$ approximations of more complicated portfolios with some non-linear value function $\Pi(X_{1,t}, \cdots, X_{n+1,t}, t)$. We will make the additional assumption that Π is *delta-hedged at* $t = 0$. The restriction to Δ-hedged portfolios is mainly made for computational simplicity, but note that these include the in practice very important class of hedging portfolios made up of derivatives and their underlying. In such a case, letting $S_{j,t}$ be the time-t price of the j-th underlying asset, we would

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typically take the log-return $X_{j,t} = \log(S_{j,t}/S_{j,0})$ as the j-th risk factor. The numerical example we consider at the end of the paper will be of this kind. A further assumption we will make is that $\mathbb{X}_t$ has zero mean, which in practice will be approximately satisfied on small time-scales t. We stress, however, that all results of this paper can be extended to general, non-hedged, quadratic portfolios with $\mathbb{X}_t$ having a non-zero mean. Indeed, in [1] this was already done on the level of the main asymptotic term for the portfolio's distribution function when $\mathbb{X}_t$ is normally distributed. However, such an extension not being entirely trivial (the more so if one wants to include error estimates as precise as those obtained in the present paper) we decided to postpone the more general case to a future paper, and first test our approach on the Δ-hedged case.

Why would one want to derive explicit analytic approximations to a portfolio's VaR when simple Monte Carlo will in principle compute this with any given precision? There are in fact a number of good reasons for wanting to do so. First of all, Monte Carlo, even when combined with various variance reduction and/or importance sampling techniques, can be notoriously slow for large portfolios. By contrast, explicit analytical expressions can in general be computed almost instantaneously, and would allow for real-time VaR evaluation[1]. Another drawback of Monte Carlo is that it the answers it provides lack transparency as regards their dependence on the various model parameters, whether these are statistical parameters underlying the portfolio model, or managerially determined ones, like portfolio loadings or choice of VaR-confidence level. Furthermore, the statistical parameters are typically obtained as point estimates, using e.g. (quasi-) maximum likelihood methods, and to obtain a more reliable and realistic picture, these point-estimates should be complemented by for example their 95% confidence intervals, reflecting the inherent uncertainty in any statistical estimation procedure. As a consequence, it becomes doubtful even whether a very precise Monte Carlo computation for a given set of parameters is meaningful, and a priori more useful and reliable than an approximate analytic answer. Moreover, to get a more realistic picture one should ideally speaking redo the VaR computation over the whole 95%-statistical confidence ranges of the parameters[2]. Doing this by Monte Carlo would involve massive computations, and therefore likely to be unfeasible in practice. On the other hand, explicit analytical expressions, even if approximate or providing bounds only, will easily permit such an analysis.

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[1] assuming of course the statistical procedure for estimating model parameters also allows for real-time updating, as for example in the case of the RiskMetric$^{\mathrm{TM}}$-methodology for estimating variances and covariances (see also [13] and [14]).

[2] possibly only over their end-points, if suitable monotonicity properties hold.

An alternative rigorous analytical approach to quadratic VaR was proposed by Cardenas et al. [2] and by Rouvinez [10]. They observed that, assuming Gaussian risk factors, the portfolio's characteristic function can be explicitly computed. Numerical Fourier-inversion will then yield the portfolio's distribution function and, consequently, its quantiles or VaR. This method was extended to jump-diffusions in Duffie and Pan [3]. Note that it is only semi-explicit, in that it still requires the numerically non-trivial step of Fourier inversion (although good algorithms are available for this). This would be a disadvantage for analyzing parameter-dependence. Moreover, explicit computation of the characteristic function is only possible when $\mathbb{X}_t$ is normally distributed[3], and the method does not generalize to the non-Gaussian risk-factors we are considering here. See also [15].

Two further papers dealing with non-Gaussian quadratic VaR are Jahel, Perraudin and Sellin [6], who assume $\mathbb{X}_t$ follows a stochastic volatility processes, and Glasserman, Heidelberger and Shahabuddin [5], who consider Student-$\mathfrak{t}$ distributed $\mathbb{X}_t$. Both papers are characteristic function based, [5] exploiting the relation of $\mathfrak{t}$-distributions with quotients of independent Gaussians, and [6] employing the characteristic function of the process to compute the moments of the portfolio distribution function, and subsequently fitting a parametric distribution from the Pearson or Johnson family to these moments (this last step introduces an uncontrolled approximation).

To begin describing our main results, consider a portfolio with non-linear *Profit and Loss (or P & L) function*[4] $\Pi = \Pi(x_1, \cdots, x_{n+1}, t)$ over the time-interval $[0, t]$. In particular, $\Pi(0, 0) = 0$, assuming (without loss of generality) that $\mathbb{X}_0 = 0$. We suppose moreover that the portfolio is delta-hedged at time 0, implying that its gradient in 0 vanishes: $\nabla\Pi(0, 0) = 0$. Let

$$\Theta := \frac{\partial \Pi}{\partial t}(0), \tag{1}$$

the rate of change of the portfolio's time value, and

$$\Gamma = (\Gamma_{ij})_{1\leq i,j\leq n+1} := \left(\frac{\partial^2 \Pi}{\partial x_i \partial x_j}(0)\right)_{1\leq i,j\leq n+1}, \tag{2}$$

the portfolio's Gamma. Suppose that we dispose of some probabilistic model for $\mathbb{X}_t$, where $t > 0$ is some small fixed later time (typically of the order of 1 day, or 1/252 in the natural unit of one financial year). To compute VaR, and related risk-measures like Expected Shortfall, we need to know the P&L's cumulative distribution function:

$$F_{\Pi_t}(x) = \mathbb{P}\left(\Pi(\mathbb{X}_t, t) < x \right), \tag{3}$$

[3]to include jumps, [3] first condition on the number of jumps

[4]we use the P & L rather than the value function; this is of course just a question of normalization

$\mathbb{P}$ standing for the objective probability. Since the distribution function (3) is in general impossible to evaluate analytically, and, for big n and complicate $\Pi(x_1, \cdots, x_{n+1}, t)$, time-consuming to compute numerically by Monte Carlo, one usually performs a preliminary quadratic approximation:

$$\begin{aligned} \Pi(\mathbb{X}) &\simeq \Theta\, t + \frac{1}{2}\mathbb{X}_t \Gamma \mathbb{X}_t^t \\ &= \Theta\, t + \frac{1}{2}\sum_{j,k} \Gamma_{ij} X_i X_j, \end{aligned} \tag{4}$$

where there is no linear term since Π is assumed to be Δ-hedged. Here, and below, we will use the following notational conventions for vectors and matrices: $x = (x_1, \cdots, x_{n+1})$ and $\mathbb{X} = (X_1, \cdots, X_{n+1})$ will designate row vectors, and their transposes x^t, $\mathbb{X}^t$ will therefore be column vectors, on which matrices like $\Gamma = (\Gamma_{ij})_{i,j}$ act by left multiplication

As of now we assume that $\mathbb{X}_t$ has a *centered multi-variate Generalized Laplace Distribution* or GLD, with parameter α. That is, $\mathbb{X}_t$ has probability density of the form:

$$f_{\mathbb{X}_t}(x) = \frac{C_{\alpha,n+1}}{\sqrt{\det\ (\mathbb{V}(t))}} \exp\left(-c_{\alpha,n+1}(x\mathbb{V}(t)x^t)^{\alpha/2}\right), \tag{5}$$

where $\alpha > 0$ and where $\mathbb{V}(t)$ is a positive definite matrix; $\mathbb{V}(t)$ will precisely be $\mathbb{X}_t$'s variance-covariance matrix, provided we choose the normalization constants $C_{\alpha,n+1}$ and $c_{\alpha,n+1}$ as

$$c_{\alpha,n+1} = \left(\frac{\Gamma\left(\frac{n+3}{\alpha}\right)}{(n+1)\Gamma\left(\frac{n+1}{\alpha}\right)}\right)^{\alpha/2}, \tag{6}$$

and

$$C_{\alpha,n+1} = \frac{\alpha}{2\pi^{(n+1)/2}}\left(\frac{\Gamma\left(\frac{n+3}{\alpha}\right)}{(n+1)\Gamma\left(\frac{n+1}{\alpha}\right)}\right)^{(n+1)/2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{\alpha}\right)}, \tag{7}$$

cf. Appendix A. Multi-variate GLD distributions with $\alpha < 2$ should be seen as an alternative to multi-variate $\mathfrak{t}$-distributions, possessing like these heavier-than-Gaussian tails, and allowing a more realistic fit to empirical asset returns around the center; they are called Generalized Exponential Distributions in [8].

The fact of only including a single time-derivative in (4) needs an explanation: if we make the, for small times t, reasonable assumption that $\mathbb{V}(t)$ grows linearly with time[5], then (4) consists of all terms of order less than or equal 1 in t (remember that there are no terms of order 1 in $\mathbb{X}$, since Π is Δ-edged at 0).

[5] an assumption which is for example satisfied if we estimate $V(t)$ using RiskMetric's EWMA-method on smaller time-intervals t/N

The Value-at-Risk at (risk-managerial) confidence level $1-p$ is defined by

$$\mathrm{VaR}_p^{\Pi_t} = \sup\{V : F_{\Pi_t}(-V) \geq p\}; \tag{8}$$

Note that, because of the minus-sign, the Value at Risk will be recorded as a positive number when it corresponds to a loss. If the distribution function of $\Pi_t = \Pi(\mathbb{X}_t, t)$ is continuous, we can replace the inequality sign on the right by an equality, and if it is moreover strictly increasing, then we simply have that $\mathrm{VaR}_p^{\Pi} = -F_{\Pi_t}^{-1}(p)$. We will assume that a reasonable approximation to VaR_p will be given by the quadratic or Γ-Value-at-Risk, $\mathrm{VaR}_p^{\Gamma_t}$, defined as in (8), but with F_{Π_t} replaced by

$$F_{\Gamma_t}(-V) = \mathbb{P}(\ \Theta t + \frac{1}{2}\mathbb{X}_t\ \Gamma\ \mathbb{X}_t^t \leq -V\ ). \tag{9}$$

In our case, the distribution function F_Γ will be strictly increasing, so the definition of Γ-VaR simplifies to $F_{\Gamma_t}^{-1}(p)$. The approximation $\mathrm{VaR}_p^{\Pi_t} \simeq \mathrm{VaR}_p^{\Gamma_t}$ can be justified by a general result, stated and proved in Appendix B, that under reasonable assumptions on the portfolio $\Pi(x,t)$,

$$\mathrm{VaR}_p^{\Pi_t}/\mathrm{VaR}_p^{\Gamma_t} \to 1, \quad t \to 0,$$

with an error which is $O(\sqrt{t})$. Also observe that if for example $\Pi(x,t) \geq \Theta_t + \frac{1}{2}x\Gamma x^t$ for all x, then of course $\mathrm{VaR}_p^{\Pi_t} \leq \mathrm{VaR}_p^{\Gamma_t}$, and similarly with all inequality signs reversed. From now on, we will take t sufficiently small but fixed, and make no distinction any more between $\mathrm{VaR}_p^{\Pi_t}$ and $\mathrm{VaR}_p^{\Gamma_t}$, that is, we will effectively suppose that $\Pi(x,t)$ is a quadratic Δ-hedged portfolio. We will also systematically drop all suffixes t, to simplify notations, and simply write $\mathbb{X}$ for $\mathbb{X}_t$, F_Γ and VaR_p^Γ for F_{Γ_t} respectively $\mathrm{VaR}_p^{\Gamma_t}$, etc. We will also simply write Θ for Θt.

Our main task will then be to compute $F_\Gamma(-V)$, or more precisely its inverse. This is still a non-trivial problem if we are looking for an analytic solution (which we are, for though Monte Carlo works faster for quadratic portfolios, it will still be slow if the portfolio is big). Our strategy will be to approximate $F_\Gamma(-V)$ for large values of V by an explicit analytic expression, with explicit error bounds. This will then allow an approximate inversion.

To state our main result, we need to introduce a certain amount of notation. Write the variance-covariance $\mathbb{V}$ as

$$\mathbb{V} = \mathbb{H}\,\mathbb{H}^t,$$

where we can for example take $\mathbb{H}$ upper- or lower-triangular, in which case this is the Cholesky decomposition (another possibility would of course be to take the spectral square-root, $\mathbb{V}^{1/2}$, and one chooses whichever can be computed fastest). Next introduce the *sensitivity-adjusted variance-covariance matrix*, $\mathbb{H}\Gamma\mathbb{H}^t$, which we diagonalize:

$$\mathbb{H}\Gamma\mathbb{H}^t = \mathbb{O}\mathbb{A}\mathbb{O}^t, \tag{10}$$

with $\mathbb{O}$ orthogonal, and $\mathbb{A}$ diagonal. In the present situation of a Δ-hedged portfolio and mean-0 risk-factors it is not necessary to know anything about $\mathbb{O}$, whose columns are precisely the eigenvectors of $\mathbb{H}\Gamma\mathbb{H}^t$; this changes, however, when one of these two conditions is not met: cf. [1] for the Gaussian case. It is also important to observe that $\mathbb{H}\Gamma\mathbb{H}^t$ is not necessarily definite, except if Γ is. We can write $\mathbb{A}$ as

$$\mathbb{A} = \begin{pmatrix} -D^-_{n_-} & 0 \\ 0 & D^+_{n_+} \end{pmatrix},$$

where

$$D^\epsilon_{n_\epsilon} = \begin{pmatrix} a^\epsilon_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & a^\epsilon_{n_\epsilon} \end{pmatrix}, \quad \epsilon = \pm 1,$$

with a^+_j, $a^-_j \geq 0$ for all j. We will from now on suppose that $\mathbb{H}\Gamma\mathbb{H}$ is non-singular, with strictly negative lowest eigenvalue of multiplicity 1:

$$-a^-_1 < -a^-_2 \leq \ldots - a^-_{n_-} < 0 < a^+_1 \leq \cdots \leq a^+_{n_+}. \tag{11}$$

Using these data we define a constant $A_{\rm pc}$ by

$$A_{\rm pc} := 2\alpha^{-\frac{n}{2}-1}(2\pi)^{n/2} c_{\alpha,n+1}^{-\frac{n+1}{\alpha}} C_{\alpha,n+1} \frac{(a^-_1)^{\frac{n}{2}}}{\sqrt{\prod_2^{n_-}(a^-_1 - a^-_j)\ \prod_1^{n_+}(a^-_1 + a^+_j)}}. \tag{12}$$

Definition 1.1. The *principal component approximation* $F_{\Gamma,\rm pc}(-V)$ to $F_\Gamma(-V)$ is defined to be:

$$F_{\Gamma,\rm pc}(R) = A_{\rm pc}\, \Gamma\left(\frac{n+1}{\alpha} - \frac{n}{2}, \left(\frac{R}{\sqrt{a^-_1}}\right)^\alpha\right), \tag{13}$$

where V and R are related by $R^2 = 2c_{n+1,\alpha}^{2/\alpha}(V + \Theta)$.

