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How to use Rosen's normalised equilibrium to implement a constrained Pareto-efficient solution

Jacek B. Krawczyk* Mabel Tidball†

Abstract

We consider a situation, in which a regulator believes that constraining a good, created jointly by competitive agents, is socially desirable. Individual levels of outputs, which generate the constrained amount, can be computed as a Pareto-efficient solution of the agents' joint utility maximisation problem. However, generically, a Pareto-efficient solution is not an equilibrium. We suggest the regulator should calculate a Nash-Rosen coupled-constraint equilibrium (or a "generalised" Nash equilibrium) and use the coupled-constraint Lagrange multiplier to formulate a threat, under which the agents would play a *decoupled* Nash game. An equilibrium of this game will possibly coincide with the Pareto-efficient solution. We focus on situations when the constraint is saturated, and examine under which conditions a match between an equilibrium and a Pareto solution is possible. We illustrate our findings using a model for a *coordination* problem, in which firms' outputs depend on each other and where the output levels are important for the regulator.

Keywords: Coupled constraints; generalised Nash equilibrium; Pareto-efficient solution
JEL: C6, C7, D7

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1 Introduction

The aim of this paper[1] is twofold. First, we want to formulate sufficient conditions, under which a two-player concave game with coupled constraints has an equilibrium à la Rosen (see Rosen (1965)) that is also a Pareto-optimal solution of a centralised problem. In that we apply Aumann's idea of *game engineering*, see Aumann, R. J. (2008) *i.e.*, we want to construct a game whose equilibrium coincides with a prescribed outcome. Secondly, we want to illustrate this result by analysing a stylised real-life game where two competitive players contribute to a public good.

We also want to contribute to *welfare economics*. We contend that our work is inspired by the *second welfare theorem*[2]. As known, lump-sum transfers that are required for the implementation of an efficient outcome, are difficult to arrange. Also, economies are often composed of agents that exert externalities on each other and their outputs are subjected to constraints imposed externally (*e.g.*, by social or environmental pressure groups). None of the above features has been included in the welfare theorems' specification. Our results, obtained for a two-agent economy, overcome these shortcomings: (1) the agents are allowed to interact through externalities (including negative), (2) the regulator seeks an efficient outcome *subject to* a constraint and (3) an efficient outcome is obtained through the threat of *nominal* (never collected) taxes.

More specifically, we are interested in situations, in which a regulator wants to control competitive agents so that their jointly created externality satisfies constraints. Typically, the constraints correspond to exogenous standards and concern the combined strategy space of all players.. The constraints can be imposed from above (*e.g.*, on pollution emitted by a cluster of pulp mills, see *e.g.*, Haurie and Krawczyk (1997) or Krawczyk (2005) or by thermal generators, see *e.g.*, Contreras et al. (2007)), or from below *e.g.*, on the amount of some public good like transportation capacity or hospital beds, available to a local population. The former constraints concern a negative externality, which needs to be restricted; the latter are imposed to satisfy a level deemed necessary.

Also, problems involving competition for a scarce resource, demanded by independent operators that depend on some legislation (like private fishermen operating

[1]This paper draws from and extends Krawczyk and Tidball (2009).

[2]See *e.g.* Groves, T. and Ledyard, J. (1977). Briefly, the two welfare theorems assert that under suitable conditions: (i) every competitive allocation is Pareto-efficient and (ii) every Pareto-efficient allocation is competitive for some distribution of endownements, realisable through transfers.

on a fishery or internet users logging in to a server, see *e.g.*, Kesselman et al. (2005)) can be analysed using the same framework and are thus of interest to us.

In all these situations the joint restrictions are likely to be saturated. This might be because agents' individually optimal solutions limit the amount of the positive externality, because not emitting negative externality is costly or because the contested resources are scarce. Notice that if the constraints had some slack, there would be "no problem" for the regulator[3]. In this paper, we will assume that the regulator is dealing with the "interesting" case of saturated constraints and wishes to know how to apportion the responsibility for the constraints' satisfaction among the agents so that an equilibrium is achieved.

Individual levels of outputs (and inputs) that generate the desired amount of the externality can be computed as a Pareto-efficient *constrained* solution to the agents' joint utility maximisation problem. The regulator might use an arbitrary weight $\alpha \in (0, 1)$ to balance the agents' utility functions or seek $\hat{\alpha}$ that maximises the sum.

As known, generically, a Pareto-efficient solution is not an equilibrium hence not self-enforcing, *thus* of problematic use in a competitive environment. We suggest the regulator calculates a Rosen coupled-constraint equilibrium (Nash normalised), see *e.g.*, Rosen (1965) or Krawczyk (2005), (or a "generalised" Nash equilibrium as this type of equilibrium is called in *e.g.*, Pang and Fukushima (2005)) and uses the coupled-constraint Lagrange multiplier to formulate a threat, under which the agents will play a *decoupled* Nash game. An equilibrium of this game will possibly coincide with the desired Pareto-efficient *constrained* solution. If so, the Pareto outcome will be achieved as a Nash equilibrium, hence self-enforcing.

In the paper, we examine, under which conditions a match between those two solutions is possible. This is the line of research started in Tidball and Zaccour (2005). Here, we generalise the results obtained in the above paper for an environmental problem. We illustrate our findings using a model for a *coordination problem*, in which firms' outputs depend on each other and where the output levels are important for the regulator.

The model considered in this paper is deterministic and information is "symmetric". We notice that should any of these assumptions not be satisfied, the regulator might assign penalty functions that would prompt the agents to produce the externality amounts that are not optimal for the agents. If so, they would start trading out the excess amounts until they became individually optimal. However,

[3]Unless the regulator would like to improve welfare from a status quo situation.

such a trading problem surpasses the scope of this paper.

What follows is a brief outline of what this paper contains. In Section 2, we describe a model in which two firm outputs are coordinated by a regulator. This motivates our search for a map between Patero-efficient solutions and competitive equilibria. In Sections 3 and 4, we develop the solution concepts for the coordination problem and revise the mathematics needed for the uniqueness of equilibrium. We develop the sufficient conditions for the map's existence in Section 5. We apply these conditions to the motivating example in Section 6. The concluding remarks summarise our findings, which include a socio-economic interpretation of the results.

2 A "public" good delivery

The mathematics of the model described below is taken from the seminal paper by Rosen (1965). The interpretations and intuitions are ours.

2.1 A model

Consider two competitive agents whose outputs are $x_1 \geq 0$ and $x_2 \geq 0$, respectively. Maintaining the outputs is expensive; the cost function of the first agent is $\frac{x_1^2}{2}$ and x_2^2 of the second.

The revenue of the first agent can only be created using the second agent's output (*positive* externality) and, in this case, it equals $x_1 x_2$. However, the goods produced by the second agent "suffer" when are utilised by the first agent (*negative* externality) so, the revenue of the second agent is $-x_1 x_2$.

In absence of regulation, the unique equilibrium of the Nash-Cournot game

$$\left.\begin{array}{rcl}\max\limits_{x_1}\Big(\phi_1(x) & = & -\frac{1}{2}x_1^2 + x_1 x_2\Big)\\[2ex] \max\limits_{x_2}\Big(\phi_2(x) & = & -x_2^2 - x_1 x_2\Big)\\[2ex] g_1(x) & = & x_1 \geq 0\\[2ex] g_2(x) & = & x_2 \geq 0\end{array}\right\} \qquad (1)$$

"played" among the agents is (0,0).

