Economic Implications of Extreme and Rare Events

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Abstract

A central role for economic policy involves understanding and reducing the impact of unexpected, extreme events. In this paper, we develop a simple economic framework with latent regime switches. This framework explains why investors and policymakers can decide not to hedge against extreme events, even those that are exogenous, with well understood probabilities and consequences. We also examine endogenous probabilities, where the consequences are less well understood. Our most striking finding is that the benefits of sustained optimal investment are bounded and small. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if there are informational costs to learning the structure of the financial environment. We also discover that the benefits of leverage represent a large percentage of income. These results obtain both in the theoretical model and upon calibration to the last half-century of US economic experience.

Keywords: Endogenous Probabilities; Extreme Events; Financial Environment; Informational Costs; Regime Shifts

JEL Classification: E44 G01 G11 G32

1 Introduction and motivation

> "We all learn by experience, and your lesson this time is that you should never lose sight of the alternative." Sherlock Holmes: *The Adventure of Black Peter.*

Unexpected economic events can have massive, disruptive effects on a nation.[1] The experience of crises in the 1990s and 2000s has stimulated researchers' interest in understanding extreme events in the US economy. When such events occur, they tend to do so in multiple settings, which amplifies their impact.[2] For example, the collapse of a major lending institution affects many households, and can cause total insurance claims to increase geometrically, since several classes are affected, including property loss and job loss.[3] Such correlated outcomes are interesting not only for their practical relevance, but also economically, since they resemble results from a broad class of theoretical research on herding and strategic complementarities.[4]

The main goal of this paper is to develop a simple economic framework with rare extreme events, in order to understand their impact and ramifications.[5] Our model delivers insight into how individuals respond to extreme events in terms of hedging and asset demands. Interestingly, we find that agents may rationally choose to ignore information about extreme events, if this information is costly. Such a finding ties our work closely to research on rational inattention, including Wilson (1975), Sims (2003), and Veldkamp and Van Nieuwerburgh (2009). There are two other important areas of research intersection. First is the recent flurry of work on extreme events, largely in response to the economic crisis.[6] Much of this research analyzes systemic instability.[7] Second, historically there is a long literature examining financial crises and bubbles, in both rational and behavioral frameworks.[8] The quantitative models in most of these research areas focus on

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[1] For evidence on welfare costs of extreme events, see Chatterjee and Corbae (2007), Barro (2009), and the references therein.

[2] See Barro (2006) and Barro (2009). Also, see Horst and Scheinkman (2006), and Krishnamurthy (2009) for economic explanations of such amplifications.

[3] For details on insurance during periods of economic disruptions, see Jaffee and Russell (1997); Jaffee (2006); and Ibragimov, Jaffee, and Walden (2009b).

[4] See Wilson (1975); Bikhchandani, Hirschliefer, and Welch (1992); Cooper (1999); and Vives (2008), chapter 6.

[5] By extreme, we refer to events that have a high impact on the particular system. This impact can be in terms of financial or social cost, or in terms of disruption of equilibrium. By rare, we refer to events that are not observed frequently, as in Table 1.

[6] For overviews of the crisis, see Acharya and Richardson (2009); Brunnermeier (2009); Reinhart (2008); and Reinhart and Rogoff (2009).

[7] See Caballero and Krishnamurthy (2008); Ibragimov, Jaffee, and Walden (2009b); Ibragimov, Jaffee, and Walden (2009a); and Shin (2009).

[8] See Fisher (1933); Keynes (1936); Blanchard (1979); Minsky (1982); Friedman and Laibson (1989); Shleifer and Vishny (1997); Kindleberger (2000); Abreu and Brunnermeier (2003); Allen and Gale (2007); and Hong, Scheinkman, and Xiong (2008).

a stationary environment. Evidently the economic climate is subject to sudden shifts.[9] Despite the clear policy and academic relevance, little existing research examines the economic impact of regime shifts in the probability of encountering extreme events. Therefore our research fills a much-needed role, by incorporating a simple model of regime shifts in extreme events. We find that the existence of such shifts may help explain the experience of unhedged extreme events in the US economy, both theoretically and empirically.

The remainder of the paper is as follows. In Section 2 we review theoretical and empirical literature on extreme events. In Section 3 we develop and calibrate a simple model of risky choice, where extreme events undergo exogenous regime shifts. Section 4 extends this model to endogenous extreme events, and Section 5 concludes.

2   Background and related literature

The paper builds on three strands of research, related to extreme events and crises, information choice, and regime shifts. Regarding extreme events, previous research includes behavioral work such as Kunreuther and Pauly (2006), who focus on the role of individual myopia in precipitating catastrophes. It also includes research on bubbles by Abreu and Brunnermeier (2003), and Blanchard (1979), among others.[10] There is still no consensus modeling approach for analysis of extremes. A major challenge is that it is unclear how individuals behave towards extreme or low probability events. Initial evidence by Allais (1953) and Kahneman and Tversky (1979) suggested that agents overweight low-probability events. However, more recent research has uncovered three additional results. First, there is evidence that agents underweight low probability events in realistic situations where they must estimate probabilities based on experience, documented by Barron and Erev (2003), Hertwig, Barron, Weber, and Erev (2005), and Rabin (2002). Second, econometrically there is a bias to under-estimate rare events, examined by King and Zeng (2001), and de Haan and Sinha (1999). Third, expected utility does not effectively incorporate low probabilities, a phenomenon studied by Bhide (2000) and Chichilnisky (2000). The finding that agents may systematically under-estimate low probability events is particularly interesting, and suggests a systematic lack of knowledge that is not possible to address in current economic frameworks such as robust control and the theory of ambiguity aversion. These frameworks typically presume