As we will presently see,

$$F_\Gamma(-V) \simeq F_{\Gamma,\rm pc}(-V), \quad V = \frac{1}{2} c_{n,\alpha}^{-2/\alpha} R^2 - \Theta \to \infty.$$

A major pre-occupation of this paper will be to obtain as precise an estimate as possible for the error. To this effect we next introduce

$$\lambda_{\min}(Q) := \frac{\alpha}{(a^-_1)^{\frac{\alpha}{2}}} \min\left(\frac{a^-_1}{a^-_2} - 1, \frac{a^-_1}{a^+_{n_+}} + 1\right); \tag{14}$$

as the notation already indicates, $\lambda_{\min}(Q)$ is the smallest eigenvalue of a certain auxiliary matrix which will be introduced in the proof of the main theorem below.. Furthermore, given a γ such that $0 < \gamma < 1$, we let

$$R_\gamma^2 := \min\left(\frac{1}{(a_1^-)^{\frac{\alpha}{2}}}\frac{4\alpha(1-\gamma)}{|2-\alpha|}, \frac{\lambda_{\min}(Q)}{2}\right). \tag{15}$$

Using these we now introduce three further constants $K_1^\pm, K_2^\pm$ and K^0, which will turn up in the estimate for the error term[6]. These will involve a further choice of a parameters ε and λ_0, and of a C^1 cut-off function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $0 \leq g \leq 1$, supp $g \subseteq [0, 1]$ and $g(s) = 1$ on a neighborhood of 0. Let

$$K_1^\pm := n(a_1^-)^{-\frac{\alpha}{2}} \cdot \left(\frac{\sqrt{2}}{\lambda_{\min}(Q)} \cdot \left(1 + \frac{||g'||_\infty}{R_\gamma^2}\right) + \frac{||g'||_\infty}{R_\gamma^2}\right). \tag{16}$$

and

$$K_2^\pm := \sqrt{2}\, n(n+2)(a_1^-)^{-\frac{\alpha}{2}}\left(\frac{2-\alpha}{8\alpha}\right)\gamma^{-\frac{n}{2}-2}. \tag{17}$$

For any explicit computations we will take for g a member of the family of functions g_a $(0 < a < 1)$, defined by

$$g_a(x) = \begin{cases} 1 & \text{si } x \leq a \\ 1 - \frac{1}{2}\left(\frac{2}{1-a}\right)^2 (x-a)^2 & \text{si } a \leq x \leq \frac{a+1}{2} \\ \frac{1}{2}\left(\frac{2}{1-a}\right)^2 (x-1)^2 & \text{si } \frac{a+1}{2} \leq x \leq 1 \\ 1 & \text{si } x \geq 1 \end{cases}; \tag{18}$$

a is left as a further free parameter. Note that $g_a = 1$ on $[0, a]$, and that $\|g_a'\|_\infty = \frac{2}{1-a}$, which is the only information we really need.

To define our third and final constant, $K^0 = K^0(\varepsilon, \lambda_0)$, let $0 < \varepsilon < 1$ and $\lambda_0 > 0$, and introduce

$$n_\varepsilon := (1-\varepsilon)\left((a_1^-)^{-1} + \alpha^{-1}a(a_1^-)^{\frac{\alpha}{2}-1}R_\gamma^2\right)^{\frac{\alpha}{2}} + \varepsilon(a_1^-)^{\alpha/2}. \tag{19}$$

Then:

$$\begin{aligned} K^0 \ &:= \ \alpha^{-1}\pi^{\frac{n+1}{2}}\left(c_{\alpha,n+1}n_\varepsilon\right)^{-\frac{n+1}{\alpha}} C_{\alpha,n+1}\,\frac{e^{\varepsilon\lambda_0(a_1^-)^{-\alpha/2}}}{(\varepsilon\lambda_0)^{n/\alpha}} \\ &\quad \cdot\left\{2\,||\,|\mathbb{A}|^{-1}||^{1/2}\,\frac{\Gamma\left(\frac{n+\alpha}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)} + \frac{2|n_- - n_+| + 10}{\alpha(\varepsilon\lambda_0)^{1/\alpha}}\,\frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}. \end{aligned} \tag{20}$$

The matrix norm is of course simply $||\,|\mathbb{A}|^{-1}|| = \max\left(1/a_{n_-}^-, 1/a_1^+\right)$.

The origin of all these constants will become clear from the proof. For practical purposes, what is important is that, although complicated in appearance, they can straightforwardly be computed from $\mathbb{A}$, for any choice of $\gamma, \varepsilon, \lambda_0$ and a (limiting ourselves to g's given by (18)).

[6]The choice of the sub- and superscripts was made to facilitate keeping track of the constants in the various proofs below, as will become clear in sections 2, 4 and 5.

We now state the main result of this paper. Recall the definition of the incomplete Γ-function:

$$\Gamma(z,w)=\int_w^\infty e^{-s}s^{z-1}ds, \tag{21}$$

Then:

Theorem 1.2. *Suppose that* $\alpha\leq 2$. *Given* $V>-\Theta$, *let*

$$R^2:=2c_{n+1,\alpha}^{2/\alpha}\,(V+\Theta) \tag{22}$$

Then

$$F_{\Gamma,\mathrm{pc}}(R)-\mathcal{E}_L(R)\leq F_\Gamma(-V)\leq F_{\Gamma,\mathrm{pc}}(R)+\mathcal{E}_U(R), \tag{23}$$

with

$$\begin{aligned}\mathcal{E}_L(R) &= A_{\mathrm{pc}}\cdot\Bigg\{K_1^{\pm}\Gamma\left(\frac{n+1}{\alpha}-\frac{n}{2}-1,\left(\frac{R}{\sqrt{a_1^-}}\right)^\alpha\right)\\ &\quad+(a_1^-)^{\frac{\alpha n}{4}}\frac{\Gamma\left(\frac{n}{2},R\right)}{\Gamma\left(\frac{n}{2}\right)}\,\Gamma\left(\frac{n+1}{\alpha},\left(\frac{R}{\sqrt{a_1^-}}\right)^\alpha\right)\Bigg\}\ ,\end{aligned} \tag{24}$$

for all $R>0$. *Moreover, if* $R^\alpha\geq\lambda_0$, *we can take*

$$\begin{aligned}\mathcal{E}_U(R) &= A_{\mathrm{pc}}\cdot\Bigg\{K_1^{\pm}\Gamma\left(\frac{n+1}{\alpha}-\frac{n}{2}-1,\left(\frac{R}{\sqrt{a_1^-}}\right)^\alpha\right)\\ &\quad+K_2^{\pm}\Gamma\left(\frac{n+1}{\alpha}-\frac{n}{2}-2,\left(\frac{R}{\sqrt{a_1^-}}\right)^\alpha\right)\Bigg\}\\ &\quad+K^0\Gamma\left(\frac{n+1}{\alpha},n_\varepsilon R^\alpha\right).\end{aligned} \tag{25}$$

Remark 1.3. Although this is perhaps not clear at first sight, (23) is a one-term asymptotic expansion with remainder, in the sense that the main term will dominate the error terms for sufficiently large R. For since $\Gamma(z,w)=w^{z-1}e^{-w}+O(w^{z-2}e^{-w})$ as $w\to\infty$, it follows that

$$\frac{\Gamma(z-k,w)}{\Gamma(z,w)}\simeq w^{-k}\to 0,\quad w\to\infty,$$

which shows that the terms involving $K_1^{\pm}$ and $K_2^{\pm}$ have a relative decay, with respect to the principal term, of $(R/\sqrt{a_1^-})^{-1}$ and $(R/\sqrt{a_1^-})^{-2}$, respectively. The second term on the right hand side of (24) has a relative exponential decay, due to the $\Gamma(n/2,R)$ in front. The same is true for the final term of (25), since for any $k,\eta>0$,

$$\frac{\Gamma\left(z,(1+\eta)w\right)}{\Gamma(z-k,w)}\simeq w^k e^{-\eta w},\quad w\to\infty.$$

It then suffices to apply this with $z=(n+1)/\alpha$, $k=n/2$ and $\eta=n_\varepsilon-(a_1^-)^{-\alpha/2}>0$.

Remark 1.4. If we expand the incomplete Γ-function of the main term in (23), we find that, asymptotically as $R = \sqrt{2}c_{\alpha,n+1}^{1/\alpha}\sqrt{(V+\Theta)} \to \infty$,

$$F_\Gamma(-V) \simeq A_{\rm pc}\Gamma\left(\frac{n+1}{\alpha} - \frac{n}{2}, \left(\frac{R}{\sqrt{a_1^-}}\right)^\alpha\right)$$
$$\simeq A_{\rm pc}\left(\frac{R}{\sqrt{a_1^-}}\right)^{n+1-\frac{n\alpha}{2}-\alpha} e^{-(R/\sqrt{a_1^-})^\alpha}.$$

If $\alpha = 2$, then

$$A_{{\rm pc},\alpha=2} = \frac{1}{\sqrt{\pi}}\frac{(a_1^-)^{n/2}}{\sqrt{\Delta(\mathbb{A})}},$$

where we have put

$$\Delta(\mathbb{A}) := \prod_2^{n_-}(a_1^- - a_j^-)\prod_1^{n_+}(a_1^- + a_1^+),$$

and therefore, in the case of normally distributed risk factors,

$$F_\Gamma(-V) \simeq \frac{1}{\sqrt{\pi}}\frac{(a_1^-)^n}{\sqrt{\Delta(\mathbb{A})}}\,\Gamma\left(\frac{1}{2}, \frac{R^2}{a_1^-}\right)$$
$$\simeq \frac{1}{\sqrt{\pi}}\frac{(a_1^-)^{(n+1)/2}}{\sqrt{\Delta(\mathbb{A})}}\,\frac{e^{-R^2/a_1^-}}{R}, \quad R = \sqrt{V+\Theta}.$$

This is essentially theorem 4.2 of [1] (with n replaced by $n+1$), except for two errors in the statement of that theorem, which we take the opportunity to correct here: the numerical factor in the constant C_0 of that theorem should have been $\pi^{-1/2}$ instead of $2(2\pi)^{(n-1)}/2$, and the exponent should have read $\exp(-R^2/a_1^-)$ instead of $\exp(-R^2/2a_1^-)$.

Keeping the incomplete Γ-functions, instead of expanding them using their own asymptotic expansions, a priori leads to a more accurate approximation, even when $\alpha = 2$.

Theorem 1.2 can be used as follows to solve our initial problem of finding good approximations and bounds for VaR_p^Γ. Let us define the *principal component* Γ-*VaR* of our quadratic portfolio as the unique solution $V = \mathrm{VaR}_p^{\Gamma,\rm pc}$ of the equation

$$F_{\Gamma,\rm pc}\left(c_{\alpha,n+1}^{1/\alpha}\sqrt{2(V+\Theta)}\right) = p. \tag{26}$$

Theorem 1.2 then suggests, as a first approximation,

$$\mathrm{VaR}_p^\Gamma \simeq \mathrm{VaR}_p^{\Gamma,\rm pc},$$

a relation which is asymptotically exact as $p \to 0$. For a given small but non-zero $p > 0$ this is, as it stands, just an uncontrolled approximation, but we can use the error bounds of theorem 1.2 to determine a rigorous interval in which VaR_p^Γ must lie. For a given $p \in (0,1)$, let $R_L = R_L(p)$ and $R_U = R_U(p)$ solve, respectively:

$$F_{\Gamma,\mathrm{pc}}(R_L) - \mathcal{E}_L(R_L) = p, \tag{27}$$

and

$$F_{\Gamma,\mathrm{pc}}(R_L) + \mathcal{E}_U(R_U) = p. \tag{28}$$

Put

$$V_j(p) := \frac{1}{2} c_{\alpha,n+1}^{-2/\alpha} R_j(p)^2 - \Theta, \quad j = L, U. \tag{29}$$

Since the lower bound (24) holds for all $R > 0$, we will always have that $V_L(p) \leq \mathrm{VaR}_p^\Gamma$. On the other hand, $\mathrm{VaR}_p^\Gamma \leq V_U(p)$ will only hold once we know that $c_{n+1,\alpha} 2^{\alpha/2} \left(\mathrm{VaR}_p^\Gamma + \Theta\right)^{\alpha/2} \geq \lambda_0$. This will certainly be satisfied if we choose:

$$\lambda_0 = R_L(p)^\alpha \tag{30}$$

Summarizing, we then have the following estimate on quadratic VaR:

Corollary 1.5. *For a given choice of parameters $p, a, \gamma \in (0,1)$ let $\lambda_0 = R_L(p)^\alpha$, where $R_L(p)$ is the solution of (27). Furthermore, for given $\varepsilon \in (0,1)$, let $R_U(p)$ be the solution of (28)*[7]*. Let $V_L(p), V_U(p)$ be defined by (29). Then:*

$$\mathrm{VaR}_p^\Gamma \in [V_L(p), V_U(p)].$$

Remark 1.6. Once we have fixed λ_0 by (30), we can look for a $\varepsilon \in (0,1)$ which minimizes $K^0(\varepsilon, \lambda_0)$. This can be done numerically. An alternative approximate analytic procedure, which works when $R_L(p)^\alpha > n(a_1^-)^{\alpha/2}/\alpha$, would be to choose

$$\varepsilon\lambda_0 = (a_1^-)^{\alpha/2} \frac{n}{\alpha}, \tag{31}$$

which minimizes part of K^0: cf. remark 4.4 below. This is allowed as long as $1 > \varepsilon = n(a_1^-)^{\alpha/2}/\alpha\lambda_0 = n(a_1^-)^{\alpha/2}/\alpha R_L(p)^\alpha$, whence the condition above. With this choice of $\varepsilon\lambda_0$, K^0 then becomes, very explicitly,

$$K^0 = \alpha^{-1} \pi^{\frac{n+1}{2}} \left(c_{\alpha,n+1} n_\varepsilon\right)^{-\frac{n+1}{\alpha}} C_{\alpha,n+1} \left(a_1^-\right)^{-\frac{n}{2}} \left(\frac{\alpha}{n}\right)^{\frac{n}{\alpha}} e^{\frac{n}{\alpha}}$$
$$\cdot \left\{ 2||\,|\mathbb{A}|^{-1}\,|| \frac{\Gamma\left(\frac{n+\alpha}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)} + \frac{2|n_- - n_+| + 10}{\alpha} \left(a_1^-\right)^{-1/2} \left(\frac{\alpha}{n}\right)^{1/\alpha} \frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right\},$$

an expression which, due to the $n^{n/\alpha}$ in the denominator, will tend to 0 as the portfolio dimension n tends to infinity. This suggests that for large portfolios we can sometimes simply leave out the term involving K^0 from $\mathcal{E}_U(R)$.