However, maintaining some positive combination of levels x_1 and x_2 may be important for the regulator. Mathematically, the regulator may want the outputs to satisfy

$$h(x) = x_1 + x_2 - 1 \geq 0\,. \tag{2}$$

Throughout this paper $\phi_i(x)$, $i = 1, 2$ are assumed continuous in all arguments and concave in x_i. The common constraint $h(\cdot)$ will then be assumed such that the constraint set

$$x \in X \equiv \{(x_1, x_2)\,:\, x_1 \geq 0, x_2(x) \geq 0, h(x) \geq 0\} \tag{3}$$

is a convex, closed and bounded subset of $\mathbb{R}^2$.

The problem of how to entice the agents to satisfy constraint (2) boils down into two subproblems:

a. What should be the levels of x_1 and x_2?

b. What should the regulator do to induce the players to choose these levels?

Briefly, the levels x_1 and x_2 can be established as a (constrained) Pareto-efficient solution and implemented as Rosen (Nash normalised) equilibrium of a *decoupled* game (to be defined). In the rest of this paper we study the mathematical conditions that $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $h(\cdot)$ need to satisfy for the Pareto and Rosen solutions to exist and coincide.

2.2 Interpretations and intuitions

To focus attention we suggest that the above mathematical problem can have the following socio-economic origin.

Consider a rail network owned by a public firm and a private firm responsible for rolling stock and transportation.

Let x_1 be the tonnage of the goods transported through the network; let x_2 be the length of the tracks owned by the tracks' owner. The revenue of the transportation firm is proportional to the tonnage and to the tracks' length $\beta_1 x_1\, x_2$.

In absence of a discount price for super-large trains, perhaps due to an imperfect state of the tracks, a reasonable approximation of the cost function to the transportation firm may be $-\frac{\alpha_1 x_1^2}{2}$: the more goods to transport, the more hardware needs to be maintained.

In brief, the operator of the rolling stock has variable revenue $\beta_1 x_1 x_2$ and costs $-\frac{\alpha_1 x_1^2}{2}$ whose combination approximates the firm's profit $\phi_1(x)$.

The public firm operating the tracks is paid a fixed amount, which is normalised to zero. The costs of maintaining the tracks at level x_2 is $\beta_2 x_1 x_2 + \alpha_2 x_2^2$ (where the first term is motivated by the destruction caused by tonnage x_1). Hence $\phi_2(x) = -\alpha_2 x_2^2 - \beta_2 x_1 x_2$.

For social reasons, the government wants transportation activity $\gamma_1 x_1 + \gamma_2 x_2$ to be above level M. This can be written as $\gamma_1 x_1 + \gamma_2 x_2 - M \geq 0$.

The above provides motivation for model (1), (2), in which $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \gamma_2 = M = 1$.

3 Solution concepts

3.1 Pareto-efficient constrained solutions

We will establish a solution to question (a.) from Section 2.1 *i.e.*, we will compute what output levels the regulator may want the agents to produce.

Consider model (1), (2).

The regulator is typically interested in a Pareto-efficient solution $\hat{x} = \big(\hat{x}_1, \hat{x}_2\big)$ *i.e.*, such that

$$\text{if } \phi_i(x_1, \hat{x}_2) > \phi_i(\hat{x}_1, \hat{x}_2) \text{ then } \phi_{-i}(x_1, \hat{x}_2) < \phi_{-i}(\hat{x}_1, \hat{x}_2) \;\; i = 1, 2 \tag{4}$$

where the subindex $_{-i}$ indicates the player.

If $\alpha \in (0, 1)$ and $\phi_1(x_1, \cdot)$ and $\phi_2(\cdot, x_2)$ are concave, differentiable and $x \in X$ then

$$\hat{x} = \big(\hat{x}_1, \hat{x}_2\big) = \arg\max_{x \in X} \{\alpha\phi_1(x) + (1 - \alpha)\phi_2(x)\} \; . \tag{5}$$

To stress that $\hat{x}$ depends on α we will write $\hat{x}(\alpha) = \big(\hat{x}_1(\alpha), \hat{x}_2(\alpha)\big)$. Notice that the larger α, the more importance that regulator assigns to the payoff of the first player. We can say that α is a marginal rate of substitution between the two players' payoffs.

If there is no particular reason for the regulator to prefer one specific value of α, solving the following problem will deliver the "best" $\hat{\alpha}$ and the best output pair $\big(\hat{x}_1(\hat{\alpha}), \hat{x}_2(\hat{\alpha})\big)$

$$\hat{\alpha} = \arg\max_{\alpha \in (0,1)} \{\alpha\phi(\hat{x}_1(\alpha), \hat{x}_2(\alpha)) + (1 - \alpha)\phi(\hat{x}_1(\alpha), \hat{x}_2(\alpha))\} \; . \tag{6}$$

In this paper we assume that the regulator is interested not (only) in $\hat{\alpha}$ and the corresponding outputs and payoffs but (also) in the full array of Pareto-efficient solutions $\big(\hat{x}_1(\alpha), \hat{x}_2(\alpha)\big)$ and the corresponding payoffs $\phi_i(\hat{x}_1(\alpha), \hat{x}_2(\alpha))$, $i = 1, 2$. This helps answer question (a.) from Section 2.1.

3.2 Coupled constraints equilibria

Rosen Rosen (1965) introduced *coupled constraints equilibrium* (CCE), also known as *generalised Nash equilibrium* (see *e.g.*, Harker (1991) or Pang and Fukushima (2005)) for games with constraints in the combined strategy space of all agents. In these games, the regulator may seek a solution that can be adopted by competitive players and such that guarantees fulfilment of the constraints, which depend on the actions undertaken by all agents. For the history of this solution concept and examples of use see Rosen (1965), Haurie (1994), Haurie and Krawczyk (1997), Krawczyk (2007) and Drouet et al. (2008); also Pang and Fukushima (2005).

We will exploit the CCE politico-economic appeal. Once the regulator establishes a desired CCE (explained below), the equilibrium implementation is straightforward. The equilibrium Karush-Kuhn-Tucker multipliers associated with the joint constraints need be used as penalty tax rates for the constraints' violation and the players have to allow for these penalties in their payoffs. The players will then "play" a *decoupled* game whose solution coincides with the desired equilibrium.

Knowledge that a CCE exists and is *unique* is crucial for the above enticement mechanism. It suffices to say that without the equilibrium uniqueness, the tax effectiveness could not be established. However, in general, there is a plethora of equilibria when *joint* actions of the players are restricted.

Rosen Rosen (1965) allows for a discriminatory treatment of players through the introduction of *weights* $r_i > 0$, $i = 1, 2$, with which the regulator can appraise each agent's payoff (*e.g.*, from a view point of the community). On the other hand, the weights help control which equilibrium is established. This is so because, given sufficient concavity of the payoffs, an equilibrium that corresponds to a particular $r = [r_1, r_2]$ is unique. (We notice that one of them may be the Pareto-efficient solution $\big(\hat{x}_1(\alpha), \hat{x}_2(\alpha)\big)$.)