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[9]For empirical research on regime shifts in the economy, see Hamilton (1989); Hamilton and Lin (1996); Ang and Bekaert (2002); Ang and Bekaert (2004); and Ang and Bekaert (2005). For theoretical modelling of regime shifts, see Reitz (1988); Evans (1996); Bekaert, Hodrick, and Marshall (2001); and Angeletos, Hellwig, and Pavan (2007).

[10]Other relevant research includes Jaffee (2006); Ibragimov, Jaffee, and Walden (2009b); and Lorenzoni (2008).

that agents are aware of their lack of knowledge. By contrast, the most devastating types of rare events involve situations where agents are unaware of their lack of knowledge, which we may term meta-ignorance.[11]

Regarding information choice, work by Morris and Shin (2002), Sims (2003), Veldkamp and Wolfers (2007), Skreta and Veldkamp (2009), and Veldkamp and Van Nieuwerburgh (2009) shows that agents do not always use all available information. This approach appeals to costs of information processing, so that agents choose to ignore potentially valuable, available information. However, these papers generally do not specify the form and size of costs. The information choice approach has been able to explain a number of anomalies in economics, including the home bias puzzle, asymmetric business cycles, portfolio under-diversification, and ratings inflation. Regarding regime shifts, there is ample evidence that the economic structure of major economic and financial variables is subject to sharp breaks. Hamilton (1989) develops the modern methodology of regime shifts, and shows its applicability to the macroeconomy. In financial markets, evidence of regime shifts is documented by Hamilton and Lin (1996), Ang and Bekaert (2002), Ang and Chen (2002), Ang and Bekaert (2004), and Ang and Bekaert (2005). Recent theoretical research has also examined economic foundations for regime changes, such as Angeletos, Hellwig, and Pavan (2007). Recent economic experience suggests that an impediment to market performance is *lack of knowledge* about how to forecast and hedge extreme events. This lack of knowledge reflects non-stationarity of the economic environment, which we embed in our model with the device of regime shifts.

2.1   Contribution of our paper

Our paper contributes to the literature in several important ways. First, we examine extreme events using a simple well-understood porfolio choice framework, with constant relative risk aversion and lognormal returns. We therefore obtain stylized facts about the impact of extreme events, in a transparent, rational setting. Second, based on theoretical and empirical considerations, we incorporate latent regime switches in the likelihood of extreme events, which may be exogenous or endogenous. Our paper appears to be the first to analyze the economic impact of extreme events using this framework. Finally, we provide support for the information choice literature of Sims

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[11] Negative examples of meta-ignorance include the current financial market crises of fall 2008, climate change, impact of new technology and natural catastrophes. See Bazerman and Watkins (2004) and Taleb (2005). Positive examples could include discovery of North Sea oil in the 1960s.

(2003) and Veldkamp and Van Nieuwerburgh (2009), since we give evidence on the size of costs needed to make agents ignore information about important extreme events.

3   Risky choice with exogenous extremes

In this section, we describe risky choice of an individual, faced with rare extreme events. There are three basic ingredients in our setup. First, the base model features a lognormal distribution with constant relative aversion (CRRA) utility. This CRRA-lognormal approach is very tractable and replicates key features of financial data. Therefore it is commonly used for macroeconomics, portfolio choice and asset pricing, as in the work of Campbell (1994), Campbell (1996), and Campbell and Viceira (2002).[12] Second, our framework consists of a single representative agent. This framework allows us to simplify analysis of a situation where large numbers of similar investors are engaged in risky borrowing, by studying their average behavior.[13] The representative agent approach is typical of modern finance research in the tradition of Lucas (1978). Third, the case of rare events is handled by a regime switch approach. Regime switches have been shown to characterize both economic and financial data, by Hamilton (1989) and Hamilton and Lin (1996). Regime switches are also empirically significant in modelling stock market correlations and variances, as shown by Ang and Chen (2002), Dueker (1997), and Haas, Mittnk, and Paolella (2004). Moreover, regime switches have been utilized to model rare events in finance, by Evans (1996)) and Gourieroux and Monfort (2004).

Notation and Calibration. In the remainder of the paper, we will use the following notation.

- The quantity d denotes agents' demand for risky investment, relative to available wealth;

- Superscript $*$ denotes an optimum;

- Superscript E denotes a decision or wealth level during extreme periods;

- Superscript P denotes a prudent investment or wealth level;

- $d^P = 1$. That is, the prudent investor will invest a maximum of all her wealth in risky investment, and will not borrow.

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[12]For further details on the rationale and implementation of the CRRA-lognormal model, see Campbell (1994) page 469; Campbell (1996) page 304; and Campbell and Viceira (2002) Chapter 2. Other textbooks that use this approach are Huang and Litzenberger (1988) and Lyons (2001).