[7] that is, with this choice of parameters in the expressions for $\mathcal{E}_L(R), \mathcal{E}_U(R)$

There is further scope for minimization of the error terms over the other two parameters, γ and a, both restricted to $(0, 1)$. In the numerical example treated in section 6, we have simply made an ad-hoc choices for these.

2. **Probability distribution of a quadratic portfolio**

If $\mathbb{X}$ as a multi-variate GLD distribution (5), then (9) is given by

$$C \int_{\{\Theta + 1\frac{1}{2}x\Gamma x^t \leq -V\}} e^{-c(x\mathbb{V}^{-1}x^t)^{\alpha/2}} \frac{dx}{\sqrt{\det(\mathbb{V})}} \tag{32}$$

where $C = C_{\alpha,n+1}$ and $c = c_{\alpha,n+1}$ are the two normalization constants (6) and (7). We decompose $\mathbb{V}$ as $\mathbb{V} = \mathbb{H}\ \mathbb{H}^t$, and let $\mathbb{H}\Gamma\mathbb{H}^t = \mathbb{O}\mathbb{A}\mathbb{O}^t$ with $\mathbb{O}$ orthogonal, and $\mathbb{A}$ diagonal, cf. (10). After some elementary changes of variables, (32) becomes

$$C \int_{\{\frac{1}{2}(|x_+|^2 - |x_-|^2) \leq -(V+\Theta)\}} e^{-c(x|\mathbb{A}|^{-1}x^t)^{\alpha/2}} \frac{dx}{\sqrt{\det(\mathbb{A})}} \tag{33}$$

(note that $\det \mathbb{A} = \det\,(\mathbb{H}\Gamma\mathbb{H}^t) = \det\,(\Gamma)\,\det\,(\mathbb{V})$). Here $x = (x_+, x_-)$ is the decomposition of $\mathbb{R}^{n+1}$ into the positive, respectively negative subspace of $\mathbb{H}\Gamma\mathbb{H}^t$ using its eigenbasis. After a further change of variables $x \to c^{-1/\alpha}x$ in (33) we arrive at the following expression for F_Γ,which will be the starting point of our analysis:

$$F_\Gamma(-V) = C' \int_{\{|x_-|^2 - |x_+|^2 \geq R^2\}} e^{-(x|\mathbb{A}|^{-1}x^t)^{\alpha/2}} dx, \tag{34}$$

with

$$C' = c^{-\frac{n+1}{\alpha}} \cdot \frac{C}{\sqrt{\det(\mathbb{A})}} \tag{35}$$

and

$$R^2 := 2c^{2/\alpha}(V + \Theta), \tag{36}$$

and where will assume from now on that $V + \Theta \geq 0$.

The next step will be to rewrite (34) as an integral of surface integrals over the level sets of the function $\eta(x)$, defined on $\{x : |x_+| < |x_-|\}$ by

$$\eta(x) = \sqrt{|x_-|^2 - |x_+|^2}.$$

Observe that the region of integration of (34) is included in the domain of η. Recall that a *Liouville form* of η is, by definition, any n-form L_η satisfying

$$d\eta \wedge L_\eta = dx_1 \wedge \ldots \wedge dx_{n+1}.$$

Although a Liouville form is not unique, its restriction to any level set $\{x : \eta(x) = r\}$ of η is[8]. A classical choice for L_η is:

$$L_\eta = \frac{1}{|\nabla\eta|^2} \sum_{j=1}^{n+1} (-1)^{j-1} \frac{\partial\eta}{\partial x_j} dx_1 \wedge \ldots \wedge [j] \wedge \ldots \wedge dx_{n+1}, \tag{37}$$

$|\nabla\eta|$ being the euclidian norm of the gradient of η, and the symbol [j] meaning that the term dx_j is deleted. Another possible choice, valid there where $\partial\eta/\partial x_1 \neq 0$, is

$$L_\eta = \left(\frac{\partial\eta}{\partial x_1}\right)^{-1} dx_2 \wedge \cdots \wedge dx_{n+1}. \tag{38}$$

Both formulas will be used in this paper. As mentioned, although different as forms on $\mathbb{R}^{n+1}$, the restrictions of (37) and (38) on any level-set of η coincide.

We now have, for any integrable function $g = g(x)$, that

$$\int_{\{\eta(x)\geq R\}} g(x)\, dx = \int_R^\infty \left(\int_{\{\eta=r\}} g(x)\, L_\eta(x) \right) dr.$$

Applying this to (34), and using that L_η is homogeneous of order n with respect to multiplication by r (that is, $\phi_r^*(L_\eta) = r^n L_\eta$, where $\phi_r(x) = r \cdot x$ and where the * indicates pull-back: this follows from η being homogeneous of degree 1), we see that the integral (34) can be written as

$$F_\Gamma(-V) = C' \int_R^\infty r^n \Big(\int_{\{\eta(x)=1\}} e^{-r^\alpha (x|\mathbb{A}|^{-1}x^t)^{\alpha/2}} L_\eta(x)) \Big) dr. \tag{39}$$

Letting

$$\Sigma := \{x : \eta(x) = 1\}, \tag{40}$$

our strategy will be to first derive an asymptotic formula with explicit error estimate for

$$I(\lambda) := \int_\Sigma e^{-\lambda (x|\mathbb{A}|^{-1}x^t)^{\alpha/2}} L_\eta(x), \tag{41}$$

as $\lambda \to \infty$. From this an asymptotic formula for (39) will follow, simply by taking $\lambda = r^\alpha$, and integrating from R to ∞.

Recall our hypothesis (11) on the eigenvalues of $\mathbb{A}$, and in particular our assumption that $-a_1^-$, the lowest eigenvalue of $\mathbb{A}$, is of multiplicity 1. By classical theory, the main contribution to the integral (41) as $\lambda \to \infty$ will come from those points on the surface Σ where the function $x|\mathbb{A}|^{-1}x^t$ has an absolute minimum. Stationary points of a function on $\Sigma = \{\eta = 1\}$ are simply points of Σ where the gradient of the function is proportional to the gradient of $\eta(x)$, and one easily verifies

[8] the restriction of a Liouville form should be carefully distinguished from the induced (Euclidian) surface measure on the level set, which is obtained by dividing L_η by the length of the gradient of η

that $x|\mathbb{A}|^{-1}x^t$ attains its absolute minimum on Σ in the two points $(\pm e_1^-, 0) \in \mathbb{R}^{n_-} \times \mathbb{R}^{n_+}$, where $e_1^- := (1, 0, \cdots, 0) \in \mathbb{R}^{n_-}$. We next write (41) as a sum three integrals, using a C^2 partition of unity $\chi_+ + \chi_0 + \chi_- = 1$ on Σ, where $0 \leq \chi_\pm, \chi_0 \leq 1$, and where $\chi_\pm = 1$ near $(\pm e_1, 0)$ (implying that $(\pm e_1^-, 0) \notin \mathrm{supp}(\chi_0)$):

$$I(\lambda) = I_-(\lambda) + I_0(\lambda) + I_+(\lambda) \tag{42}$$

with

$$I_\nu(\lambda) = \int_\Sigma \chi_\nu(x) e^{-\lambda (x|\mathbb{A}|^{-1}x^t)^{\alpha/2}} L_\eta(x), \quad \nu = \pm, 0. \tag{43}$$

The supports of the χ_ν will be chosen in a special way related to the local geometry of the phase function near the two critical points. The main step in our analysis will be to determine the contribution of the two absolute minima in $(\pm e_1^-, 0)$. By symmetry, it suffices to concentrate on one of these, say $(e_1^-, 0)$ (provided we of course also choose $\chi_\pm$ symmetrical). Using $x' = (x'_-, x_+) := (x_{2,-}, \cdots, x_{n_-,-}, x_{1,+}, \cdots, x_{n_+,+})$ as local coordinates on Σ near $(e_1^-, 0)$, with

$$x_{1,-} = \sqrt{1 - x_{2,-}^2 - \ldots - x_{n_-,-}^2 + x_{1,+}^2 + \ldots + x_{n_+,+}^2},$$

and observing that in these coordinates L_η restricted to Σ is given by

$$\begin{aligned} L_\eta(x) &= \left(\frac{\partial \eta}{\partial x_{1,-}}\right)^{-1} dx_{2,-} \wedge \ldots \wedge dx_{n_-,-} \wedge dx_{1,+} \wedge \ldots \wedge dx_{n_+,+} \\ &= x_{1,-}^{-1} \quad dx', \end{aligned}$$

(where we used that $\eta = 1$ on Σ) we see that

$$I_+(\lambda) = \int_{\mathbb{R}^n} \widetilde{\chi}_+(x') \; e^{-\lambda(c_1 + q(x'))^{\alpha/2}} \; \left(1 - |x'_-|^2 + |x_+|^2\right)^{-1/2} \; dx', \tag{44}$$

where we put

$$c_1 := \frac{1}{a_1^-}, \tag{45}$$

and

$$\begin{aligned} q(x') \; = \; & \left(\frac{1}{a_2^-} - \frac{1}{a_1^-}\right) x_{2,-}^2 + \ldots + \left(\frac{1}{a_{n_-}^-} - \frac{1}{a_1^-}\right) x_{n_-,-}^2 + \\ & \left(\frac{1}{a_1^+} + \frac{1}{a_1^-}\right) x_{1,+}^2 + \ldots + \left(\frac{1}{a_{n_+}^+} + \frac{1}{a_1^-}\right) x_{n_+,+}^2 \; , \end{aligned} \tag{46}$$

and with $\widetilde{\chi}_+(x') := \chi_+(\sqrt{1 - |x'_-|^2 + |x_+|^2}, x'_-, x_+)$. In the next section we will make a careful study of the asymptotic behavior, for big λ, of integrals like (44).

3. Sharp estimates for Laplace integrals

In this section we derive precise estimates for a general n-dimensional Laplace-type integral:

$$J(\lambda) = \int_{\mathbb{R}^n} a(x) e^{-\lambda\psi(x)} dx, \tag{47}$$

with C^2 amplitude a and C^4 phase function ψ satisfying the following hypotheses:

- *(i)* $\psi(x) \geq 0$, and ψ has a unique minimum in on supp(a) in $x = 0$, with $\psi(0) = 0$.
- *(ii)* The hessian $Q = \left(\frac{\partial^2\psi}{\partial x_i \partial x_j}(0)\right)_{i,j=1,\dots,n}$ is non-degenerate (and therefore strictly positive).
- *(iii)* $\psi(x) = \frac{1}{2} xQx^t + R(x)$ with $R(x) = O(|x|^4)$.
- *(iv)* $\nabla a(0) = 0$.

Hypothesis (iv) is made for convenience rather than necessity, since it will anyhow be satisfied by the amplitude of (44), and simplifies some of the estimates below. A further hypothesis on a will be introduced in the next paragraph: cf. (v) below.

The philosophy behind our estimates for (47) is to express all constants in terms of Q and its geometry, by means the associated distance, $d_Q(x) = \sqrt{xQx^t}$. A first example will be given by the final hypothesis, on supp (a), which we will state now. Let $\psi(x) = \frac{1}{2}xQx^t + R(x)$, as above, and let $R_-(x) = \max(-R(x), 0)$, the negative part of the 4^{th}-order remainder. If $0 < \gamma < 1$ is a constant, to be chosen arbitrarily, then clearly $\frac{1}{2}xQx^t - R_-(x)$ will dominate $\frac{1}{2}\gamma xQx^t$ on some neighborhood of 0. We give a more precise quantitative form to this observation by introducing

$$r_\gamma := \sup\{r : \frac{1}{2}xQx^t - R_-(x) \geq \frac{\gamma}{2}xQx^t,\ x \in B_Q(0,r)\}, \tag{48}$$

where $B_Q(0,r) = \{x : xQx^t \leq r^2\}$, the Q-ball of radius r. We then add as our final hypothesis that

- *(v)* supp $(a) \subseteq B_Q(0, r_\gamma)$.

To simplify notations, we will often write $Q(x)$ for xQx^t. We next define two constants $\|R/Q^2\|_{\infty,r_\gamma}$ and $\|a/Q\|_{\infty,r_\gamma}$ by

$$\|R/Q^2\|_{\infty,r_\gamma} := \max_{B_Q(0,r_\gamma)} \left(\frac{R(x)}{Q(x)^2}\right), \tag{49}$$

and, letting $\rho_2(x) := a(x) - a(0) - \nabla a(0)x^t = a(x) - a(0)$ (in view of condition (iv)), the remainder term in the first order Taylor expansion of a,

$$\|\rho_2/Q\|_{\infty,r_\gamma} := \max_{B_Q(0,r)} \left(|\frac{\rho_2(x)}{Q(x)}|\right). \tag{50}$$

Observe that both quantities are finite, since $R(x) = O(|x|^4)$ and $\rho_2(x) = O(|x|^2)$, and since Q(x) is positive definite. We can now formulate the main theorem of this section:

Theorem 3.1. *For a given γ, $0 < \gamma < 1$, and under the assumptions (i)-(v), we have that*

$$J(\lambda) = \frac{a(0)}{\sqrt{det(Q)}}\Big(\frac{2\pi}{\lambda}\Big)^{n/2} + E(\lambda),$$

with the following estimate for the error-term:

$$|E(\lambda)| \leq$$

$$\frac{1}{\sqrt{det\ (Q)}}\Big(\frac{2\pi}{\lambda}\Big)^{n/2}\left(\frac{n\|\rho_2/Q\|_{\infty,r_\gamma}}{\lambda} + \frac{n(n+2)\|a\|_\infty\ \|R/Q^2\|_{\infty,r_\gamma}}{\lambda^2\gamma^{n/2+2}} + \frac{|a(0)|\Gamma(\frac{n}{2},\frac{\lambda r_\gamma^2}{2})}{\Gamma(\frac{n}{2})}\right),$$

where $\Gamma(z,w)$ is the incomplete Γ-function defined by (21).

Proof. We split $J(\lambda)$ as

$$\begin{aligned} J(\lambda) &= \int_{\mathbb{R}^n} a(x)e^{-\lambda Q(x)/2}dx + \int_{\mathbb{R}^n} a(x)(e^{-\lambda R(x)} - 1)e^{-\lambda Q(x)/2}dx \\ &=: J_1 + J_2, \qquad (51) \end{aligned}$$

and estimate J_1 and J_2 separately.