The main role of the weighs in controlling the agents' behaviour is that they can modify the Karush-Kuhn-Tucker multipliers and adjust the tax rates among players to entice them to choose actions that lead to a desired equilibrium outcome.

Below we will review the mathematics of CCE and its implementation, including the uniqueness conditions; for details see Rosen (1965), Haurie (1994) or Krawczyk (2007).

4 Existence and uniqueness of coupled constraints equilibrium

We will adopt the literature results to the two-person game (1) with one joint constraint (2). For the proofs see Rosen (1965).

4.1 Introductory remarks

The solution to game (1) with the joint constraint (2) can be written as

$$x^* = \text{equil}_{y_i|x^*_{-i}\in X} \{\phi_1(x), \phi_2(x)\} \ , \tag{7}$$

which means that $\phi_i(x^*), i = 1, 2$ satisfy

$$\phi_i(x^*) = \max_{y_i|x^*_{-i}\in X} \phi_i(y_i|x^*_{-i}), \ \ i = 1, 2 \tag{8}$$

where $y_i|x_{-i} \equiv (y_i, x_{-i})$ denotes a collection of actions when the i-th agent "tries" y_i while the other agent is playing $x_{-i}, i = 1, 2$.

At x^* no player can improve their own payoff by a unilateral change in his (or her) strategy, which keeps the combined vector in $X \subset \mathbb{R}^2$. In general, the strategy set X is assumed a convex, closed and bounded subset of $\mathbb{R}^2$ and $X \equiv \{(x_1, x_2) \, : \, x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq M\}$, as in (3).

Game (7) shall be called a *coupled constraints game* (à la Rosen, see Rosen (1965)). The *coupling* refers to the fact that one player's action affects what the other players' actions can be. In the special case where $X = X_1 \times X_2$ *i.e.*, each player's action is individually constrained, the game is said to have *uncoupled* constraints.

If each payoff function ϕ_i is multiplied by weight $r_i > 0$, then

$$x^*(r) = \text{equil}_{y_i|\mathbf{x}^*_{-\mathbf{i}}\in X} \{r_1\phi_1(x), r_2\phi_2(x)\} \ , \tag{9}$$

where $r = [r_1, \, r_2] \in \mathbb{R}^2_+$. Our aim is to examine when $x^*(r)$ can match $\hat{x}(\alpha)$ for a given $\alpha \in (0, \, 1)$.

4.2 Definition

We know from Rosen (1965)[4] that an equilibrium exists and is unique if the game is *diagonally strictly concave* (*DSC*).

[4]Also see Krawczyk et al. (1998) or Krawczyk (2007) for some applications.

The economic interpretation of *DSC* is quite simple. A game that is *DSC*, is one in which each player has more control over his own payoff than the other players have over it. This is a desired, and rather common, feature of many economic models.

Mathematically, let us call $f(x, r) \equiv r_1\phi_1(x) + r_2\phi_2(x)$ the joint payoff function. A ("smooth") game is *DSC* if the so-called pseudo-Hessian of the joint payoff function (*i.e.* Jacobian of pseudo-gradient of $f(x, r)$, see *e.g.*, Rosen (1965) or Krawczyk et al. (1998)) for the given game is negative definite.

Theorem 4.1. *In a game with uncoupled constraints, if the joint payoff function $f(x, r)$ is DSC for some $r > 0$, then there exists a unique Nash equilibrium.*

When the constraints are *coupled*, there are no such guarantees, and a special type of equilibrium must be defined.

For that purpose, assume that the constraint set X is defined through (2). (In general, X needs to be defined by a collection of concave functions, see Rosen (1965) or Krawczyk (2007) and such that the constraint qualification conditions are satisfied.)

Denote the constraint shadow price vector for player i by $\lambda_i^* \geq 0$. Then, $x^* \in X$, is a *coupled constraint equilibrium* point if and only if it satisfies the following Karush-Kuhn-Tucker conditions:

$$h(x^*) \geq 0 \tag{10}$$

$$\lambda_i^* h(x^*) = 0 \tag{11}$$

$$\phi_i(x^*) \geq \phi_i(y_i|x^*) + \lambda_i^* h(y_i|x^*) \tag{12}$$

for all $i = 1, \dots n$ and where $y_i|x$ was defined in (7).

The above conditions establish a solution to (7) under the adopted differentiability and qualification assumptions. We notice that conditions (10)-(12) define x^* as a vector of non improvable strategies if $x^* \in X$, hence Nash.

In general, the multipliers[5] λ_1 and λ_2 will not be related to each other. However, we shall consider a special kind of equilibrium, which can reflect the different levels of agent responsibility for the constraint satisfaction (expressed by the vector r) and is unique.

Definition 4.1. *An equilibrium point x^* is a Rosen (Nash normalised[6]) equilibrium point if, for some vectors $r > 0$ and $\lambda^* \geq 0$, conditions (10)-(12) determine x^* and*

[5] They will be vectors if there were more coupled constraints to satisfy.

[6] "Normalisation" means in this context that both players face the same constraint shadow price λ if $r = 1$. For $r > 1$, the first player's constraint shadow price is $\dfrac{\lambda}{r}$.

are satisfied for

$$\lambda_i^* = \frac{\lambda^*}{r_i} \tag{13}$$

for each i. [7]

For shortness we have dropped *coupled constraints* from the equilibrium definition.

Now, we can better understand the role of the weights r_i. If an agent's weight r_i (see (9)) is greater than those of his (or her) competitors, then his (or her) Lagrange multipliers are lessened, relative to the competitors'. This means that the marginal cost of the constraint's violation is lower for this agent than for their competitor. Paraphrasing, the vector r tells us of how the regulator has distributed the burden of the constraints' satisfaction among the agents.

The wording of the following theorem crucial for coupled-constraint games is a bit stronger than in Rosen (1965), see Krawczyk (2002).

Theorem 4.2. *Let the weighting $\bar{r} \in Q$ be given where Q is a convex subset of $I\!R_+^2$. Let $f(x, \bar{r})$ be diagonally strictly concave on the convex set X and such that the Karush-Kuhn-Tucker multipliers exist. Then, for the weighting $\bar{r}$, there is a unique Rosen (Nash normalised) equilibrium point.*

In other words, if a game is *DSC* for a feasible distribution of the constraint's satisfaction responsibilities, then the game possesses a unique coupled constraint equilibrium for each such distribution.

4.3 Enforcement through taxation

Here we establish a solution to subproblem (b.) from Section 2.1. In particular, we explain how specific output levels, including those desired by the regulator, can be made optimal for the agents.

In Section 3.2 we mentioned a *decoupled* game that the agents play after the regulator has modified their payoffs. The decoupling means that the players decide upon their actions without the explicit knowledge of the common constraint. Instead, they know the penalty functions for the common constraint violation and incorporate them in their payoff functions.

The penalty functions $T_i(\lambda^*, r_i, x)$, $i = 1, 2$ contain the tax rate determined by the Lagrange multiplier obtained as CCE of the constrained game and the constraint

[7] We could say that λ^* are the "objective" shadow prices while λ_i^* are the "subjective" ones.

violation term

$$T_i(\lambda^*, r_i, x) = \frac{\lambda^*}{r_i} \max(0, -h(x)) \tag{14}$$

where λ^* is the Lagrange multiplier associated with the constraint and r_i is player i's weight[8] that defines their responsibility for the constraint's satisfaction.