[13]The analysis of large numbers of similar investors is also examined by the literature on strategic complementarities, see Cooper (1999).

- Superscript L denotes a leveraged (excessive) investment or wealth level

- Subscript n denotes an endogenous investment or wealth level;

- Subscript x denotes an exogenous investment or wealth level.

In order to calibrate the various models, we use the following empirical estimates from US data.[14]

- Annualized excess stock return $\hat{\mu} = 0.081 - 0.009 = 0.072$, from Campbell (2003), page 805.

- Annualized stock market volatility $\sigma = 0.156$, from Campbell (2003), page 805.

- Annualized average borrowing rate[15] $r = 0.018$;

- Discount factor $\beta = 0.99$, from Mehra and Prescott (2003), page 907;

- Risk aversion $\gamma \in \{1, 2, ..., 10\}$, from Lewis (1999) page 576; Mehra and Prescott (2003), page 907; and Mehra and Prescott (1985) page 154;

- Annualized likelihood of an extreme event $\alpha = 0.017$, from Barro (2006) page 837.

3.1   Excessive investment in a risky asset: A general case

Much of economic research concerns the aggregate effects of excess borrowing for investment, as discussed by researchers from Fisher (1933) to Allen and Gale (2007). Such excessive borrowing is often motivated as irrational. While irrationality can certainly drive excess behavior in many settings, it is valuable to determine whether such behavior may arise in a simple, rational framework. In Proposition 1 we show that such excessive investment is consistent with rational behavior in a very general setting. Consider a general neoclassical utility function $U(W)$ that depends on wealth W.[16] Among other qualities, this utility function is strictly increasing, bounded, continuous and concave. Following the approach of Campbell and Viceira (2002), the agent is endowed with

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[14]Other sources for calibrating regime switches include Hamilton (1989), Hamilton and Lin (1996), and Mehra-Prescott (2003).

[15]We compute this as the average of the monthly (log) Prime Bank Loan rate, from 1947 to 2009. The Prime Bank Loan rate is available from the Federal Reserve Bank of St. Louis.

[16] By neoclassical utility function, we mean one that is strictly increasing and differentiably continuous, as in Allen and Gale (2007), chapter 2.

initial wealth W_0, and invests a proportion d in a risky asset with returns $r = r^f + \varepsilon$. The remainder is invested in a riskfree asset with returns r^f. Thus, $W = dW_0(1 + r) + (1 - d)W_0(1 + r^f)$, or

$$
\begin{aligned}
W &= dW_0(1 + r^f + \tilde{\varepsilon}) + (1 - d)W_0(1 + r^f) \\
&= dW_0(1 + r^f) + dW_0\tilde{\varepsilon} + (1 - d)W_0(1 + r^f) \\
&= W_0(1 + r^f) + dW_0\tilde{\varepsilon}
\end{aligned}
\tag{1}
$$

We will use the expression for the objective function in (1) for proving the propositions below. The agent maximizes utility subject to the wealth constraint, which as a strictly convex program, yields a unique solution d^*, and unique expected wealth $W^*(d^*)$. We have the following proposition and corollary.

Proposition 1 *If the investor deviates from the optimal investment strategy d^* by choosing a suboptimal investment strategy $\hat{d}$ during a small proportion α of the time, her expected utility loss is bounded above.*

Proof. See Appendix. □

Corollary 1 *If there are high enough costs to learning whether she is behaving suboptimally a small proportion α of the time, the investor will rationally choose to continue behaving suboptimally.*

Proof. See Appendix. □

Theorem 1 and Corollary 1 show that for standard expected utility functions, if agents are suboptimal some of the time and there are costs to detecting extremes, then agents can rationally choose to be suboptimal. While this insight is valuable, it is important to relate the result to observable economic parameters. In order to do so, we need to use standard parametric utility functions and return processes, which we do for the remainder of the paper.

3.2 Base model

We first consider a base model of 'typical' events, where asset returns obey a simple stochastic law. The decision environment consists of a single individual with initial wealth W_0, choosing a fraction of wealth d to invest in a risky asset. For these typical economic environments, the investor's

problem is straightforward: she maximizes expected utility by choosing the fraction d to invest in the risky assets. In order to develop the intuition of the previous subsection more concretely, we utilize an important class of preferences and return processes. In particular, we suppose that the investor's preferences exhibit constant relative risk aversion over wealth W, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, where γ is the coefficient of relative risk aversion. We also assume that the random terms in risky asset returns are lognormally distributed,

$$\tilde{r} \equiv \log(1 + \tilde{R}) \sim N(\mu, \sigma^2). \tag{2}$$

These classes of preferences and returns are widely used in financial economics, for example Campbell (1996), and Campbell and Viceira (2002). To solve the investor's problem, observe that the expectation of a lognormal variable z satisfies $\log E(z) = E(\log z) + \frac{1}{2}V(\log z)$. Then, ignoring the constant $1 - \gamma$, and exchanging logs and expectations, we can write the investor's maximization problem as

$$\max_d \log EW^{1-\gamma} = (1 - \gamma)E(w) + \frac{1}{2}(1 - \gamma)^2 V(w),$$