Estimation of J_1. Do a 2^{nd} order Taylor expansion of $a(x)$ around 0:

$$a(x) = a(0) + \nabla a(0)\ x^t + \rho_2(x),$$

with $|\rho_2(x)| \leq C|x|^2$ in supp (a) (we do not use yet that $\nabla a(0) = 0$ at this stage). Inserting this in the integral and observing that odd powers of x integrate to 0, we easily find that

$$\begin{aligned} (52)\ J_1(\lambda) &= \frac{a(0)}{\sqrt{\det\ (Q)}}\Big(\frac{2\pi}{\lambda}\Big)^{n/2} + \int_{B_Q(0,r_\gamma)} \rho_2(x)e^{-\lambda Q(x)/2}dx + \\ &\quad \int_{\mathbb{R}^n\setminus B_Q(0,r_\gamma)} \rho_2(x)e^{-\lambda Q(x)/2}dx, \end{aligned}$$

where we used the standard change of variables $x \to \lambda^{-1/2}Q^{-1/2}x$ to obtain the first term. But if $\nabla a(0) = 0$, then $\rho_2(x) = -a(0)$ on the complement of the support of a, and since supp $(a) \subseteq B_Q(0, r_\gamma)$ by hypothesis, we see that the final term in (52) equals

$$\begin{aligned} &-a(0)\textstyle\int_{\{xQx^t\geq r_\gamma^2\}} e^{-\lambda Q(x)/2}dx \\ &= -\frac{a(0)}{\sqrt{\det\ (Q)}}\int_{\{|x|^2\geq r_\gamma^2\}} e^{-\lambda|x|^2/2}dx \\ &= \frac{-a(0)|S_{n-1}|}{\sqrt{\det\ (Q)}}\int_{r_\gamma}^\infty r^{n-1}e^{-\lambda r^2/2}dr, \end{aligned}$$

where we introduced polar coordinates, and where $|S_{n-1}| = 2\pi^{n/2}/\,\Gamma(n/2)$ is the surface measure of the unit sphere in $\mathbf{R}^n$. The integral can be transformed into an incomplete Γ-function, and it will be useful, here and for later, to note the following easily proved general identity:

$$\int_R^\infty r^a e^{-br^\alpha} dr = \alpha^{-1} b^{-(a+1)\alpha^{-1}} \Gamma\left(\alpha^{-1}(a+1), bR^\alpha\right). \tag{53}$$

We then find that the last term in (52) equals

$$-\frac{a(0)}{\sqrt{\det\,(Q)}}\Big(\frac{2\pi}{\lambda}\Big)^{n/2} \cdot \frac{\Gamma(n/2\,, \lambda r_\gamma^2/2)}{\Gamma(n/2)}. \tag{54}$$

Observe that for big λ this decays exponentially as $\simeq C\lambda^{-1}e^{-\lambda r_\gamma^2/2}$, by the asymptotics of the incomplete Γ function.

As for the second term in (52), we can estimate its absolute value by

$$\max_{B_Q(0,r)}\left(\left|\frac{\rho_2(x)}{Q(x)}\right|\right)) \cdot \int_{\mathbb{R}^n} Q(x)e^{-\lambda Q(x)/2}dx = \frac{n\|\rho_2/Q\|_{\infty,r_\gamma}}{\lambda\sqrt{\det\,(Q)}}\left(\frac{2\pi}{\lambda}\right)^{n/2},$$

by the change of variables $x \to \lambda^{-1/2}Q^{-1/2}x$. Summarizing, we found that

$$J_1(\lambda) = \frac{a(0)}{\sqrt{\det\,(Q)}}\Big(\frac{2\pi}{\lambda}\Big)^{n/2} + E_1(\lambda), \tag{55}$$

where

(56)

$$|E_1(\lambda)| \leq \frac{1}{\sqrt{\det\,(Q)}}\left(\frac{2\pi}{\lambda}\right)^{n/2}\left(\frac{n\|\rho_2/Q^2\|_{\infty,r_\gamma}}{\lambda} + |a(0)|\frac{\Gamma(\frac{n}{2}\,, \frac{\lambda r_\gamma^2}{2})}{\Gamma(\frac{n}{2})}\right).$$

<u>Estimation of J_2</u>. Using the elementary inequality $|e^y - 1| \leq |y|\max(e^y, 1)$ $(y \in \mathbb{R})$ with $y = -\lambda R(x)$, we see that

$$\begin{aligned} |J_2| &\leq \lambda \int_{B_Q(0,r_\gamma)} |a(x)||R(x)|e^{-\lambda(Q(x)/2 - R_-(x))}dx \\ &\leq \lambda\|a\|_\infty \int_{B_Q(0,r_\gamma)} |R(x)|e^{-\lambda\gamma Q(x)/2}dx, \end{aligned}$$

since $\text{supp}(a) \subseteq B_Q(0, r_\gamma)$ and $\frac{1}{2}Q(x) - R_-(x) \geq \frac{\gamma}{2}Q(x)$ on $B_Q(0, r_\gamma)$. Multiplying and dividing by $Q(x)^2$, we find:

$$\begin{aligned} |J_2| &\leq \|a\|_\infty\|R/Q^2\|_{\infty,r_\gamma}\int_{\mathbb{R}^n} Q(x)^2 e^{-\lambda\gamma Q(x)/2}dx \\ &= \Big(\frac{2\pi}{\lambda}\Big)^{n/2}\frac{n(n+2)\,\|a\|_\infty\,\|R/Q^2\|_{\infty,r_\gamma}}{\lambda^2\gamma^{n/2+2}\sqrt{\det\,(Q)}}, \end{aligned}$$

where we used that

$$\int_{\mathbb{R}^n} |x|^4 e^{-|x|^2/2}dx = (2\pi)^{n/2}(n^2 + 2n).$$

Adding this to (55), we have proved theorem 3.1. QED

A closer examination of the proof of theorem 3.1 reveals that we can obtain sharper asymmetrical upper and lower bounds for $J(\lambda)$, if we have information about the signs of $a(0)$ and of $R(x)$. Specifically, if $a(0) > 0$, then (54) will be negative, and can be discarded if we are looking for an upper bound of $J(\lambda)$. Similarly, if $a(x) \geq 0$ and $R(x) \leq 0$ (as will be the case in our application to $I(\lambda)$), then $\exp(-\lambda R(x)) - 1$ will clearly be positive, and $J_2(\lambda)$ can be left out of a lower bound. We therefore have the

Corollary 3.2. *(of the proof of theorem 3.1) Under the conditions of theorem 3.1 and if, moreover,* $a(x) \geq 0$ *and* $R(x) \leq 0$ *on* $B(0, r_\gamma)$*, then*

$$-E_L(\lambda) \;\leq\; J(\lambda) - \frac{a(0)}{\sqrt{\det(Q)}} \left(\frac{2\pi}{\lambda}\right)^{n/2} \;\leq E_U(\lambda),$$

with upper and lower errors

$$E_U(\lambda) = \frac{1}{\sqrt{\det(Q)}} \left(\frac{2\pi}{\lambda}\right)^{n/2} \left(\frac{n\|\rho_2/Q\|_{\infty,r_\gamma}}{\lambda} + \frac{n(n+2)\|a\|_\infty \, \|R/Q^2\|_{\infty,r_\gamma}}{\gamma^{n/2+2}\lambda^2}\right),$$

and

$$E_L(\lambda) = \frac{1}{\sqrt{\det(Q)}} \left(\frac{2\pi}{\lambda}\right)^{n/2} \left(\frac{n\|\rho_2/Q\|_{\infty,r_\gamma}}{\lambda} + a(0)\frac{\Gamma(\frac{n}{2}, \frac{\lambda r_\gamma^2}{2})}{\Gamma(\frac{n}{2})}\right).$$

As a final observation we note that both theorem 3.1 and corollary 3.2 will continue to hold if we replace r_γ by some smaller number $R_\gamma < r_\gamma$ (provided we do the same in condition *(v)*), as is clear from the proofs.

4. Estimation of $I(\lambda)$

4.1. Asymptotics of $I_\pm(\lambda)$.

We first apply the results of the previous section to $I_\pm(\lambda)$. To simplify notations, we will, in this subsection only, drop the accents, and write $x = (x_-, x_+)$ for $x' = (x'_-, x_+)$ (so that x will now be in $\mathbb{R}^n$ instead of $\mathbb{R}^{n+1}$).

We see from equations (44), (45) and (46) that $\exp c_1^{\alpha/2}\, I_+(\lambda)$ is of the form (47), with phase function

$$\psi(x) = (c_1 + q(x))^{\alpha/2} - c_1^{\alpha/2},$$

and amplitude

$$a(x) = \widetilde{\chi}_+(x)\,(1 - |x_-|^2 + |x_+|^2)^{-1/2}.$$

Here $c_1 := (a_1^-)^{-1} > 0$ and $q(x)$ is the positive definite quadratic form given by (46). If we let $f(y) = (c_1 + y)^{\alpha/2}$, then

$$f(y) = c_1^{\alpha/2} + \frac{\alpha}{2} c_1^{\frac{\alpha}{2}-1} y + \frac{\alpha}{4}(\frac{\alpha}{2} - 1)(c_1 + \theta_y y)^{\frac{\alpha}{2}-2} y^2$$

with $0 < \theta_y < 1$. Hence,

$$\psi(x) = \frac{\alpha}{2}\, c_1^{\frac{\alpha}{2}-1} q(x) \ + \ R(x) = \frac{1}{2} Q(x) + R(x)$$

with $Q(x) = \alpha c_1^{\frac{\alpha}{2}-1} q(x)$ and, if $\alpha \leq 4$,

$$|R(x)| \leq \frac{|\alpha(\alpha-2)|}{8} c_1^{\frac{\alpha}{2}-2} q(x)^2 = \frac{|\alpha - 2|}{8\alpha} c_1^{-\alpha/2} Q(x)^2.$$

Observe that $R(x) \leq 0$, and $R = -R_-$, if $\alpha \leq 2$, which is the interesting range of α's for applications to portfolio risk. It follows that $\frac{1}{2}Q - R_- \geq \frac{\gamma}{2} Q$ is the true whenever

$$\frac{|\alpha - 2|}{8\alpha} c_1^{-\alpha/2} Q(x) \leq \frac{1-\gamma}{2},$$

which, recalling (48), implies that

$$r_\gamma^2 = \frac{4\alpha(1-\gamma)}{|\alpha - 2|} \cdot c_1^{\frac{\alpha}{2}}. \tag{57}$$

These estimates also show that

$$\|R/Q^2\|_\infty \leq \frac{|\alpha - 2|}{8\alpha} c_1^{-\alpha/2} = \frac{|\alpha - 2|}{8\alpha} (a_1^-)^{\alpha/2},$$

on all of $\mathbb{R}^n$.

We next turn to the amplitude. We will choose our cut-off function $\widetilde{\chi}_+$ of the form

$$\widetilde{\chi}_+(x) = g\left(\frac{Q(x)}{R_\gamma^2}\right),$$

with suitably chosen $R_\gamma \leq r_\gamma$, and with $g : \mathbb{R}_{\geq 0} \to [0,1]$ a C^1 cut-off function supported in $[0,1]$ and equal to 1 on a neighborhood of 0. For any explicit computations below we will take g equal to g_a defined by (18), in which case $\|g_a'\|_\infty = \frac{2}{1-a}$. Letting $h(y) := (1-y)^{-1/2}$, we can write $a(x) = g(Q(x)/r_\gamma^2) h(|x_-|^2 - |x_+|^2)$, and since $\nabla a(0) = 0$, we obtain from the 0-th order Taylor expansions with remainder of g_a and h that:

$$\begin{aligned}
\rho_2(x) &= a(x) - 1 \\
&= -\frac{1}{2} \frac{|x_-|^2 - |x_+|^2}{(1 - \theta(|x_-|^2 - |x_+|^2))^{3/2}} + \frac{Q(x)}{r_\gamma^2} g'(\theta' \frac{Q(x)}{r_\gamma^2}) \\
&\quad -\frac{1}{2} \frac{|x_-|^2 - |x_+|^2}{(1 - \theta(|x_-|^2 - |x_+|^2))^{3/2}} \cdot \frac{Q(x)}{r_\gamma^2} g'(\theta' \frac{Q(x)}{r_\gamma^2}),
\end{aligned}$$

for suitable $\theta = \theta_x, \theta' = \theta_x' \in (0,1)$. We now pick R_γ such that $R_\gamma < r_\gamma$ and such that $|\, |x_-|^2 - |x_+|^2| \leq 1/2$ on $B(0, R_\gamma)$. To do this explicitly, simply observe that

$$\big|\, |x_-|^2 - |x_+|^2 \big| \leq |x|^2 \leq \lambda_{\min}(Q)^{-1} Q(x),$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q. Hence it suffices to choose

$$R_\gamma^2 = \min\left(r_\gamma^2, \frac{1}{2}\lambda_{\min}(Q)\right). \tag{58}$$

Straightforward estimates then show that

$$||\frac{\rho_2(x)}{Q(x)}||_\infty \leq \frac{\sqrt{2}}{\lambda_{\min}(Q)}\left(1+\frac{||g'||_\infty}{R_\gamma^2}\right)+\frac{||g'||_\infty}{R_\gamma^2}. \tag{59}$$

With this choice of R_γ also have that

$$\|a\|_\infty \leq \sup_{B(0,R_\gamma)} (1-|x_-|^2+|x_+|^2)^{-1/2} \leq \sqrt{2}.$$

Let us define constants $\widehat{K}_1^\pm, \widehat{K}_2^\pm$ by:

$$\widehat{K}_1^\pm := n\cdot\left\{\frac{\sqrt{2}}{\lambda_{\min}(Q)}\left(1+\frac{||g'||_\infty}{R_\gamma^2}\right)+\frac{||g'||_\infty}{R_\gamma^2}\right\}, \tag{60}$$

and

$$\widehat{K}_2^\pm := \sqrt{2}\,\frac{n(n+2)}{\gamma^{\frac{n}{2}+2}}\frac{|\alpha-2|}{8\alpha}(a_1^-)^{\alpha/2}. \tag{61}$$

Corollary 3.2 then implies the following intermediary result, which we state as a lemma, for future reference:

Lemma 4.1.

$$\begin{aligned} -\widehat{E}_L^\pm(\lambda) &\leq I_+(\lambda)+I_-(\lambda)-\frac{2}{\sqrt{\det(Q)}}\left(\frac{2\pi}{\lambda}\right)^{n/2} e^{-\frac{\lambda}{(a_1^-)^{\alpha/2}}} \\ &\leq \widehat{E}_U^\pm(\lambda), \end{aligned} \tag{62}$$

with

$$\widehat{E}_U^\pm(\lambda) = \frac{2}{\sqrt{\det(Q)}}\left(\frac{2\pi}{\lambda}\right)^{n/2} e^{-\frac{\lambda}{(a_1^-)^{\alpha/2}}}\left(\frac{\widehat{K}_1^\pm}{\lambda}+\frac{\widehat{K}_2^\pm}{\lambda^2}\right),$$

and

$$\widehat{E}_L^\pm(\lambda) = \frac{2}{\sqrt{\det(Q)}}\left(\frac{2\pi}{\lambda}\right)^{n/2} e^{-\frac{\lambda}{(a_1^-)^{\alpha/2}}}\left(\frac{\widehat{K}_1^\pm}{\lambda}+\frac{\Gamma\left(\frac{n}{2},\frac{\lambda R_\gamma^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right).$$