Hence, if the weight for player i is, for example $r_i > 1$ and the weight for the other player is 1, then the responsibility of player i for the constraints' satisfaction is lessened.

The players' payoff functions, so modified, will be

$$\underline{\phi}_i(x) = \phi_i(x) - T_i(\lambda^*, r_i, x)\,. \tag{16}$$

Notice that under this taxation scheme the penalties remain "nominal" (*i.e.*, zero) if all constraints are satisfied.

The Nash equilibrium of the new unconstrained game with payoff functions $\underline{\phi}_i$ is implicitly defined by the equation

$$\underline{\phi}_i(x^{**}) = \max_{y_i \in \mathbb{R}^+} \underline{\phi}_i(y_i | x^{**}) \qquad \forall\, i, \tag{17}$$

(compare with equation (8)).

We can easily see that the equilibrium conditions for x^{**} are equivalent to (10)-(12) and conclude that $x^{**} = x^*$.(See Krawczyk and Uryasev (2000), Krawczyk (2005) or Krawczyk (2007) for a discussion and examples).

5 A relationship between Pareto-efficient solutions and Rosen's equilibria

5.1 Pareto efficiency first order conditions

Consider the regulator problem (5) of dealing with two economic agents whose outputs need to be controlled for social reasons. We repeat the mathematical model for this problem, which is:

$$\max_{x \in X} \{\alpha \phi_1(x_1, x_2) + (1 - \alpha)\phi_2(x_1, x_2)\}\ . \tag{18}$$

[8]If the weights r were identical $[1\,, 1\,, \dots 1]$ then the penalty term for the constraint is the same for each player f

$$T_i(\lambda^*, 1, x) = \lambda^* \max(0, -h(x))\,. \tag{15}$$

.

We will use $P(\cdot,\cdot)$ or simply P to refer to the contents of the curly brackets above.

As in Section 2, $\phi_i(\cdot,\cdot)$, $i = 1, 2$ are differentiable payoff functions concave in the player's own decision variable and X is a convex set of output combinations that the optimal solutions need to satisfy.

We will assume that the regulator is interested in optimal solutions that saturate the constraint $h(x_1, x_2) = 0$. (*e.g.*, because of the good's scarcity, or abundance of pollution).

The Lagrangean is:

$$L^P = \alpha\phi_1(x_1, x_2) + (1-\alpha)\phi_2(x_1, x_2) + \mu h(x_1, x_2). \tag{19}$$

The first order conditions for a Pareto optimal solution (when $h(x_1, x_2) = 0$) are:

$$\left.\begin{aligned} \frac{\partial L^P}{\partial x_1} &= \alpha\frac{\partial\phi_1(x_1, x_2)}{\partial x_1} + (1-\alpha)\frac{\partial\phi_2(x_1, x_2)}{\partial x_1} + \mu\frac{\partial h(x_1, x_2)}{\partial x_1} = 0, \\ \frac{\partial L^P}{\partial x_2} &= \alpha\frac{\partial\phi_1(x_1, x_2)}{\partial x_2} + (1-\alpha)\frac{\partial\phi_2(x_1, x_2)}{\partial x_2} + \mu\frac{\partial h(x_1, x_2)}{\partial x_2} = 0, \end{aligned}\right\} \tag{20}$$

Given concavity of the payoff functions and convexity of the constraint set, the above conditions are also sufficient for a solution $\hat{x}_1(\alpha), \hat{x}_2(\alpha)$ to (20), to be a Pareto optimal solution to (18).

5.2 Rosen's equilibrium first order conditions

It is well known that a Pareto optimal (efficient) solution *i.e.*, the pair $x_1(\alpha), x_2(\alpha)$ that solves problem (18) is not a generic Nash equilibrium. Consequently, it does not have the self-enforcing properties that the latter solution concept enjoys.

On the other hand the regulator knows from Section 4.3 (also, see Haurie and Krawczyk (1997), Krawczyk (2005) or Krawczyk (2005)) that it is possible to control competitive agents, who share a common constraint, to satisfy this constraint. This is achieved through a threat function (14), which results from a CCE. This equilibrium does possess the features of Nash equilibrium.

Mathematically, the regulator may then seek $x_1(r), x_2(r)$ that satisfy:

$$\left.\begin{aligned} &\max_{x_1} r\,\phi_1(x_1, x_2) \\ &\max_{x_2} \phi_2(x_1, x_2) \\ &h(x_1, x_2) = 0. \end{aligned}\right\} \tag{21}$$

where $r \geq 1$ is a weight, which the regulator attaches to the first player's payoff relative[9] to the second player's payoff.

The player Lagrangeans are:

$$L_1^R = r\phi_1(x_1, x_2) + \lambda h(x_1, x_2), \quad L_2^R = \phi_2(x_1, x_2) + \lambda h(x_1, x_2)\,. \tag{22}$$

Following (10)-(12), (13) and when $h(x_1, x_2) = 0$, a pair $x_1(r), x_2(r)$ is a *normalised equilibrium*, called Rosen's, of game (21) if it satisfies the following first order conditions (KKT):

$$\left.\begin{aligned} \frac{\partial L_1^R}{\partial x_1} &= r\frac{\partial \phi_1(x_1, x_2)}{\partial x_1} + \lambda\,\frac{\partial h(x_1, x_2)}{\partial x_1} = 0, \\ \frac{\partial L^P}{\partial x_2} &= \frac{\partial \phi_2(x_1, x_2)}{\partial x_2} + \lambda\,\frac{\partial h(x_1, x_2)}{\partial x_2} = 0, \end{aligned}\right\} \tag{23}$$

If the joint payoff function is *diagonally strictly concave* then the pair $x_1(r), x_2(r)$, which satisfies (23), is the unique normalised (Rosen) equilibrium of game (21), see Theorem 4.2 .

5.3 Relations between α and r

We want to find a relationship between α and r such that the solutions for the two problems (Pareto and Rosen) are identical *i.e.*, $x^*(r) = \hat{x}(\alpha)$.

Assume that

$$\mu = K\lambda. \tag{24}$$

The multipliers μ and λ need be positive so, if we find $K > 0$ that satisfies this equation then the regulator will be able to use a Rosen's equilibrium to enforce a Pareto optimal solution.

If solutions $x_1(\alpha), x_2(\alpha)$ and $x_1(r), x_2(r)$ are to be the same, then (20) and (23) imply:

$$Kr\frac{\partial \phi_1(x_1, x_2)}{\partial x_1} = \alpha\frac{\partial \phi_1(x_1, x_2)}{\partial x_1} + (1-\alpha)\frac{\partial \phi_2(x_1, x_2)}{\partial x_1}, \tag{25}$$

$$K\frac{\partial \phi_2(x_1, x_2)}{\partial x_2} = \alpha\frac{\partial \phi_1(x_1, x_2)}{\partial x_2} + (1-\alpha)\frac{\partial \phi_2(x_1, x_2)}{\partial x_2}. \tag{26}$$

[9]So, we have scaled $r_2 = 1$ and set $r = r_1$. See Appendix A for a proof that, for the case of two players, Theorem 4.2 is true when the regulator uses just one r to apprise the first player's payoff relative to the second player's, instead of using two "absolute" weights r_1 and r_2.