subject to $w = r + w_0$, where $w = \log W$, $r = \log(1 + R)$, and $w_0 = \log W_0$. To evaluate the above objective function, we therefore must compute the mean and variance of portfolio returns. The mean excess return is $E[r - f^f] = d[E(r) - r^f] + \frac{1}{2}d(1 - d)V[r]$. The variance of the portfolio return is $d^2 V[r]$. Using equation (2), and standard algebraic manipulation as in Chapter 2 of Campbell and Viceira (2002), we can rewrite the investor's problem as

$$\max_d d[E(r) - r^f] + \frac{1}{2}d(1 - d)V[r] + \frac{1}{2}(1 - \gamma)d^2 V[r] \tag{3}$$

$$= d\hat{\mu} + \frac{1}{2}d(1 - d)\sigma^2 + \frac{1}{2}(1 - \gamma)d^2\sigma^2,$$

where $\hat{\mu} = [E(r) - r^f]$. Taking derivatives yields first order conditions $\hat{\mu} + \frac{1}{2}(1 - 2d)\sigma^2 + (1 - \gamma)d\sigma^2 = 0$, or $d[\sigma^2 - (1 - \gamma)\sigma^2] = \hat{\mu} + \frac{1}{2}\sigma^2$. The optimal solution is therefore

$$d^* = \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2} = \frac{2\hat{\mu} + \sigma^2}{2\gamma\sigma^2}. \tag{4}$$

Equations (3) and (4) represent the basic form of objective function and optimum, which we shall use throughout the remainder of this paper. Intuitively, the optimal risky investment is increasing in expected returns and decreasing in risk aversion and variance.

3.3 A model of exogenous extremes

Now we consider the case of rare extreme events. Following the literature on peso problems, we model this situation as a small-probability regime switch in risky asset returns. Specifically, the structure of the problem is unchanged from above, except that the risky return now obeys (2) most of the time, but a small fraction α of the time, there is a regime shift to a period of larger tail events:

$$
\begin{aligned}
\tilde{r} \quad &\sim \quad N(\mu, \sigma^2), \text{ with probability } 1 - \alpha \ \text{ (Typical regime)} \\
&\sim \quad N(\mu, \frac{\sigma^2}{\alpha}), \text{ with probability } \alpha \ \text{ (Extreme regime)},
\end{aligned}
\tag{5}
$$

where α is small.[17][18] In this subsection, we examine two levels of investor awareness about the stochastic environment: complete knowledge, and complete misunderstanding.

Agent completely understands the environment First, consider a situation where the individual knows the stochastic environment. At the very beginning of each period, she knows which regime prevails, and just solves for the optimal demand in each regime.[19] Using the same optimization approach as before, the optimal demand will now depend on the regime, and is a vector. Now the investor accounts for the greater variance in the extreme regime, and her optimal investment is a vector $d^* = (d^L, d^E)$. Leverage-friendly times occur with probability $1 - \alpha$ and extreme periods occur with probability α. Therefore the optimal demand vector is

$$
\begin{aligned}
d^L \quad &= \quad \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma \sigma^2} = \frac{2\hat{\mu} + \sigma^2}{2\gamma\sigma^2}, \text{ with probability } 1 - \alpha \\
d^E \quad &= \quad \frac{\hat{\mu} + \frac{\sigma^2}{2\alpha}}{\gamma \frac{\sigma^2}{\alpha}} = \frac{2\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2} \\
&= \quad \alpha d^L + \frac{1 - \alpha}{2\gamma}, \text{ with probability } \alpha.
\end{aligned}
\tag{6}
$$

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[17] In this simple specification, the probability of the rare event is inversely proportional to its impact: the lower the probability, the higher the impact on variance. Therefore it is an easy way to deliver a low probability, high impact event. This specification is similar to that of Gourieroux and Jasiak (2001).

[18] We might want to compare this to empirical research on the proportion of funds invested in the US, eg Blume and Friend (1976), Polkovnichenko (2007).

[19] The individual does not know the value of risky returns, just the distribution from which they come. Observe that the mixture of log-normals is not restrictive on the unconditional distribution. Conditionally, each regime satisfies log-normality, but unconditionally, a mixture of normals can approximate most empirically observed return distributions arbitrarily closely. For more details on normal mixtures, see McLachlan and Peel (2000).

This is the basic form of investment demand with exogenous extremes in our framework.[20]

Properties of the Solution   We can note two things about the solution in (6). First, d^E depends positively and linearly on the probability α of extremes. Second, for positive excess returns $\hat{\mu}$, it is the case that $d^L > d^E$, which is intuitive.[21]

To glean a quantitative sense of this differential, we calibrate expression (6) to US data, displayed in Table 1. d^L always greatly exceeds d^E, as expected. For example, with risk aversion $\gamma = 5$, we find that $d^L = 0.69$ and $d^E = 0.11$. Thus the risky demand in extreme times is around 6 times smaller than in typical times. This result is qualitatively intuitive, if we think of the extreme regimes as high volatility, disaster periods, where most investors hold small amounts of risky assets, and typical regimes as good or boom periods, when it is relatively more attractive to hold a large position in risky assets.