4.2. **Estimation of $I_0(\lambda)$.** We now re-instate the accented variables, and let $x=(x_1,x')\in\mathbb{R}^{n+1}$, as before, in section 2. To complete our asymptotic formula for $I(\lambda)$, we have to estimate the contribution of

$$I_0(\lambda) = \int_\Sigma e^{-\lambda\varphi(x)}\chi_0(x)L_\eta(x), \tag{63}$$

where we have put

$$\varphi(x) := \left(x|\mathbb{A}|^{-1}x^t\right)^{\alpha/2}. \tag{64}$$

Recall that φ assumes its absolute minimum $c_1^{\alpha/2} = (a_1^-)^{-\alpha/2}$ on Σ in the two points $\pm e_1 := (\pm e_1^-, 0)$, which are both outside of the support of χ_0; $I_0(\lambda)$ will therefore have an exponential decrease with respect to $I_\pm(\lambda)$, and the only point is to give a precise quantitative form to this observation. If Σ is compact, that is, if $n_+ = 0$, then the integral $I_0(\lambda)$ can be trivially estimated by

$$|I_0(\lambda)| \leq \exp\left(-\lambda \min_{\mathrm{supp}\chi_0} (\varphi - \varphi(e_1))\right) |S_{n-1}| \, e^{-\lambda\varphi(e_1)},$$

since for the unit sphere the Liouville measure of $\eta(x) = |x_-|$ is equal to the surface measure. However, in the general case the total Liouville measure of Σ will be infinite, and we will use a fraction of the exponential to arrive at a convergent integral. Let therefore $\varepsilon \in (0,1)$. Then

$$\begin{aligned} e^{\lambda\varphi(e_1)} I_0(\lambda) &= \int_\Sigma e^{-\lambda(\varphi(x)-\varphi(e_1))} \chi_0 L_\eta \\ &\leq \max_{\mathrm{supp}\ \chi_0} \exp\left(-\lambda(1-\varepsilon)(\varphi - \varphi(e_1))\right) \\ &\quad \cdot \int_\Sigma e^{-\varepsilon\lambda(\varphi-\varphi(e_1))} \ \chi_0 \ L_\eta, \end{aligned} \tag{65}$$

whose absolute value will, for λ's bigger than some $\lambda_0 > 0$, be bounded by

$$\exp\left(-\lambda(1-\varepsilon)) \min_{\mathrm{supp}\ \chi_0} (\varphi - \varphi(e_1))\right) \cdot e^{\varepsilon\lambda_0\varphi(e_1)} \cdot \int_\Sigma e^{-\varepsilon\lambda_0\varphi} |L_\eta|;$$

here λ_0 is to be chosen conveniently in concrete applications. We therefore can estimate, for $\lambda \geq \lambda_0$,

$$|I_0(\lambda)| \leq K_{\varepsilon,\lambda_0} \cdot e^{-\lambda(\varphi(e_1) + m_\epsilon)}, \tag{66}$$

where

$$m_\varepsilon = (1-\varepsilon) \min_{\mathrm{supp}\ \chi_0} (\varphi - \varphi(e_1)) > 0, \tag{67}$$

and

$$K_{\varepsilon,\lambda_0} = e^{\varepsilon\lambda_0\varphi(e_1)} \left| \int_\Sigma e^{-\varepsilon\lambda_0\varphi} L_\eta \right| \ < \ \infty. \tag{68}$$

This shows, as announced, that $I_0(\lambda)$ is exponentially decreasing with respect to $I_\pm(\lambda)$, as $\lambda \to \infty$. To obtain a precise quantitative form of this, we now bound the two constants m_ε and $K_{\varepsilon,\lambda_0}$, with special attention to the dependence on the GLD-parameter α and on the choice of R_γ in the estimate for $I_\pm(\lambda)$. We begin with $K_{\varepsilon,\lambda_0}$. We will use Stokes' theorem to convert the integral over the hyper-surface Σ into one over the exterior domain, and for this we first compute the exterior derivative of L_η.

Lemma 4.2. *Let $\eta = \eta(x)$ and $v = v(x)$ be a C^2, respectively C^1, function, defined on some open subset of $\mathbb{R}^{n+1}$ on which $\nabla\eta$ is nowhere vanishing. Then*

$$d\left(vL_\eta\right) = g(x)dx_1 \wedge \cdots \wedge dx_{n+1},$$

where

$$g(x) = \frac{1}{|\nabla\eta|^2}\left(\nabla v \cdot \nabla\eta + v\Delta\eta\right) - \frac{2v}{|\nabla\eta|^4}\sum_{j,k}\frac{\partial^2\eta}{\partial x_j \partial x_k}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_k}.$$

The proof, a straightforward differentiation exercise, is left to the reader.

In our case, $\eta(x) = \sqrt{|x_-|^2 - |x_+|^2}$, and therefore

$$\nabla\eta(x) = \eta(x)^{-1}\left(x_-, -x_+\right).$$

In particular, $|\nabla\eta(x)| = |x|/|\eta(x)|$. We next compute

$$\frac{\partial^2\eta}{\partial x_j \partial x_k} = \epsilon_j \frac{\delta_{jk}}{\eta(x)} - \epsilon_j\epsilon_k \frac{x_j x_k}{\eta(x)^3},$$

where $\epsilon_j = 1$ if $1 \leq j \leq n_-$, and $\epsilon_j = -1$ if $n_-+1 \leq j \leq n_-+n_+ = n+1$. It follows that $\Delta\eta = (n_- - n_+)/\eta - |x|^2/\eta^3$ and also that,

$$\sum_{j,k}\frac{\partial^2\eta}{\partial x_j \partial x_k}\frac{\partial\eta}{\partial x_j}\frac{\partial\eta}{\partial x_k} = \left(\sum_j \epsilon_j \frac{x_j^2}{\eta^3}\right) - \frac{|x|^4}{\eta^5}.$$

By Cauchy-Schwarz,

$$\frac{|\nabla v \cdot \nabla\eta|}{|\nabla\eta|^2} \leq \frac{|\nabla v|}{|\nabla\eta|},$$

and we easily find that since $|\nabla\eta| = |x|/|\eta|$,

$$\begin{aligned} |g(x)| &\leq \frac{|\nabla v|}{\nabla\eta} + |v| \cdot \left\{ \frac{|n_- - n_+|}{\eta|\nabla\eta|^2} + \frac{|x|^2}{\eta^3|\nabla\eta|^2} + \frac{2|x|^2}{\eta^3|\nabla\eta|^4} + \frac{2|x|^4}{\eta^5|\nabla\eta|^4} \right\} \\ &= \frac{|\nabla v|}{|\nabla\eta|} + |v| \cdot \left\{ |n_- - n_+| \frac{\eta}{|x|^2} + \frac{1}{\eta} + \frac{2\eta}{|x|^2} + \frac{2}{\eta} \right\}. \end{aligned}$$

Hence, if $\eta(x) \geq 1$ then, since $\eta(x) \leq |x|$ and $|\nabla\eta(x)| \geq 1$,

$$|g(x)| \leq |\nabla v| + |v| \cdot \left(|n_- - n_+| + 5\right);$$

(this could have been slightly sharpened[9]). Taking $v(x) = \exp\left(-\varepsilon\lambda_0\varphi(x)\right)$ with $\varphi(x) = \left(x|\mathbb{A}|^{-1}x^t\right)^{\alpha/2}$ and using Stokes' theorem applied to the exterior domain, we find that

$$\begin{aligned} &|\textstyle\int_\Sigma e^{-\varepsilon\lambda_0\varphi}L_\eta| = |\textstyle\int_{\{\eta\geq 1\}} d\left(e^{-\varepsilon\lambda_0\varphi}L_\eta\right)| \\ &\leq \int_{\mathbb{R}^n}\left(\alpha\,\varepsilon\lambda_0\left(x|\mathbb{A}|^{-1}x^t\right)^{\alpha/2-1} \left|\,|\mathbb{A}|^{-1}x\right| + \left(|n_- - n_+| + 5\right)\right)e^{-\varepsilon\lambda_0\varphi}dx. \end{aligned}$$

[9] namely to: $|g(x)| \leq |\nabla v| + |v| \cdot (|n_- - n_+| + 5)/\eta(x)$; however, for large n, the extra decay of $\eta(x)^{-1}$ will not make a huge difference after integration over $\{\eta(x) \geq 1\}$.

After the change of variables $x \to (\varepsilon\lambda_0)^{-1/\alpha}|\mathbb{A}|^{1/2}x$, the right hand side of becomes:

$$\frac{\sqrt{\det|\mathbb{A}|}}{(\epsilon\lambda_0)^{n/\alpha}}\int_{\mathbb{R}^{n+1}}\left\{\alpha|x|^{\alpha-2}\left|\,|\mathbb{A}|^{-1/2}x\right| + \frac{|n_- - n_+| + 5}{(\varepsilon\lambda_0)^{1/\alpha}}\right\}\, e^{-|x|^\alpha}dx.$$

Now $\left|\,|\mathbb{A}|^{-1/2}x\right| \leq ||\,|\mathbb{A}|^{-1}\,||^{1/2}\,|x|$,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} e^{-|x|^\alpha}\,dx &= \frac{1}{\alpha}|S_n|\,\Gamma\left(\frac{n+1}{\alpha}\right)\\ &= \frac{2}{\alpha}\pi^{(n+1)/2}\frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} |x|^{\alpha-1}e^{-|x|^\alpha}\,dx &= \frac{1}{\alpha}|S_n|\,\Gamma\left(\frac{n+\alpha}{\alpha}\right)\\ &= \frac{2}{\alpha}\pi^{(n+1)/2}\frac{\Gamma\left(\frac{n+\alpha}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.\end{aligned}$$

Collecting all terms we find that $K_{\epsilon,\lambda_0} \leq \widehat{K}_0(\varepsilon\lambda_0)$, where

$$\begin{aligned}\widehat{K}^0(\varepsilon\lambda_0) \quad &:= \quad \pi^{(n+1)/2}\,\sqrt{\det|\mathbb{A}|}\,\frac{e^{\varepsilon\lambda_0(a_1^-)^{-\alpha/2}}}{(\varepsilon\lambda_0)^{n/\alpha}}\\ &\cdot\left\{2\,||\,|\mathbb{A}|^{-1}||^{1/2}\,\frac{\Gamma\left(\frac{n+\alpha}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)} + \frac{2|n_- - n_+| + 10}{\alpha(\varepsilon\lambda_0)^{1/\alpha}}\,\frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}. \quad (69)\end{aligned}$$

Finally, we compute m_ε, given by (67). A moment's thought will show that the minimum will be attained at a point $x = (x_1, x') \in \Sigma$ of supp$\chi_\pm$ where $Q(x') = aR_\gamma^2$, a as in (18). Since $\varphi(x)|_\Sigma = (c_1 + q(x'))^{\alpha/2}$, and $q(x') = \alpha^{-1}c_1^{1-\frac{\alpha}{2}}Q(x')$, we find

$$m_\epsilon = (1-\epsilon)[(c_1 + \alpha^{-1}c_1^{1-\frac{\alpha}{2}}a\cdot R_\gamma^2)^{\alpha/2} - c_1^{\alpha/2}],. \quad (70)$$

This completes our estimation of $I_0(\lambda)$. Summarizing, and recalling that $\varphi(e_1) = (a_1^-)^{-\alpha/2}$, we have shown:

Lemma 4.3. *For $\lambda \geq \lambda_0$,*

$$|I_0(\lambda)| \leq \widehat{K}^0(\varepsilon\lambda_0)\,e^{-n_\varepsilon\lambda},$$

with

$$n_\varepsilon := m_\varepsilon + (a_1^-)^{-\alpha/2},$$

and with $\widehat{K}^0(\varepsilon\lambda_0)$ and m_ε given by (69) and (70), respectively.

Remark 4.4. In practice, we would want to choose $\epsilon\lambda_0$ such that $\widehat{K}^0(\varepsilon\lambda_0)$ is minimal. An exact minimization involves computing the minimum of a function of the form

$$z \to \left(z^{-k} + Cz^{-k-l}\right)e^{cz},$$

with parameters $k, l, c, C > 0$; for us, $z = \varepsilon\lambda_0 > 0$, $k = n/\alpha$ and $l = 1/\alpha$. Putting the derivative equal to 0 leads to the equation

$$cz^{l+1} - kz^l + cCz - (k+l)C = 0,$$

which, in general, cannot be solved explicitly. An exception is when $l = 1$, corresponding to $\alpha = 1$, in which case the positive root is given by $z = (2c)^{-1}\left(k - cC + \sqrt{(k-cC)^2 + 4cC(k+1)}\right) \simeq k/c$, for large k. A simple upper bound for the minimum can be obtained by only exactly minimizing one of the two terms making up $\widehat{K}^0(\varepsilon\lambda_0)$, using that a function of the form $z \to z^{-k}e^{cz}$ $(c > 0)$ attains its minimum $c^k k^{-k} e^k$ on $z > 0$ in the point $z = k/c$. Applying this with $k = n/\alpha$ and $c = (a_1^-)^{-\alpha/2}$, we obtain that

$$\min_{\varepsilon\lambda_0} \widehat{K}^0(\varepsilon\lambda_0) \leq \pi^{(n+1)/2}\sqrt{\det|\mathbb{A}|}\,(a_1^-)^{-n/2}\,\frac{e^{n/\alpha}}{(n/\alpha)^{n/\alpha}} \cdot \left\{2||\,|\mathbb{A}|^{-1}\,||\frac{\Gamma\left(\frac{n+\alpha}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)} + \frac{2|n_- - n_+| + 10}{\alpha}\,(a_1^-)^{-1/2}\left(\frac{\alpha}{n}\right)^{1/\alpha}\frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\right\}. \tag{71}$$

The above suggests $n(a_1^-)^{\alpha/2}/\alpha$ as a reasonable choice for $\varepsilon\lambda_0$. A lower bound on λ_0 will then determine ε and, consequently, n_ε.