Conditions (25) and (26) give two equations for the two unknown K and r. Solving these equations yields

$$r(\alpha) = \frac{\frac{\partial \phi_2}{\partial x_2}}{\frac{\partial \phi_1}{\partial x_1}} \, \frac{\alpha \frac{\partial \phi_1}{\partial x_1} + (1-\alpha)\frac{\partial \phi_2}{\partial x_1}}{\alpha \frac{\partial \phi_1}{\partial x_2} + (1-\alpha)\frac{\partial \phi_2}{\partial x_2}} \tag{27}$$

and

$$K(\alpha) = \frac{\alpha \frac{\partial \phi_1}{\partial x_2} + (1-\alpha)\frac{\partial \phi_2}{\partial x_2}}{\frac{\partial \phi_2}{\partial x_2}}. \tag{28}$$

The derivatives in equations (27) and (28) are evaluated at $x_1(\alpha), x_2(\alpha)$ hence, $r = r(\alpha)$, $K = K(\alpha)$ *i.e.*, they are functions of α.

Note that the numerator $\alpha \dfrac{\partial \phi_1}{\partial x_1} + (1-\alpha)\dfrac{\partial \phi_2}{\partial x_1}$ in the expression for r (27) can be negative or zero if $\phi_2(x_1, x_2)$ decreases in x_1 (*i.e.*, if x_1 is a negative externality in the problem) and if α is small (*i.e.*, if the second player's payoff is somehow preferred).

Also, the denominator $\alpha \dfrac{\partial \phi_1}{\partial x_2} + (1-\alpha)\dfrac{\partial \phi_2}{\partial x_2}$ can be negative or zero if $\phi_1(x_1, x_2)$ decreases in x_2 (*i.e.*, if x_2 is a negative externality in a problem) and if α is large (*i.e.*, the first player's payoff is somehow preferred). It follows from the above that if there are negative externalities, then $r(\alpha)$ can have breaks in domain and attain negative values that preclude the existence of a Nash equilibrium, which could implement the desired Pareto solution. We can say that:

Theorem 5.1.
(a) For $\alpha \in (0, 1)$ such that a solution to (18) exists with $\lambda > 0$ and if $0 < r(\alpha) < \infty$ the regulator can implement a desired Pareto-efficient solution as a Rosen (Nash normalised) equilibrium. In particular, formula (27) determines the level of responsibility of the first player for the constraint satisfaction relative to the level of the second player, for a specific level of α.

For the situations when there are no negative externalities and when there are no externalities at all, we have the respective corollaries.

Corollary 5.1. *If there are no negative externalities i.e., if $\dfrac{\partial \phi_i}{\partial x_j} > 0$ then $0 < r(\alpha) < \infty$. Hence, for a given value of α, the corresponding Pareto-efficient solution can be made optimal for individual agents.*

Corollary 5.2. *If $\phi_i(x_i, x_j) = \phi_i(x_i)$, ($\phi_i$ concave en x_i), then*

- *Rosen equilibria exist for any $r > 0$. This is so, because the pseudo-hessian:*

$$2\begin{pmatrix} r\phi_1'' & 0 \\ 0 & \phi_2'' \end{pmatrix}.$$

has positive determinant and trace negative hence is negative definite. So, the game is DSC.

- *For all* $\alpha \in (0,1)$ *such that a solution to (18) exists with* $\lambda > 0$*, there exists a bijection between* $\alpha \in (0,1)$ *and* $r \in (0,\infty)$ *given by*

$$r(\alpha) = \frac{\alpha}{1-\alpha}.$$

This means that for this case, the set of Rosen equilibria coincides with the set of Pareto optima with constraints.

In the next section we continue the motivating example from Section 6 of two agents exerting *negative externalities* on each other. We will establish the values of α such that a solution to (18) exists with $\lambda > 0$ and that verifies $0 < r(\alpha) < \infty$.

6 Realisation of a public good delivery

We now analyse the public good's delivery model (1), under the delivery condition (2).

6.1 Does status quo need be modified?

The regulator needs to establish whether the solution $\overline{x}$ to the *unconstrained* game, presumably "played" at present (hence "status quo"),

$$\overline{x} = \text{equil}_{y_i|\overline{x}_{-i} \in \mathbb{R}^2} \{\phi_1(x), \phi_2(x)\} \,, \tag{29}$$

generates scarcity

$$\overline{x}_1 + \overline{x}_2 - 1 \leq 0\,. \tag{30}$$

If there is abundance of the joint good $\overline{x}_1 + \overline{x}_2 - 1 > 0$, then there is "no problem"[10] for the regulator to solve because the constraint is satisfied in a Nash equilibrium. Condition (30) implies that, in a *constrained* equilibrium, $\lambda > 0$ if such an equilibrium exists.

As we said in Section 2.1, a solution to the unconstrained game (1) is $\overline{x}_1 = 0, \overline{x}_2 = 0$. This clearly satisfies (30) hence, the regulator's problem of how to assure satisfaction of the constraint in an equilibrium is real.

[10]See footnote 3. In this paper, we assume that the regulator's main concern is the constraint satisfaction.

6.2 Which Pareto-efficiency programmes can be enforced

6.2.1 The necessary conditions

The regulator has to verify that the Pareto-efficiency programme (18) when $X = \mathbb{R}^2$, also generates a "scarce" solution *i.e.*, that $\hat{x}_1(\alpha) + \hat{x}_2(\alpha) - 1 \leq 0$. This will imply that once the constraint (2) is enforced, a *constrained* Pareto-efficient solution will be saturated hence $\mu > 0$ in (19). Consequently, $K > 0$ in (24).

The following Karush-Kuhn-Tucker conditions formulated for programme (18) with $h(x) = x_1 + x_2 - 1 \geq 0$ and $x_i \geq 0$, $i = 1, 2$ constitute the *necessary conditions* for a constrained Pareto-efficient outcome:

$$-\alpha x_1 + (2\alpha - 1)x_2 + \mu \leq 0 \tag{31}$$

$$x_1(-\alpha x_1 + (2\alpha - 1)x_2 + \mu) = 0 \tag{32}$$

$$\alpha x_1 - (1 - \alpha)(2x_2 + x_1) + \mu \leq 0 \tag{33}$$

$$x_2(\alpha x_1 - (1 - \alpha)(2x_2 + x_1) + \mu) = 0 \tag{34}$$

$$x_1 + x_2 - 1 \geq 0 \tag{35}$$

$$\mu(x_1 + x_2 - 1) = 0 \tag{36}$$

Their solution results in several threads.

First, $\mu > 0$.

a. $x_1 > 0$, $x_2 > 0$

From (32), (34), (36)

$$\mu(\alpha) = \frac{6\,\alpha - 6\,\alpha^2 - 1}{3\alpha} > 0 \quad \text{for } \alpha \in \left(\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6}\right), \tag{37}$$

or, approximately, for $\alpha \in (0.211,\ 0.789)$.