We also examine another perspective on investors' risk positions, since a central part of our paper concerns the propensity of individuals to spend more than they can reasonably repay.[22] In terms of our model above, the ratio d of individuals' borrowing to their available, disposable income increases over time, and is close to 1 or exceeds 1. In order to see whether this situation obtains for the US economy as a whole, we calculate an empirical version of d in two ways. First, we measure d as the ratio of total US consumer credit outstanding to available, real disposable income.[23] The results are illustrated in figure 1. Evidently, this ratio is increasing over time, and has consistently exceeded unity since July 1986. Second, we measure d as the ratio of total US household credit market debt to available real disposable income. This quantity is shown in figure 2. Once again

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[20]To see the third row, note that

$$
\begin{aligned}
d^E &= \frac{\hat{\mu} + \frac{\sigma^2}{2\alpha}}{\gamma \frac{\sigma^2}{\alpha}} = \frac{\alpha\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2} = \alpha d^L + \frac{\frac{\sigma^2}{2} - \frac{\alpha\sigma^2}{2}}{\gamma\sigma^2} \\
&= \alpha d^L + \frac{\frac{\sigma^2}{2}(1-\alpha)}{\gamma\sigma^2} = \alpha d^L + \frac{1-\alpha}{2\gamma}.
\end{aligned}
$$

[21] To see this, observe that the condition for $d^L > d^E$ can be written, using expression (6), as $d^L > \alpha d^L + \frac{1-\alpha}{2\gamma}$, or $d^L > \frac{1}{2\gamma}$. Substituting in the definition of d^L yields $\frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2} > \frac{1}{2\gamma}$, which simplifies to $2\hat{\mu} > 0$.

[22]This propensity is related to the concept of "over-borrowing," used by Fisher (1933) in the context of financial crises. For related research on excessive expansion of credit, see Abreu and Brunnermeier (2003); Lorenzoni (2008); and Shin (2009).

[23]In order to capture the total amount of income that is available to consumers to repay their borrowing, we subtract real consumption from real income. We call this quantity available, real disposable income.

the ratio is increasing, and consistently exceeds unity since 1959. Thus, the historical experience of the US economy indicates that d has been large and growing throughout the last half century.[24]

Agent misunderstands the environment In the preceding example, the investor was aware of the extreme risks she faced. By contrast, some of the most significant extreme events in history have been *unknown and unforeseen* by the public at large.[25] One way to model our ex ante ignorance about such extremes is to use a hidden regime shift.[26] Specifically, although the true risky return distribution features a regime shift as in (5), the investor believes that $\tilde{\varepsilon} \sim N(\mu, \sigma^2)$ with probability 1. Accordingly, she demands $d = d^L$ with probability 1, instead of probability $1 - \alpha$ as in equation (6). The investor is therefore over-levered $\alpha\%$ of the time, investing d^L instead of the optimal d^E.

We may ask two important questions about the investor's behavior. First, how much does this suboptimal investment hurt her? This question is natural in light of Proposition 1 because the suboptimality only occurs a small percentage of the time. Second, if there are costs associated with learning about extremes, would the investor change her suboptimal strategy? We summarize the answers to these questions in Proposition 2 and Corollary 2, below.

Proposition 2 *The cost to investors of suboptimal behavior during extremes is bounded above by a constant K, which is proportional to squared, standardized excess returns $\left(\frac{\hat{\mu}}{\sigma}\right)^2$.*

Proof. See Appendix. □

Corollary 2. *If the costs of learning about extreme events are above a finite threshold, the investor will prefer to over-invest during extreme periods.*

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[24]Theoretically, we can also show that d^L involves leverage. This means we need to show that $d^L > 1$, or using definition (6), this means $\frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma \sigma^2} > 1$. By positivity of γ and σ^2, we can write this as

$$
\begin{aligned}
2\hat{\mu} + \sigma^2 &> 2\gamma\sigma^2 \qquad\qquad\qquad\qquad\qquad (7)\\
\implies \frac{\hat{\mu}}{\sigma^2} &> \frac{2\gamma - 1}{2}.
\end{aligned}
$$

Given a risk aversion of 2, for example, expression (7) says that leverage is optimal when the Sharpe ratio exceeds 1.5.

[25]In addition to 2008's financial crisis, other negative examples include the Black Death of 1348; the 1929 US stock market crash; the set of events leading up to the creation of the atomic bomb; global warming; and the devastation of 2005's Hurricane Katrina. Positive examples include the invention of the wheel; signing of the first US copyright law in 1790; the Wright brothers' 1903 flight; and the record-breaking US stock market levels of the 1990s.

[26]To the best of our knowledge, this formulation of hidden extreme events is novel to the current paper. A parallel framework is used by Gourieroux and Jasiak (2001), who provide an asset demand application, although they do not consider hidden regimes, nor endogenous extremes.

Proof. See Appendix. □

Thus, if there are large enough costs to learning about extremes, the investor's strategy is insensitive to rare extreme events. This is true even when extremes deliver a large effect on return volatility.[27] To summarize this subsection, we have shown that in an environment of exogenous extremes, a knowledgeable investor will invest much more in normal times than in extreme times. We have also provided a bound on the utility loss from suboptimal behavior by investors who do not understand the economic environment. The existence of this bound is consistent with the literature on global games, rational attention and information choice.[28] It suggests that even if agents were informed of the suboptimality of their investment strategy, a high enough level of costs associated with learning about extremes will prevent them from shifting their strategy.