5. **Proof of theorem 1.2**

It remains to replace λ by r^α in the estimates of the previous section, integrate from R to ∞ with respect to $r^n dr$, and multiply by $C' = c^{-(n+1)/\alpha}C\sqrt{\det|\mathbb{A}|}$ with $c = c_{\alpha,n+1}$, $C = C_{\alpha,n+1}$; cf. (34), (35). This is basically a book-keeping exercise, but we will still indicate the main steps of the computations, for convenience of the reader. We first observe that the $\sqrt{\det|\mathbb{A}|}$ in the denominator of the constant C', formula (35), and the $\sqrt{\det Q}$ in the denominators in lemma 4.1 combine to yield an overall factor of

$$(|\det\mathbb{A}|\,\det Q)^{-1/2} = \alpha^{-n/2}(a_1^-)^{\frac{\alpha n}{4}-\frac{1}{2}}\,\Delta(\mathbb{A})^{-1/2},$$

where we have put

$$\Delta(\mathbb{A}) := \prod_2^{n_-}(a_1^- - a_j^+)\prod_1^{n_+}(a_1^- + a_j^+).$$

(Recall that $Q = \alpha^{n/2}(a_1^-)^{-(\frac{\alpha}{2}-1)}\,q$, with q given by (46).) Using (39), the principal term of $I_+ + I_-$ in (62) (that is, $2(\det(Q))^{-1/2}(2\pi\lambda^{-1})^{n/2}$ $\exp(-\lambda(a_1^-)^{-\alpha/2})$) will then give rise to a principal term of $F_\Gamma(-V)$ of

$$\begin{aligned} F_{\Gamma,\mathrm{pc}}(-V) \;&:=\; 2\alpha^{-n/2}(2\pi)^{n/2}c^{-\frac{n+1}{\alpha}}C\,(a_1^-)^{\frac{\alpha n}{4}-\frac{1}{2}}\,(\Delta(\mathbb{A}))^{-1/2} \\ &\cdot\int_R^\infty r^{n(1-\frac{\alpha}{2})}\exp\left(-(a_1^-)^{-\alpha/2}r^\alpha\right)dr, \end{aligned}$$

where we recall that $V = \frac{1}{2}c^{-2/\alpha}R - \Theta$. Using (53), we find that

$$F_{\Gamma,\mathrm{pc}}(-V) = A_{\mathrm{pc}}\,\Gamma\left(\frac{n+1}{\alpha} - \frac{n}{2}, \left(\frac{R}{\sqrt{a_1^-}}\right)^{\alpha}\right), \tag{72}$$

with $A_{\mathrm{pc}} = 2\alpha^{-\frac{n}{2}-1}(2\pi)^{n/2}c^{-\frac{n+1}{\alpha}}C\ (a_1^-)^{n/2}\Delta(\mathbb{A})^{-1/2}$, which establishes the main term approximation (13), (12).

The estimates for the upper and lower error terms can be found similarly. We begin with the latter. It is important to observe that for a lower bound for $I(\lambda)$ we can leave out the $I_0(\lambda)$-term altogether[10]. We therefore have, using lemma 4.1,

$$\begin{aligned}
& F_{\Gamma}(-V) - F_{\Gamma,\mathrm{pc}}(-V) \\
\geq\ & -C'\int_R^{\infty} r^n \widehat{E}_L(r^{\alpha})\,dr \\
=\ & -2(2\pi)^{n/2}\frac{C'}{\sqrt{\det Q}}\Bigg\{\ \widehat{K}_1^{\pm}\int_R^{\infty} r^{n-\alpha(\frac{n}{2}+1)}\exp\left(-\frac{r^{\alpha}}{(a_1^-)^{\alpha/2}}\right)dr \\
& + \frac{1}{\Gamma(n/2)}\int_R^{\infty}\Gamma\left(\frac{n}{2}, \frac{1}{2}R_{\gamma}^2\,r^{\alpha}\right)\exp\left(-\frac{r^{\alpha}}{(a_1^-)^{\alpha/2}}\right)r^n\,dr\Bigg\} \quad .
\end{aligned}$$

The first integral on the right can again be evaluated using (53), yielding

$$\widehat{K}_1^{\pm}\ \alpha^{-1}(a_1^-)^{\frac{n+1}{2}-\frac{\alpha n}{4}-\frac{\alpha}{2}}\,\Gamma\left(\frac{n+1}{\alpha} - \frac{n}{2} - 1, \left(\frac{R}{\sqrt{a_1^-}}\right)^{\alpha}\right).$$

The second integral can be treated as follows: inserting the definition of the incomplete Γ-function, and interchanging order of integration, it is found to equal

$$\frac{1}{\Gamma(n/2)}\int_R^{\infty} s^{\frac{n}{2}-1}e^{-s}\int_R^{(2R_{\gamma}^{-2}s)^{1/\alpha}} r^n\exp\left(-\frac{r^{\alpha}}{(a_1^-)^{\alpha/2}}\right)dr\,ds. \tag{73}$$

Since this is a term which will be exponentially small for large R, we won't evaluate this integral explicitly (but see Sadefo [11]), but contend ourselves with an upper bound, by extending the inner integral over $[R,\infty)$. The double integral then becomes a product, equal to

$$\alpha^{-1}(a_1)^{\frac{n+1}{2}}\,\frac{\Gamma\left(\frac{n}{2}, R\right)}{\Gamma\left(\frac{n}{2}\right)}\,\Gamma\left(\frac{n+1}{\alpha}, \left(\frac{R}{\sqrt{a_1^-}}\right)^{\alpha}\right).$$

10 As already noted in the introduction, this is extremely helpful, since it will imply a λ_0-independent lower bound on F_{Γ} which, in turn can be used to find a lower bound on λ_0 when computing VaR_p^{Γ} for a given p.

Finally, the overall coefficient in front equals $2(2\pi)^{n/2}\alpha^{-n/2}c^{-(n+1)/\alpha}C\cdot(a_1^-)^{\frac{\alpha n}{4}-\frac{1}{2}}\Delta(\mathbb{A})^{-1/2}$, and combining all terms gives the lower bound

$$\begin{aligned}-\mathcal{E}_L(R) \quad := \quad & -(a_1^-)^{-\alpha/2}\, A_{\mathrm{pc}}\, \widehat{K}_1^{\pm}\, \Gamma\left(\frac{n+1}{\alpha}-\frac{n}{2}-1, \left(\frac{R}{\sqrt{a_1^-}}\right)^{\alpha}\right) \\ & -(a_1^-)^{\frac{\alpha n}{4}}\, A_{\mathrm{pc}}\, \frac{\Gamma\left(\frac{n}{2},R\right)}{\Gamma\left(\frac{n}{2}\right)}\, \Gamma\left(\frac{n+1}{\alpha}, \left(\frac{R}{\sqrt{a_1^-}}\right)^{\alpha}\right),\end{aligned}$$

which proves one half of theorem 1.2.

The upper error can be bounded in the same way. By the other half of lemma 4.1, and lemma 4.3, we find that if $R^\alpha \geq \lambda_0$, then

$$F_\Gamma(-V) - F_{\Gamma,\mathrm{pc}}(-V) \leq \mathcal{E}_U(R),$$

where

$$\mathcal{E}_U(R) := C' \int_R^\infty r^n \left(\widehat{E}_U(r^\alpha) + \widehat{K}^0(\varepsilon\lambda_0)\, e^{-n_\varepsilon r^\alpha}\right)\, dr.$$

The two integrals can be treated as before, and we find after some computations that

$$\begin{aligned}\mathcal{E}_U(R) \quad = \quad & A_{\mathrm{pc}}\cdot \sum_{j=1,2}(a_1^-)^{-j\alpha/2}\widehat{K}_j^{\pm}\, \Gamma\left(\frac{n+1}{\alpha}-\frac{n}{2}-j, \left(R/\sqrt{a_1^-}\right)^{\alpha}\right) \\ & + K^0(\varepsilon\lambda_0)\Gamma\left(\frac{n+1}{\alpha}, n_\varepsilon R^\alpha\right),\end{aligned}$$

with

$$K^0(\varepsilon\lambda_0) = \alpha^{-1}|\det(\mathbb{A})|^{-1/2}\,(cn_\varepsilon)^{-\frac{n+1}{\alpha}}\, C\, \widehat{K}^0(\varepsilon\lambda_0),$$

which is equal to (20). This completes the proof of theorem 1.2. QED

6. Numerical application

We constructed a Δ-hedged portfolio that contains respectively $n = n_1+n_2$ equities with $n_1 = 30$ European short call options on n_1 equities and $n_2 = 15$ European long put options from the *French CAC 40 Market* (January 05, 2005 to October 17, 2005), with data plotted in Figures 1 and 2. The price of the portfolio is given by

(74)

$$\Pi(t,S(t)) = \sum_{i=1}^{30}[-C_i(t,S_i(t))+\Delta^i\cdot S_i(t)] + \sum_{j=31}^{45}[P_j(t,S_j(t))+\Delta^j\cdot S_j(t)],$$

where S_i is an equity price i, with $S(t) = (S_1(t),\ldots,S_n(t))$, and $C_i(t,S_i(t))$ is the price of European call option i on equity i. Δ is known in the literature as a gradient portfolio sensitivity vector. Our portfolio has been chosen so that $\Delta = 0$, with $\Delta^i = \frac{\partial C_i}{\partial S_i}(S_i(0))$, and $\Delta = (\Delta^1,\ldots,\Delta^n)$. We defined the volatility σ_i of the underlying stock i as the sample standard deviation of the log return of stock i. We set the maturity time $T = 1/4$ years, the interest rate $r = 0.1, 0.05,$

$c = [c_1; c_2]$ and used $E_i = (1 + c_1\ \sigma_i)\ S_i(0)$ for the exercise price of call i with $i = 1, \ldots, 30$ and c_1 a parameter to be chosen. We also set $E_j = (1 + c_2\ \sigma_j)\ S_j(0)$ for the exercise price of put i with $j = 1, \ldots, 15$ and c_2 a parameter to be chosen. The parameter α of the multivariate Generalized Laplace distribution (MGLD) can be estimated via maximum likelihood, using the EWMA covariance matrix and the sample log-returns (see Sadefo-Kamdem and Genz [15] for details.)

In the following Tables: $\mathrm{VaR}^{\Pi}_{MC,NSIM}$ denotes the VaR of the full portfolios (without approximation) obtained with $NSIM$ Monte Carlo simulation using MGLD random with $\alpha = 2$; $\mathrm{VaR}^{\Theta-\Gamma}_{MC,NSIM}$ denotes the VaR of the quadratic approximation portfolios obtained with $NSIM$ Monte Carlo simulations. The following Tables provide computed R and $VaR_{p,\alpha}$ values for $\alpha = 2$, for selected p's and $c = [c_1, c_2]$ values. $NSIM$ denotes the number of simulations for Monte Carlo.

- **Portfolio with $c_1 = 1$ and $c_2 = -1$:** $\Pi(0) = 830.7588383380647$, $\breve{\Theta}_t = 0.18908531206273$, $\alpha = 2$, $\epsilon = 0.4798$, $\lambda_0 = R^{\alpha}_{LB}$, $a = 0.5$ and $\gamma = 0.01$, we obtained

p	0.01	0.001	0.0001	0.00001
$R_{p,LB}$	0.42217492	0.52435910	0.61145949	0.688559884
$R_{p,PC}$	0.42225041	0.52439849	0.611484333	0.68857728
$R_{p,UB}$	0.42234784	0.52445154	0.61151859	0.68860163
$\mathrm{VaR}_{p,UB}$	-0.01070761	0.08596410	0.18486967	0.28508690
$\mathrm{VaR}_{p,PC}$	-0.01078991	0.08590847	0.18482778	0.28505335
$\mathrm{VaR}_{p,LB}$	-0.01085365	0.085867158	0.18479740	0.28502940

Remark 6.1. In the preceded table we can see that the VaR is positive when, when p tends to 0. In the next part, we consider the confidence level $p = 0.00001$.

- **Portfolio with $c_1 = 1$ and $c_2 = -1$:** $\Pi(0) = 830.7588383380647$, $\breve{\Theta}_t = 0.18908531206273$, $p = 0.00001$, $\lambda_0 = R^{\alpha}_{LB}$, $a = 0.5$ and $\gamma = 0.01$, we obtained

α	1.50	1.70	1.80	1.95
$R_{p,LB}$	0.938002	0.838618	0.786274	0.711358
$R_{p,PC}$	1.080324	0.889633	0.810993	0.715394
$R_{p,UB}$	2.421969	1.006289	0.828150	0.718946
$\mathrm{VaR}_{p,UB}$	1.216434	0.291861	0.241510	0.275642
$\mathrm{VaR}_{p,PC}$	0.090560	0.186818	0.223854	0.271061
$\mathrm{VaR}_{p,LB}$	0.021733	0.144943	0.199065	0.265884
$\mathrm{VaR}_{p,UB} - \mathrm{VaR}_{p,LB}$	1.194702	0.146918	0.042446	0.009758

- **Portfolio with $c_1 = 1$ and $c_2 = 0$:** $\Pi(0) = 830.7588383380647$, $\breve{\Theta}_t = 0.18908531206273$, $p = 0.00001$, $\lambda_0 = R^{\alpha}_{LB}$, $a = 0.5$ and $\gamma = 0.01$, we obtained

α	1.90	1.93	1.96	1.99
$R_{p,LB}$	0.732863	0.718536	0.703535	0.687662
$R_{p,PC}$	0.743132	0.725392	0.708533	0.692503
$R_{p,UB}$	0.751234	0.731080	0.716014	0.698472
$\text{VaR}_{p,UB}$	0.263035	0.269421	0.280773	0.287512
$\text{VaR}_{p,PC}$	0.253309	0.262296	0.270981	0.279380
$\text{VaR}_{p,LB}$	0.241134	0.253781	0.2644959	0.272836
$\text{VaR}_{p,UB} - \text{VaR}_{p,LB}$	0.021901	0.015641	0.0152774	0.0146759

- For $\Pi(0) = 85.6680838$, $c = [-1; 1]$, $\breve{\Theta}_t = 0.18908531206273$, p=0.001, we obtained

Limits	Lower (LB)	Principal (PC)	Upper (UB)
R	0.52436622	0.52439849	0.52445154
$\text{VaR}^{\Gamma,\alpha=2}_{Analytic}$	0.08587462	0.08590847	0.08596410
$\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,10000}$	0.083588053	0.083588053	0.083588053
$\frac{\text{VaR}^{\Gamma,\alpha=2}_{Analytic}}{\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,10000}} - 1$	0.016611073	-0.027760116	0.017008664
$\lvert\frac{\text{VaR}^{\Gamma,\alpha=2}_{Analytic}}{\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,10000}} - 1\rvert$	1,66 %	2,78 %	1,70 %
$\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,100000}$	0.091566139	0.091566139	0.091566139
$\frac{\text{VaR}^{\Gamma,\alpha=2}_{Analytic}}{\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,100000}} - 1$	-0.041355979	0.061787811	-0.040508470
$\lvert\frac{\text{VaR}^{\Gamma,\alpha=2}_{Analytic}}{\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,100000}} - 1\rvert$	4,13 %	6,17 %	4,05 %

	Computation Time-Exec
$\text{VaR}^{\Gamma,\alpha=2}_{Analytic}[LB, PC, UB)$	1.41 seconds
$\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,1000}$	1.92 seconds
$\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,10000}$	182.91 seconds
$\text{VaR}^{\Theta-\Gamma,\alpha=2}_{MC,100000}$	17003.92 seconds

- Following the precede tables, for all $L \in \{LB, PC, UB\}$, the relative error we make is

$$\left|\frac{\text{VaR}^{\Gamma,0.001}_{L} - \text{VaR}^{\Gamma,0.001}_{MC(10000)}}{\text{VaR}^{\Gamma,0.01}_{MC(10000)}}\right| \simeq 2,78\%,$$

and

$$|\frac{\mathrm{VaR}_{L}^{\Gamma,0.001}-\mathrm{VaR}_{MC(100000)}^{\Gamma,0.001}}{\mathrm{VaR}_{MC(100000)}^{\Gamma,0.01}}|\simeq 6,17\%,$$

while the relative size of the analytic estimation interval

$$[\mathrm{VaR}_{LB}^{\Gamma,0.001},\mathrm{VaR}_{UB}^{\Gamma,0.001}],$$

is approximately 0.009 % of true VaR.