We have plotted in Figure 1 the values of the constraint's "shadow price" μ implied by the Pareto weight α, as a function of the weight. The dash-dotted (blue) line shows the values of μ, for which a constrained "interior" solution exists. Here, we mean *interior* if (x_1, x_2) belong to the inside of the segment $\{(x_1, x_2) : x_2 = -x_1 + 1,\ x_1 \in (0,\ 1)\}$. We notice that the shadow price for the interior solutions diminishes when the regulator's preferences become more definite for one of the agents (*i.e.*, α is away from the centre).

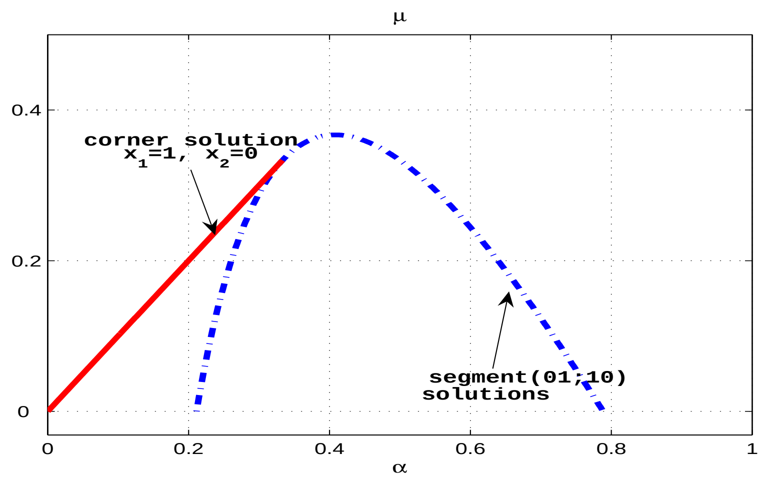

Figure 1: The shadow price in the constrained Pareto-efficiency problem as a function of α

.

The output values that maximise the Pareto program (18), under the delivery condition (2), are plotted in Figure 2 top panel. Their analytical expressions are:

$$\hat{x}_1(\alpha) = \frac{1}{3\alpha}, \qquad \hat{x}_2(\alpha) = \frac{3\alpha - 1}{3\alpha}\,. \tag{38}$$

We observe that only for $\alpha > \frac{1}{3}$ both outputs are positive, see the blue line segments in this figure. The corresponding payoffs for each player are shown in the middle panel of Figure 2. The regulator's goal value, which is a weighted sum of the agents' payoffs is shown in the bottom panel.

b. $x_1 = 0,\ x_2 > 0$

From (36) $x_2 = 1$ and from (34)

$$(1-\alpha)(-2\cdot 1) + \mu = 0 \tag{39}$$

thus $\mu = 2(1-\alpha)$. This combination of x_1, x_2, μ does not satisfy (31), hence there is no "corner" solution of the regulator's programme at $x_1 = 0,\ x_2 = 1$.

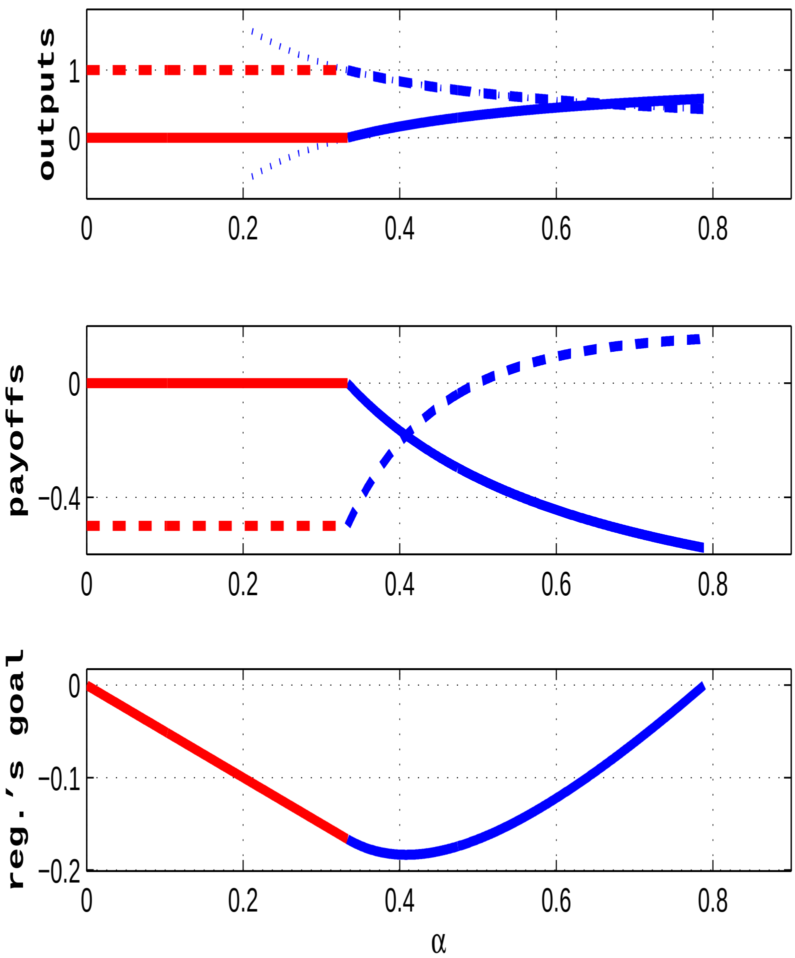

Figure 2: Pareto-efficient solutions as a function of α.

c. $x_2 = 0$, $x_1 > 0$

From (36) $x_1 = 1$ and from (32)

$$-\alpha + \mu = 0 \tag{40}$$

hence $\mu = \alpha > 0$ for any $\alpha \in (0, 1)$, see the solid (red) line in Figure 1. The KKT conditions with $\mu > 0$ are satisfied for $\alpha < \frac{1}{3}$ and we conclude that the corner solution $x_1 = 1$, $x_2 = 0$ exists for $0 < \mu = \alpha < \frac{1}{3}$.

We notice non-uniqueness of $\mu(\alpha)$ for $\alpha \in (0.211, 0.333)$ in Figure 1. However, this non-uniqueness can be resolved if the regulator requires non-negative, hence *realisable*, outputs for either agent (see (38)). If so, the solution is unique: for $\alpha \in (0,\, 0.333)\,, x_1 = 1, x_2 = 0$; for $\alpha \in (0.333,\, 0.789)$ the solution is given by (37). There though appears that there is no solution for $\alpha \in (0.789,\, 1)$.

The resulting payoffs to the players and regulator's aim values are displayed in Figure 2 in middle and bottom panels, respectively.