3.4 Calibration to the US economy

We calibrate Proposition 2 to US data using equation (12) from the Appendix. The results are displayed in Figure 3. This figure shows that the costs of excess leverage range from 2% to 6% of wealth. These costs decrease with risk aversion, since more risk averse investors would have lower leverage.

4 Risky choice with endogenous extremes

The likelihood of extreme and rare events is affected by the behavior of agents in social settings. Such endogenous extreme events include the effect of human activity on extreme climate changes, and the effect of risky borrowing on financial crises.[29] Accordingly, in this section, we consider a situation where excessive risky borrowing permanently raises the likelihood of being in the high variance regime.[30] This environment entails more complex information processing for investors,

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[27]The key to this insensitivity may be the combination of regime shifts and CRRA-lognormal framework. Insensitivity of general expected utility functionals to rare events has been examined by Chichilnisky (2000); For related contexts involving biased perception of virgin risks and fearsome risks, see Chichilnisky (2007); Chichilnisky and Heal (2003); Pavlov and Wachter (2006); Sunstein and Zeckhauser (2008); and Weber (2006).

[28]See Morris and Shin (2002) Sims (2003); Skreta and Veldkamp (2009); Veldkamp and Van Nieuwerburgh (2009).

[29]For climate change, see the cover story of Time, March 30, 2007; and Stern (2007). For risky borrowing, see Grossman (1988), Fisher (1933), and various issues of the Economist in March 2007 and October 2008.

[30]It is possible to account for endogenous regimes in a less draconian way, for example if the probability of extremes lowers after a few periods of prudent behavior. The role of excess borrowing in precipitating extreme financial market behavior has been motivated in many ways. One approach emphasizes heightened investor and bank fragility due to

since their returns depend on the likelihood of extremes, which in turn depends on their investment strategies. Similar to the literature on information choice of Sims (2003), such processing costs may lead investors to ignore potentially important information. A further layer of complexity concerns complete lack of knowledge, when individuals are unaware of their collective impact on the likelihood of unforeseen extremes.[31] In light of these considerations, we formalize endogenous extremes by considering an investor who believes the risky return comes from a single distribution as in equation (2), while in truth, the distribution switches endogenously. Optimally the investor should use a cutoff level for risky investment, as we showed in (6). However, unaware of the consequences, she follows the approach of (4) and just chooses risky demand equal to d^L.

Once more we may ask two questions. First, does this situation harm the investor? In order to answer this question, we compute the expected wealth from behaving optimally and suboptimally. Optimal investment involves a cutoff rule, with potentially non-constant d, while suboptimal investment involves a constant d^L. Therefore this situation can in principle hurt the investor if α is large enough, since the elevated extremes are permanent. Second, under what conditions will she learn? It turns out that if costs are high enough, there is nothing in the model to alert the investor to extreme events. Therefore a risk averse individual can ignore endogenous, high-impact regime shifts.

4.1 A Two-period model

There is a lot of evidence that excessive credit and risky borrowing are related to extreme financial events.[32] We summarize this evidence by saying that there are two periods in the economy, with the consequences of first-period investment choices being felt in the second period. In particular, if the investor is too leveraged in the first period, then the likelihood of extremes is increased to α_n in the second period.[33] For simplicity, we set $\alpha_n = 2\alpha$. Thus, in this endogenous extreme

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lack of liquidity. Prominent examples are the cases of LTCM in 1998 and Lehman Brothers in 2008. Such firms and investors are especially susceptible to even small liquidity shocks and margin calls, see Shleifer and Vishny (1997). Another approach is taken by the research on bubbles and financial crises, see Allen and Gale (2000) and Blanchard (1979).

[31]Examples include climate change, or stock market bubbles. This class of extreme events is related to rare events of Taleb (2005), and oblivious ignorance of Bhide (2000). In geopolitics, an instance of unknown endogenous extremes could be the set of events in the early cold war that culminated in the Cuban missile crisis of 1962. This resembles a reverse peso problem: by failing to account for their own ignorance, rational individuals do not anticipate extreme events, which they themselves precipitated.

[32]See Fisher (1933); Bernanke (1983); (Allen and Gale (2007); Lorenzoni (2008); and Shin (2009).

[33]For ease of notation, we will use the terms "prudent" and "leveraged" to denote an investor who is unlevered and who is over-levered, respectively. The prudent levels of wealth and investment are denoted by a superscript P, and the leveraged levels are denoted by a superscript L.

events model, leverage-friendly times occur with probability $1 - 2\alpha$ and extreme periods occur with probability 2α. Using the same approach as in Section 2, the optimal demand vector is

$$d^L = \frac{2\hat{\mu} + \sigma^2}{2\gamma\sigma^2}, \quad \text{with probability } 1 - 2\alpha \tag{8}$$

$$d^E_{2\alpha} = \frac{\hat{\mu} + \frac{\sigma^2}{4\alpha}}{\gamma\frac{\sigma^2}{2\alpha}} = \frac{4\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}$$

$$= 2\alpha d^L + \frac{1 - 2\alpha}{2\gamma}, \quad \text{with probability } 2\alpha.$$

We will use this expression to calculate the effect of endogenous extremes on risky behavior.[34]

<u>First Period</u>: The first period is a typical regime. We assume that the investor's optimal demand d^L exceeds her wealth, $d^L > 1$. If the investor wishes to be prudent, she can instead invest $d^P = 1$. Now her investment choice is more involved since she also has to consider credit market effects. She can invest $d^P = 1$, which has the benefit of ensuring a constant level of extremes and the cost of foregone returns; or she can borrow to invest $d^L > 1$, which has the benefit of higher possible returns and the cost of increased danger of extremes.