- The R's values were determined solving the equations $P = F_{\Gamma,PC}(R) - \mathcal{E}_L(R) = LB(R)$, $p = F_{\Gamma,PC}(R)$, $p = F_{\Gamma,PC}(R) + \mathcal{E}_U(R) = UB(R)$ for $p = 0.001$ using a bisection method. The estimation of $\mathrm{VaR}_{PC}^{\Gamma,0.001}$, $\mathrm{VaR}_{LB}^{\Gamma,0.001}$ and $\mathrm{VaR}_{UB}^{\Gamma,0.001}$ takes 1.42 seconds. For Monte Carlo VaR calculation the times of execution were respectively 182.91 for 10000 simulations and 17003.92 seconds for 100000 simulations.
- On a more powerful computer the time for performing the MC simulations could of course be significantly shorter. Observe, however, that our example portfolio is not particularly big.

Remark 6.2. Our method gives the following VaR estimates at the 99% confidence level, where the computations were done in Matlab on a Pentium IV, with 512 MHZ of RAM and 1.5 GHz of processor; the zeros were found using the bisection algorithm.

7. Conclusions

In this paper we have considered the problem of the analytical approximate of Value-at-Risk. Given a specified confidence level p, and assuming a generalized Laplace distribution for the joint log returns, our approach is designed to supplement the usual Monte-Carlo techniques, by providing an asymptotic formula for the quadratic portfolio's cumulative distribution function, together with explicit error-estimates.

We illustrated the use of this analytical method with several examples based on real data taken from the French CAC 40 Market. We have shown that appropriately chosen analytical method can efficiently provide accurate results for these problems, with a very good computer speed.

We expect that the type of results in this paper can be generalized for mixture of generalized Laplace distributions risk factors (see [11, 12]). An important result of this paper is the analytical approximation of the distribution function of a nonlinear (e.g. quadratic) of generalized Laplace distribution random vectors with explicit error-estimates. Our methods were derived assuming a quadratic Taylor approximation for the portfolio price, but our methods could also be used for other quadratic approximations (those developed by Studer[16], for example. See appendix E). Note that Studer [16] and Mina [7] describe procedures by which the quadratic approximation is estimated by least squares methods. These methods produce fairly accurate and fast delta–gamma approximations to "true" VaR. Even though we have use the RiskMetrics EWMA for our computation, an improvement is possible with our method by using DCC (see Engle [4]) or a regime switching volatility approach (see Pelletier [9]). An application to assessing the quality of an approximate distribution by using the Kullback-Leibler information measure is possible. Our method is applicable to portfolios of bonds and also to a portfolio of mortgage backed securities.

Appendix A. **Normalization constants of the GLD**

The k-dimensional GLD with $\mathbb{V} = I$ is given by (cf. (5)):

$$f(x) = C\exp\left(-c|x|^{\alpha}\right),$$

where the normalization constants $c = c_{\alpha,k}$ and $C = C_{\alpha,k}$ are such that

$$\int_{\mathbb{R}^k} f(x)\,dx = \int_{\mathbb{R}^k} x_j^2 f(x)\,dx = 1. \tag{75}$$

By rotation invariance of f, the latter condition is equivalent to $\int_{\mathbb{R}} |x|^2 f dx = k$. Changing variables $x \to c^{-1/\alpha}x$ and introducing polar coordinates, we obtain the following system of equations for c and C:

$$\begin{aligned} 1 &= c^{-k/\alpha}C\,|S_{k-1}| \int_0^{\infty} r^{k-1}e^{-r^{\alpha}}dr \\ &= \alpha^{-1}c^{-k/\alpha}C|S_{k-1}|\,\Gamma\left(\frac{k}{\alpha}\right), \end{aligned} \tag{76}$$

and

$$\begin{aligned} k &= c^{-(k+2)/\alpha}C\,|S_{k-1}| \int_0^{\infty} r^{k+1}e^{-r^{\alpha}}dr \\ &= \alpha^{-1}c^{-(k+2)/\alpha}C|S_{k-1}|\,\Gamma\left(\frac{k+2}{\alpha}\right). \end{aligned} \tag{77}$$

Here $|S_{k-1}| = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere in $\mathbb{R}^k$. Dividing (76) by (77), we obtain that

$$c = c_{\alpha,k} = \left(\frac{\Gamma\left(\frac{k+2}{\alpha}\right)}{k\Gamma\left(\frac{k}{\alpha}\right)}\right)^{\alpha/2}. \tag{78}$$

(If $\alpha = 2$, this gives $c = 1/2$, as it of course should.) Substituting this in (76) then gives

$$\begin{aligned} C &= \frac{\alpha c^{k/\alpha}\Gamma\left(\frac{k}{2}\right)}{2\pi^{k/2}\Gamma\left(\frac{k}{\alpha}\right)} \\ &= \frac{\alpha}{2\pi^{k/2}}\left(\frac{\Gamma\left(\frac{k+2}{\alpha}\right)}{k\Gamma\left(\frac{k}{\alpha}\right)}\right)^{k/2}\frac{\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{k}{\alpha}\right)}. \end{aligned} \tag{79}$$

(Check: for $\alpha = 2$ this yields the normalization constant of the normal distribution, $(2\pi)^{-k/2}$.)

Appendix B. Approximating non-linear VaR by quadratic VaR

It is commonly believed that for small time-windows $[0, t]$, $\mathrm{VaR}_p^{\Pi_t}$ is well-approximated by $\mathrm{VaR}_p^{\Gamma_t}$. To our knowledge, however, there is not yet a theorem available establishing this formally. We will fill this apparent gap in the literature by proving the following general result for Δ-hedged portfolios, which will cover the situation considered in the present paper.

Theorem B.1. *Suppose that the risk factors $\mathbb{X}_t$ follow an elliptical distribution $E(0, \mathbb{V}_t, \phi)$ having finite third moments, and variance-covariance matrix $\mathbb{V}_t$ linear (or approximately linear) in t. Let $\Pi(x, t)$ be a non-linear Δ-hedged portfolio satisfying*

$$\max_{|\alpha|=3} \sup_{(x,t)} \left|\partial_{x,t}^{\alpha}\Pi(x,t)\right| < \infty \ . \tag{80}$$

Suppose that p is such that $\mathrm{VaR}_p^{\Gamma_t} > 0$ for all sufficiently small t, $t \leq t_0$. Then for any $\varepsilon > 0$,

$$\limsup_{t\to 0} \frac{\mathrm{VaR}_{p+\varepsilon}^{\Pi_t}}{\mathrm{VaR}_p^{\Gamma_t}} \leq 1 \leq \liminf_{t\to 0} \frac{\mathrm{VaR}_{p-\varepsilon}^{\Pi_t}}{\mathrm{VaR}_p^{\Gamma_t}}. \tag{81}$$

Remarks B.2. (i) We do not suppose that either $\mathbb{X}_t$ or any of the two portfolios concerned have a continuous, let alone differentiable, probability distribution function. This theorem therefore also applies to situations where X_t could have jump components, e.g. for applications to credit risk, or where the underlying $E(0, I, \phi)$-distribution is an infinitely divisible Lévy distribution.

(ii) Also, the hypothesis of having elliptically distributed $\mathbb{X}_t$ is, in itself, not crucial: what will be important is that $\mathbb{X}_1 =_d \sqrt{t}\mathbb{X}_1$ which, in our case, follows from the fact that elliptic distributions having the same ϕ are uniquely distinguished by their variance-covariance matrices and their means.

The proof of this theorem will be based on the following elementary lemma.

Lemma B.3. *Let X and Y be two real-valued random variables, with cumulative distribution functions F_X and F_Y respectively, and let $\overline{F}_Y := 1 - F_Y$. Then, for any λ with $0 < \lambda < 1$, we have:*

$$F_X(x/\lambda) - \overline{F}(-(1-\lambda)x/\lambda) \leq F_{X+Y}(x) \leq F_X(\lambda x) + F_Y((1-\lambda)x) \,. \tag{82}$$

Proof. The right and inequality is an immediate consequence of $\{X + Y \leq x\} \subseteq \{X \leq \lambda x\} \cup \{Y \leq (1-\lambda)x\}$. To prove the other inequality, write X as $X = (X+Y) - Y$. Then by what we just proved, $F_X(x) \leq F_{X+Y}(\lambda x) + F_{-Y}\left((1-\lambda)x\right)$, or

$$F_{X+Y}(\lambda x) \geq F_X(x) - F_{-Y}\left((1-\lambda)x\right).$$

Replacing x by x/λ and observing that $F_{-Y}(y) = \overline{F}_Y(-y)$, the lemma follows. QED

Proof of theorem B.1. If we do a second order Taylor expansion of $\Pi(x,t)$, then

$$\Pi(x,t) = \Theta t + \frac{1}{2} x \Gamma x^t + R(x,t),$$

with

$$R(x,t) = \sum_{j=1}^{n} \partial^2_{x_j,t} \Pi(0,0) + \frac{1}{2}\partial_t^2 \Pi(0,0) + \sum_{|\alpha=3|} \frac{1}{\alpha!} \partial^\alpha_{(x,t)} \Pi(\theta_{x,t}x, \theta_{x,t}t),$$

where $\theta_{(x,t)} \in (0,1)$. By assumption (80), if $t > 0$ stays bounded,

$$\begin{aligned} |R(x,t)| &\leq C\left(|x|t + t^2 + (|x|+t)^3\right) \\ &\leq C(t^2 + |x|t + |x|^3), \end{aligned}$$

where C is the usual type of generic constant whose numerical value may differ from line to line. If we let

$$R_t := R(\mathbb{X}_t, t),$$

then it follows that

$$\begin{aligned} \mathbb{E}(|R_t|) &\leq C\left(t^2 + t\mathbb{E}(|\mathbb{X}_t|) + \mathbb{E}(|\mathbb{X}_t|^3)\right) \\ &\leq C t^{3/2}, \end{aligned} \tag{83}$$

since $\mathbb{X}_t \sim E(0, \mathbb{V}_t, \phi)$ with $\mathbb{V}_t = t\mathbb{V}_1$. We now apply lemma B.3 with $X = \Theta t + \frac{1}{2}\mathbb{X}_t \Gamma \mathbb{X}_t^t$, and $Y = R_t = \Pi_t - X$. Then for any $V \in \mathbb{R}_+^*$,

$$\begin{aligned} F_{\Pi_t}(V) &\leq F_{\Gamma_t}(\lambda V) + F_{R_t}\left((1-\lambda)V\right) \\ &\leq F_{\Gamma_t}(\lambda V) + \frac{\mathbb{E}(|R_t|)}{(1-\lambda)V} \\ &\leq F_{\Gamma_t}(\lambda V) + \frac{Ct^{3/2}}{(1-\lambda)V} \end{aligned} \tag{84}$$

where we used Chebyshev's inequality and (83). If what follows, $q_X(p) := \inf\{x : F_X(x) \geq V\}$ will be the p-th quantile of a random variable X (so that $\mathrm{VaR}_p^X = -q_X(p)$). By hypothesis, $q_{\Gamma_t}(p)$ is strictly negative, and in particular non-zero, if $t \leq t_0$. If we now take $V = \lambda^{-1} q_{\Gamma_t}(p) - \eta |q_{\Gamma_t}(p)| = (\lambda^{-1} + \eta) q_{\Gamma_t}(p)$ with $\eta > 0$ arbitrary, then it follows from (84) and the definition of the p-th quantile of Γ_t, that

$$F_{\Pi_t}\left((\lambda^{-1}+\eta) q_{\Gamma_t}(p)\right) < p + C \frac{t^{3/2}}{(1-\lambda)(\lambda^{-1}+\eta)|q_{\Gamma_t}(p)|}.$$

(If F_{Γ_t} were continuous we could simply take $\eta = 0$, but this would not make much difference for the remainder of the proof). It follows that

$$q_{\Pi_t}\left(p + C\frac{t^{3/2}}{(1-\lambda)(\lambda^{-1}+\eta)|q_{\Gamma_t}(p)|}\right) > (\lambda^{-1}+\eta)q_{\Gamma_t}(p). \tag{85}$$

We now observe that, by the linearity of $\mathbb{V}_t$, $\Theta t + \mathbb{X}_t\Gamma\mathbb{X}_t^t/2$ is equidistributed with $t(\Theta + \mathbb{X}_1\Gamma\mathbb{X}_1^t/2)$, and therefore $q_{\Gamma_t}(p) = tq_{\Gamma_1}(p)$, with the quantile on the right independent of t. Hence, (85) implies that

$$q_{\Pi_t}\left(p + Ct^{1/2}\right) \geq \lambda^{-1}q_{\Gamma_t}(p) - \eta,$$

for some new constant $C = C(\lambda, \eta, p)$. Now take t sufficiently small, so that $Ct^{1/2} < \varepsilon$. Then for such t,

$$q_{\Pi_t}\left(p + \varepsilon\right) \geq (\lambda^{-1}+\eta)q_{\Gamma_t}(p),$$

since q_X is a non-decreasing function. Multiplying both sides of the inequality above by -1, dividing and letting $t \to 0$, we conclude that,

$$\limsup_{t\to 0}\frac{|\mathrm{VaR}_{p+\varepsilon}^{\Pi_t}|}{|\mathrm{VaR}_p^{\Gamma_t}} \leq \lambda^{-1}+\eta,$$

for any positive $\lambda < 1$ and η. Letting $\lambda \to 1$ and $\eta \to 0$, we find the first half of the statement of the theorem.