- For $\mu = 0$. We consider this case to examine the solution existence, or its lack, in the interior of the set $\{(x_1, x_2) \,:\, x_2 > -x_1 + 1,\, x_1 \geq 0,\, x_2 \geq 0\}$.

  a. The cases of $x_1 = 0,\, x_2 = 1$ and $x_1 = 1,\, x_2 = 0$ coincide with items (b.) and (c.) above, for $\alpha = 1$ and $\alpha = 0$, respectively.

  b. Assuming $x_1 > 0,\, x_2 > 0$ and using (32), (34) yields

$$-\alpha x_1 + (2\alpha - 1)x_2 = 0 \tag{41}$$

$$(2\alpha - 1)x_1 - 2(1 - \alpha)x_2 = 0\,. \tag{42}$$

  This system has the zero solution that contradicts (35).
  However, if $\alpha_s = \dfrac{1}{2} + \dfrac{\sqrt{3}}{6}$ then

$$\Delta = 6\alpha - 6\alpha^2 - 1 = 0 \tag{43}$$

  and we could have a singular solution

$$x_1^s = \frac{2\sqrt{3}}{3 + \sqrt{3}}\, x_2^s \tag{44}$$

  From (35),

$$\frac{2\sqrt{3}}{3 + \sqrt{3}}\, x_2^s + x_2^s = \frac{\sqrt{3} + 3}{\sqrt{3} + 1}\, x_2^s \geq 1 \tag{45}$$

  so $x_2^s \geq \dfrac{\sqrt{3} + 1}{\sqrt{3} + 3}$.

  If we use α_s in programme (18) and allow for $x_1 = \dfrac{2\sqrt{3}}{3 + \sqrt{3}}\, x_2$ then we obtain $P = 0$.

  However, we do not expect the regulator to have their preferences between firms set at *exactly* $\alpha = \alpha_s$ and will not delve into the singular solution.

6.2.2 Sufficient conditions

To establish, for which α, x_1 and x_2 the Pareto programme can be maximised we need to analyse the shape of $P(\cdot,\cdot)$ as a function of α. This can be done easily by examining Hessian and gradient of $P(\cdot,\cdot)$, both as functions of α. The formulae are available from the authors. Here, for transparency of the analysis we present 3D snapshots of $P(\cdot,\cdot)$, for three selected values of α: $0.5, 0.789$ and 0.9, see Figure 3. The feasible region $X = \{x : x_1 + x_2 \geq 1\}$ is on the right hand side of the coordinate system.

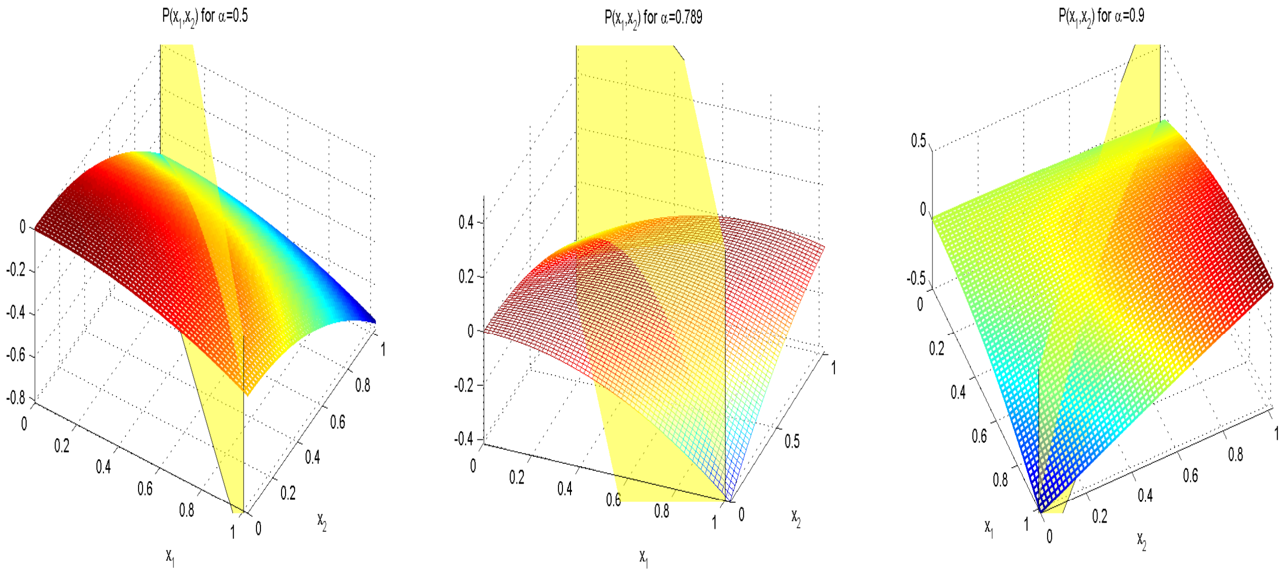

Figure 3: The shape of $P(\cdot,\cdot)$ for selected values of α $(0.5, 0.789$ and $0.9)$.

We notice that $P(\cdot,\cdot)$ is concave for all values of α. However, the location of the maximum changes. The left panel depicts the Pareto programme for $\alpha = 0.5 < 0.789$ and shows that it has a maximum in the "scarcity" region. So, the locus of the constrained maxima coincides with the constraint. (The maximum will be $x_1 = 1$, $x_2 = 0$ for $\alpha \leq 0.333$ but we do not show this figure for brevity.)

The right panel of Figure 3 depicts the Pareto programme for $\alpha = 0.9 > 0.789$. We see that the function is unbounded in the feasibility region for this value of α. We can also see in the middle panel that the programme is "symmetrical" around the region's frontier and infer that there is no finite maximum of $P(\cdot,\cdot)$ for $\alpha > 0.789$. Hence there is no unbounded solution to the Pareto programme for this interval of α. Consequently, the regulator's choices of $\alpha \in (0.789,\ 1)$ will not be implementable as coupled constraints equilibria.

We conclude that the Pareto optimal solutions for $\alpha \in (0., 0.789)$ exist and are unique. However, the solutions cannot be obtained for "large" values of α, which would have favoured the income generated by the first agent (the transportation firm).

6.3 Which coupled constraints equilibria are available

Subsequently, existence and uniqueness of a coupled constraints' equilibrium for $r > 0$ needs be established. For that purpose we compute pseudo-Hessian for the game at hand (see Section 4.2):

$$\mathcal{H} = \begin{bmatrix} -r & -\frac{1}{2} + \frac{1}{2}\,r \\ -\frac{1}{2} + \frac{1}{2}\,r & -2 \end{bmatrix} \tag{46}$$

that is negative definite for

$$\frac{5}{2}r - \frac{1}{4} - \frac{1}{4}r^2 = -r^2 + 10r - 1 > 0 \tag{47}$$

$$\textit{i.e.,} \quad 5 - 2\sqrt{6} < r < 5 + 2\sqrt{6} \quad \text{or, approximately,} \quad 0.101 < r < 9.899\,. \tag{48}$$

So, we know there exists an interval for r, for which the CCE exists and is unique.

We now compute the mapping $\alpha \rightsquigarrow r$ from (27)

$$r(\alpha) = \frac{1 - 6\,\alpha}{-2 + 3\,\alpha} \tag{49}$$

and plot it in Figure 4.

Map (49) enables us to compute the largest $\bar{\alpha}$ that corresponds to the upper end of interval (48). This is $\bar{\alpha} \approx 0.583$, the largest value of α, for which a unique equilibrium is guaranteed[11]. At the lower end of this interval is $\underline{\alpha} \approx 0.191$. We notice that $\underline{\alpha} > 0$ *i.e.*, there are $\alpha < \underline{\alpha}$, for which a Pareto solution exists. On the other hand, $\bar{\alpha} < 0.789$ *i.e.*, $\bar{\alpha}$ is below the largest α, for which the Pareto programme possesses a solution.