<u>Second Period</u>: In the second period, the probability of extremes is

$$\Pr(\text{extremes}) = \alpha, \quad \text{if investor chose } d^P$$
$$2\alpha, \quad \text{if investor chose } d^L.$$

The investor is only allowed to borrow for the first period, and if so, she repays with interest at the end of the second period. Therefore, in the second period the investor must choose d^P if there is a typical regime. Thus, depending on the investor's choices, the economy can evolve along a path with a low level α of extremes or with a high level 2α of extremes. To determine which path the investor will choose, we again consider two levels of investor awareness of the economic environment, corresponding to complete understanding and misunderstanding.

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[34]To see the third row, note that

$$d^E_{2\alpha} = \frac{\hat{\mu} + \frac{\sigma^2}{4\alpha}}{\gamma\frac{\sigma^2}{2\alpha}} = \frac{2\alpha\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2} = 2\alpha d^L + \frac{\frac{\sigma^2}{2} - \alpha\sigma^2}{\gamma\sigma^2}$$

$$= 2\alpha d^L + \frac{\frac{\sigma^2}{2}(1 - 2\alpha)}{\gamma\sigma^2} = 2\alpha d^L + \frac{1 - 2\alpha}{2\gamma}.$$

13

Agent completely understands the environment In this case, the representative investor understands that the environment features regime shifts in the likelihood of extreme events. Further, she knows that excess leverage may raise the likelihood of extreme events. We summarize the investor's strategy in Proposition 3 below.

Proposition 3 *There is a net benefit of leverage for investors who know that the environment features regime-switching in extreme events. This benefit may be expressed as a polynomial in $\hat{\mu}$ and σ^2.*

Proof. See Appendix. □

Agent misunderstands the environment In this case, the investor does not know that there are regime shifts and does not know that she can influence the likelihood of extremes. In period 1, she can demand either d^L or d^P. In period 2, she repays any borrowing, and since she mistakenly believes that the world is always in the typical regime, she demands the largest fraction she can, $d^P = 1$. We summarize the results of this investor's decisions in Proposition 4 below.

Proposition 4 *The utility loss from following a suboptimal strategy is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.*

Proof. See Appendix. □

The import of Proposition 3 is that rational investors will knowingly increase the likelihood of extreme events in the second period. In a related sense, Proposition 4 shows that investors who do not understand the environment face losses that are bounded. Therefore if the costs of learning about the environment are large enough, investors may choose to continue with a suboptimal strategy.

4.2 Calibration to the US economy

We calibrate Propositions 3 and 4 to US data, using expressions for the net benefits and costs from the Appendix.[35] The results are displayed in Figures 4 and 5. From Figure 4 the net benefit of leverage is always positive. It increases with risk aversion, because in equations 6 and (8), individuals invest more during endogenous than in exogenous extremes. This occurs because the former feature lower volatility. From Figure 5 we see that the relative benefits from optimal investment

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[35]The expressions for Propositions 3 and 4 are in equations (16), (19) and (20).

100% of the time versus $100 - \alpha\%$ of the time are low, between 2 and 11 per cent of available wealth.

5   Conclusions

In this paper we construct a simple latent regime-switching model of portfolio choice, in order to assess the implications for over-investing. Motivated by theoretical and empirical considerations, we examine the benefits and costs of leverage, and of suboptimal investment. Our most striking finding is that in both one and two-period models, the benefits of sustained optimal investment are bounded. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if there are costs to learning the structure of the financial environment. We also discover that the benefits of leverage represent a large percentage of income. Upon calibration to the US economy, we document that the costs of ignoring extreme events are small and the benefits of leverage are substantial.

Our paper therefore provides both a theoretical framework for examining extreme events, and empirical evidence on the scope of costs related to learning about extremes. From an academic perspective, our results may provide support for theoretical research on costs to information processing and rational inattention.

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Table 1: Examples of Extreme and Rare Events

	Frequent	**Rare**
Non-Extreme (Small Impact)	No war, post-1990 western Europe	↓ CO_2 pollution
Extreme (Large Impact)	↑ CO_2 Pollution	Multi-nation war, post-1990 western Europe Multi-country stock market crash, post-Great Depression

Table 2: Risky Asset Demand in Extreme and Normal Times

The table presents risky demand d^E and d^L during extreme and normal times respectively, using equation (6). The calibration is as in Section 2. The parameter γ denotes the coefficient of relative risk aversion.