To prove the other half of the theorem, the lower bound in lemma B.3 and Chebyshev's inequality imply that

$$\begin{aligned}
F_{\Pi_t}(V) &\geq F_{\Gamma_t}(\lambda^{-1}V) - \overline{F}_{R_t}\left(-\lambda^{-1}(1-\lambda)V\right)\\
&\geq F_{\Gamma_t}(\lambda^{-1}V) - \lambda\frac{\mathbb{E}(|R_t|)}{(1-\lambda)|V|}\\
&\geq F_{\Gamma_t}(\lambda^{-1}V) - Ct^{-3/2}|V|^{-1},
\end{aligned}$$

with a λ-dependent constant, by (83) again. Assuming $t \leq t_0$, so that $q_{\Gamma_t}(p) < 0$, we now take $V = \lambda q_{\Gamma_t}(p) + \eta|q_{\Gamma_t}(p)| = (\lambda-\eta)q_{\Gamma_t}(p)$ with $0 < \eta < \lambda$ arbitrary. It follows that

$$q_{\Pi_t}\left(p - Ct^{-3/2}(\lambda-\eta)|q_{\Gamma_t}(p)|^{-1}\right) \leq (\lambda-\eta)q_{\Gamma_t}(p).$$

Using, as before, that q_{Γ_t} is linear in t, we find that for any $\varepsilon > 0$ and sufficiently small $t < t(\varepsilon)$, we have that $q_{\Pi_t}(p-\varepsilon) \leq (\lambda-\eta)q_{\Gamma_t}$. This implies that

$$\liminf_{t\to 0}\frac{\mathrm{VaR}_{p-\varepsilon}^{\Pi_t}}{\mathrm{VaR}_p^{\Gamma_t}} \geq (\lambda-\eta),$$

which yields the second half of (81), after letting $\delta \to 1$ and $\eta \to 0$. QED

It is natural to ask wether one can do better, under additional hypotheses on the cumulative distribution functions on Π_t and Γ_t. For example, if we could take $\varepsilon = 0$ in (81), the conclusion could immediately be strengthened to

$$\lim_{t\to 0} \frac{\mathrm{VaR}_p^{\Pi_t}}{\mathrm{VaR}_p^{\Gamma_t}} = 1. \tag{86}$$

We conjecture this is possible if F_{Π_t} and F_{Γ_t} are continuous. Assuming this to be true, and assuming that Γ_t possesses a continuous probability density, we can sharpen theorem B.1 by giving a rate of convergence.

Theorem B.4. *With the same hypotheses as in theorem B.1, suppose moreover that F_{Π_t} is strictly increasing and continuous, and that F_{Γ_t} is continuously differentiable and strictly increasing also. Then*

$$\left| \frac{\mathrm{VaR}_p^{\Pi_t}}{\mathrm{VaR}_p^{\Gamma_t}} - 1 \right| \leq C\sqrt{t}. \tag{87}$$

Proof. By similar arguments as in the proof of theorem B.1, but interchanging the rôles of Π_t and Γ_t, one shows that

$$p - C\frac{t^{3/2}}{|q_{\Pi_t}(p)|} \leq F_{\Gamma_t}\left(q_{\Pi_t}(p)\right) \leq p + C\frac{t^{3/2}}{|q_{\Pi_t}(p)|}.$$

Applying $F_{\Gamma_t}^{-1}$, which is an increasing function, we find that

$$F_{\Gamma_t}^{-1}\left(p - C\frac{t^{3/2}}{|q_{\Pi_t}(p)|}\right) \leq q_{\Pi_t}(p) \leq F_{\Gamma_t}^{-1}\left(p + C\frac{t^{3/2}}{|q_{\Pi_t}(p)|}\right). \tag{88}$$

By Taylor's formula and the inverse function theorem,

$$F_{\Gamma_t}^{-1}\left(p \pm C\frac{t^{3/2}}{|q_{\Pi_t}(p)|}\right) \simeq q_{\Gamma_t}(p) \pm \frac{C}{F'_{\Gamma_t}\left(q_{\Gamma_t}(p)\right)}\frac{t^{3/2}}{|q_{\Pi_t}(p)|}. \tag{89}$$

Now since $\Theta t + \frac{1}{2}\mathbb{X}_t\Gamma\mathbb{X}_t^t =_d t\left(\Theta + \frac{1}{2}\mathbb{X}_1\Gamma\mathbb{X}_1^t\right)$, it follows that $F_{\Gamma_t}(x) = F_{\Gamma_1}(x/t)$ and, as we already observed before, $q_{\Gamma_t}(p) = tq_{\Gamma_1}(p)$. Hence

$$F'_{\Gamma_t}\left(q_{\Gamma_t}(p)\right) = t^{-1}F'_{\Gamma_1}\left(q_{\Gamma_1}(p)\right),$$

and

$$\frac{t^{3/2}}{|q_{\Pi_t}(p)|} = \frac{t^{3/2}}{|q_{\Gamma_t}|}\,\frac{|q_{\Gamma_t}(p)|}{|q_{\Pi_t}(p)} \leq C\frac{t^{1/2}}{|q_{\Gamma_1}|}$$

where we used (86). Using this, (89) and (88) and the fact that $q_{\Gamma_1}(p) \neq 0$, imply that, with some suitable constant $C > 0$,

$$|q_{\Pi_t}(p) - q_{\Gamma_t}(p)| \leq Ct^{3/2}|q_{\Gamma_1}(p)| = Ct^{1/2}|q_{\Gamma_t}(p)|$$

for some suitable constant $C > 0$, which implies the theorem. QED

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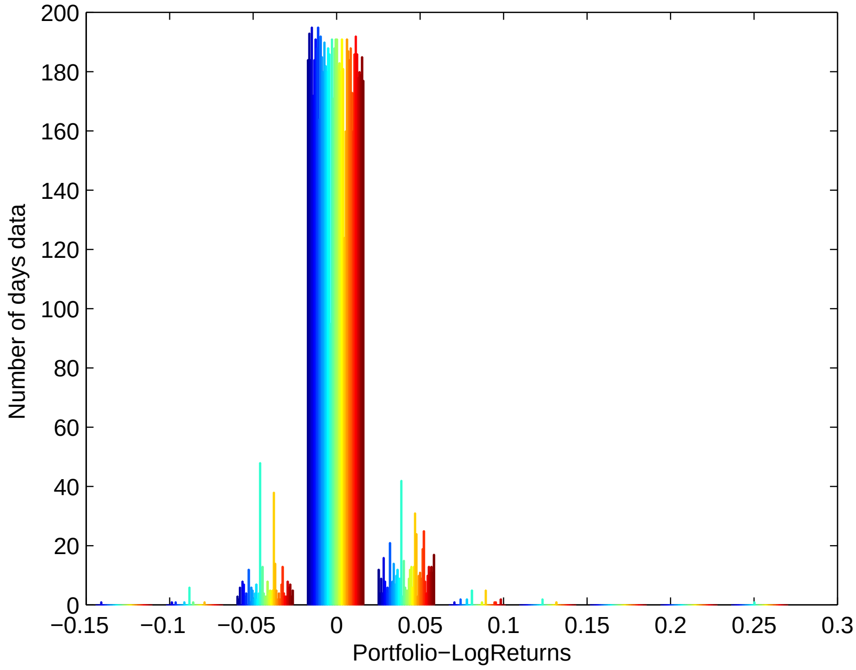

FIGURE 1. Histogram of 199 daily log-returns of 45 stocks (CAC 40)

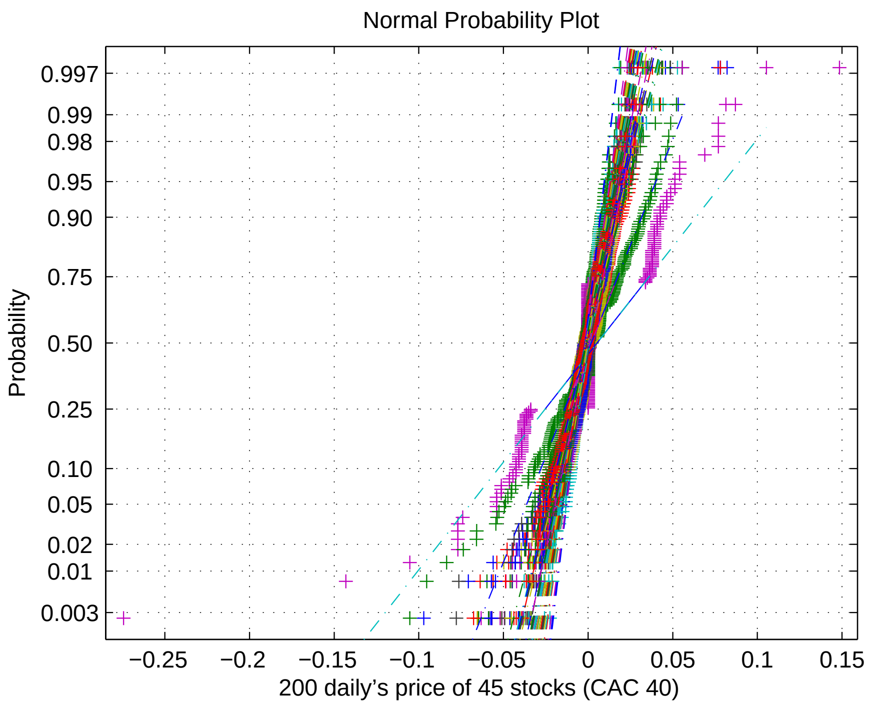

FIGURE 2. Normalplot of 199 daily log-returns of 45 stocks (CAC 40)

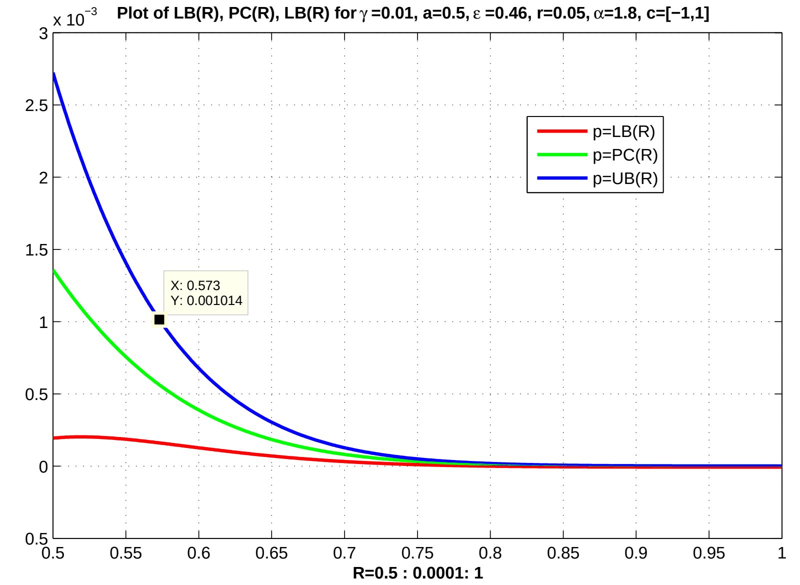

FIGURE 3. Graph of fonctions $p = F_{\Gamma,PC}(R) - \mathcal{E}_L(R) = LB(R)$, $p = F_{\Gamma,PC}(R)$, $p = F_{\Gamma,PC}(R) + \mathcal{E}_U(R) = UB(R)$, when $\alpha = 1.8$, for a given CAC 40 Δ-hedged portfolio.

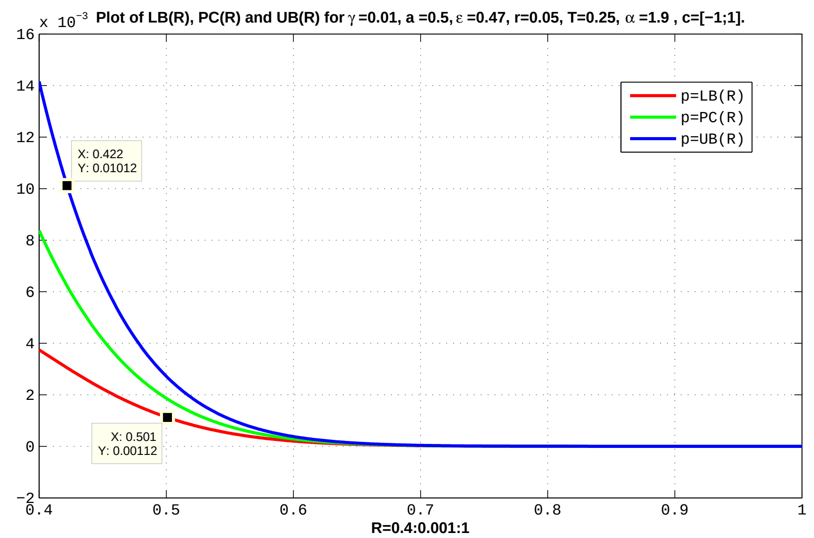

FIGURE 4. Graph of functions $p = F_{\Gamma,PC} - \mathcal{E}_L(R) = LB(R)$, $p = F_{\Gamma,PC}(R)$, $p = F_{\Gamma,PC} + \mathcal{E}_U(R) = UB(R)$, when $\alpha = 1.9$, for a given CAC 40 Δ-hedged portfolio.

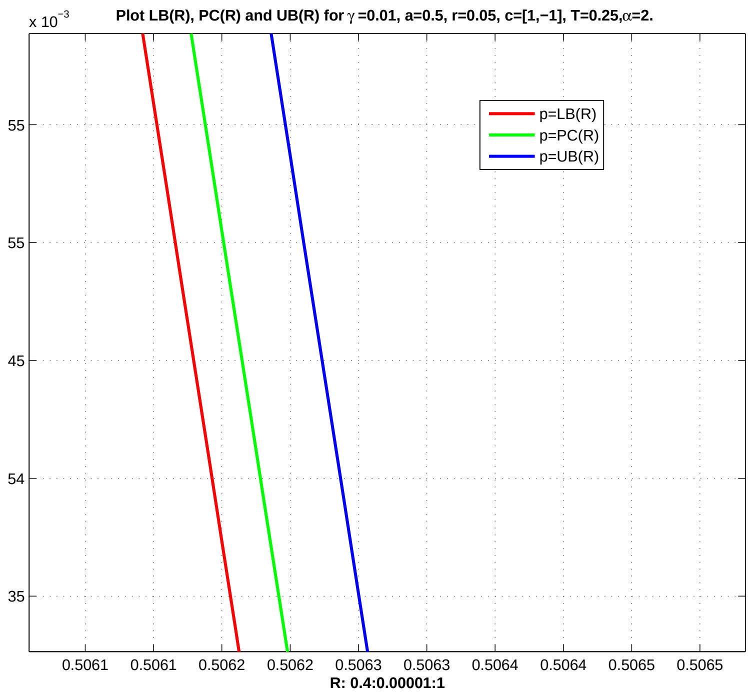

FIGURE 5. Graph of functions $p = F_{\Gamma,PC}(R) - \mathcal{E}_L(R) = LB(R)$, $p = F_{\Gamma,PC}(R)$, $p = F_{\Gamma,PC}(R) + \mathcal{E}_U(R) = UB(R)$, when $\alpha = 2$, for a given CAC 40 Δ-hedged portfolio.

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