We observe that the interval $(\underline{\alpha}, \bar{\alpha})$ is included in $\left(\frac{1}{6}, \frac{2}{3}\right)$, for which $r(\alpha) > 0$. The intersection[12] is

$$\alpha \in (\underline{\alpha}, \bar{\alpha}) \approx (0.191, 0.583) \tag{50}$$

[11]We notice that $\mathcal{H} > 0$ is a sufficient condition for uniqueness and cannot exclude that uniqueness may be achieved for $r > 0$ from outside the above interval.

[12]Notice that $\alpha = \frac{2}{3}$ is special in that $x_1\left(\frac{2}{3}\right) = x_2\left(\frac{2}{3}\right) = \frac{1}{2}$; furthermore, for $\alpha > \frac{2}{3}$ the contribution of the second firm toward the constraint satisfaction is greater than of the first firm. However, $\alpha = \frac{2}{3} > \bar{\alpha}$ that is outside the interval, for which unique equilibria are guaranteed.

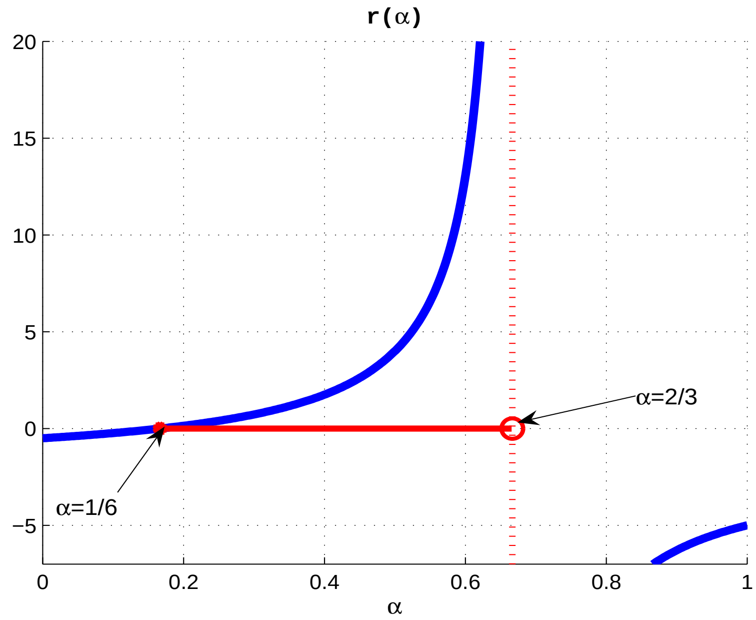

Figure 4: The map between α and r.

that defines the interval of α for which $r > 0$ and such that the uniqueness of equilibria is guaranteed.

We can see that as the regulator attaches more weight to the first firm's payoff *i.e.*, if α grows from $\underline{\alpha}$, to $\bar{\alpha}$ (*i.e.*, from 0.191 to 0.583), the preferential treatment as measured by r, which diminishes marginal cost of the constraint's violation (see (14)), becomes increasingly stronger. This appears logical: the more income from firm 1 the regulator "wants", the smaller the marginal cost this firm should face.

This enables us to see the *dual* function of weight r. On a one hand it stimulates the first firm's production by diminishing its marginal cost; on the other hand, it motivates firm 2 to produce because of the fear of punishment.

In summary, if $\alpha \in (0.191, 0.583)$, then $0.101 < r < 9.899$ and (49) defines a relationship between a Pareto solution and CCE.

Recall, a value of α "close" to 0 signifies that the second firm's payoff is of more value to the regulator than the first firm's; an α "close" to 1 means more importance attached to first firm's payoff. It becomes clear that because of the constraint (2), the regulator *cannot* prioritise the second firms' payoffs in some "extreme" fashion. However, this does not preclude existence of the "corner" solution $x_1 = 1$, $x_2 = 0$, that exists for any $\alpha \in (0.191\, 0.333)$.

We have also computed $\hat{\alpha} = \frac{\sqrt{6}}{6} \approx 0.4082 \in (0.191, \bar{\alpha})$ that *minimises* the regulator's programme (6), which is a convex function of α, see Figure 2 third panel. This suggests that the regulator might seek to implement an equilibrium that corresponds to $\bar{\alpha}$.

7 Concluding remarks

We have proved Theorem 5.1, which formulates the necessary conditions, under which a constrained Pareto-efficient solution can be supported by a coupled constraints equilibrium *à* la Rosen. Corollaries 5.1 and 5.2 provide the conditions for the situations with no negative externalities and "nil" externalities, respectively.

The above constitute the mathematical conditions for a novel approach to the solution of a politico-economic coordination problem. We have used a game theoretic framework that has allowed us to formulate this problem *naturally* as a coupled constraints equilibrium. We have illustrated how to use Theorem 5.1 to solve the problem.

We have concluded that if agents interact through positive and negative externalities, then the regulator's choices for his (or her) preferred solutions may exclude some extreme values of the marginal rate of substitution between the firms' payoffs.

Furthermore, we have obtained some quantitative results relevant to the considered example. The array of unique equilibria that can support the regulator's choices is non symmetrical with respect to $\alpha = 0.5$ and exclude solutions, in which the "public" firm's payoff would have contributed more than 58% toward the regulator's programme. On the other hand, heavy preferences of the "private" firm's payoff (*i.e.*, small α) are also excluded. This suggests that the Pareto programmes supported by coupled constraints equilibria are politically equilibrated and hence acceptable to the stockholders.

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Appendix

A Rosen's relative weights in $\mathbb{R}^2_+$

Consider a game with payoffs $\Pi_1(e), \Pi_2(e)$ that satisfy know that this game has a unique equilibrium for a choice of r_1, r_2. The equilibrium fist order conditions are

$$\left.\begin{aligned} \frac{\partial \Pi_1(e)}{\partial e_1} &= -\frac{\lambda(r_1, r_2)}{r_1} \\ \frac{\partial \Pi_2(e)}{\partial e_2} &= -\frac{\lambda(r_1, r_2)}{r_2} \end{aligned}\right\} \tag{51}$$

where $\lambda \geq 0$ is the shadow price of the common constraint of type (2).

Let us choose $r_1 = r, r \in (0, \infty)$ and $r_2 = 1$. The first order conditions (51) become now

$$\left.\begin{aligned} \frac{\partial \Pi_1(e)}{\partial e_1} &= -\frac{\lambda'(r, 1)}{r} \\ \frac{\partial \Pi_2(e)}{\partial e_2} &= -\lambda'(r, 1) \end{aligned}\right\} \tag{52}$$

where $\lambda' > 0$ is the Lagrange multiplier that corresponds to this choice or r.

We notice that conditions (51) are equivalent to (52) if

$$\left.\begin{aligned} \frac{\lambda(r_1, r_2)}{r_1} &= \frac{\lambda'(r, 1)}{r} \\ \frac{\lambda(r_1, r_2)}{r_2} &= \lambda'(r, 1) \end{aligned}\right\}. \tag{53}$$

The above is true if and only if

$$r \equiv \frac{r_1}{r_2}. \tag{54}$$

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