γ	d^L	d^E
1	3.4586	0.5503
2	1.7293	0.2751
3	1.1529	0.1834
4	0.8646	0.1376
5	0.6917	0.1101
6	0.5764	0.0917
7	0.4941	0.0786
8	0.4323	0.0688
9	0.3843	0.0611
10	0.3459	0.0550

Figure 1: Consumer Credit Ratio for US Households: 1959-2009

The figure shows the ratio of total US consumer credit to available income, where the latter is computed as real disposable income minus real consumption. All variables are available from the Federal Reserve Bank of St. Louis. The frequency is monthly, and the time period is January 1959 to June 2009.
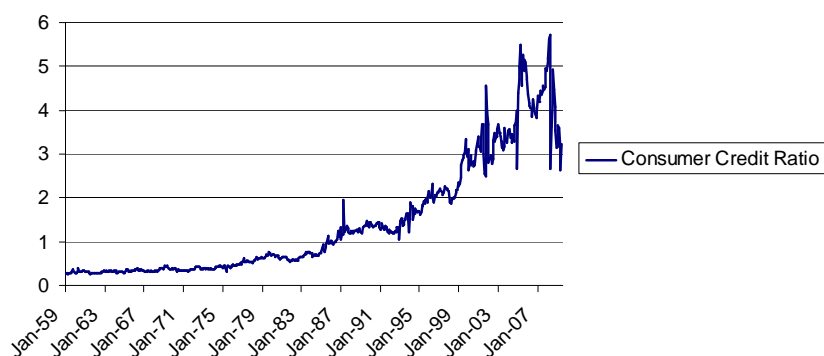

Figure 2: Credit Market Debt Ratio for US Consumers: 1953-2009

The figure shows the ratio of total US household credit market debt to available income, where the latter is computed as real disposable income minus real consumption. All variables are available from the Federal Reserve Bank of St. Louis. The frequency is quarterly, and the time period is January 1953 to January 2009.
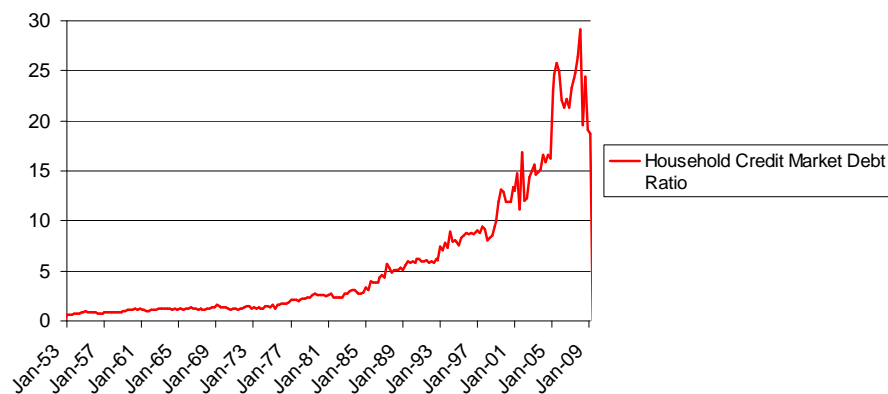
Figure 3: Investor Costs of Excess Leverage: US Stock Returns

The figure calibrates the bound from Proposition 2, using US data and the calibration of Section 2. The bound shows the cost to an investor of excess leverage during rare extreme events. According to Corollary 2 this bound may also be interpreted as the minimum cost of learning about regime shifts in the likelihood of rare events, as discussed in Section 3 of the text.
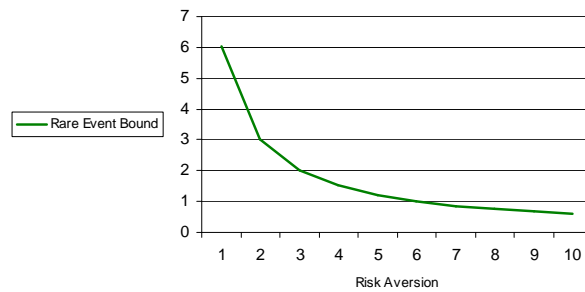

Figure 4: Net Benefits from Leverage for a US investor

The figure calibrates the bound from Proposition 3, equation (16). We use US data and the calibration values described in Section 2 of the text. The bound shows the net benefit from being leveraged during endogenous extreme regimes, as discussed in Section 3.
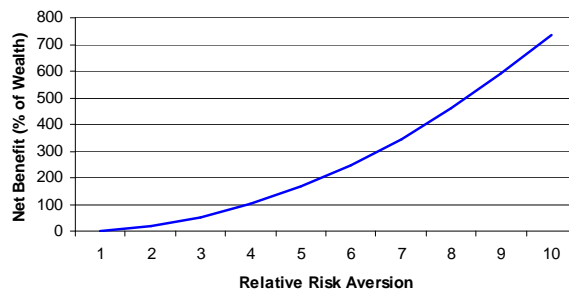
Figure 5: Benefits from optimal investment for a US investor

The figure calibrates the bounds from Proposition 4, equations (19) and (20). We use US data and the calibration values described in Section 2 of the text. The bound shows the net benefit of optimal investment relative to suboptimal investment during endogenous extreme regimes. This bound may be interpreted as the minimum cost to learning about regime shifts in the likelihood of rare events, as discussed in Section 3.
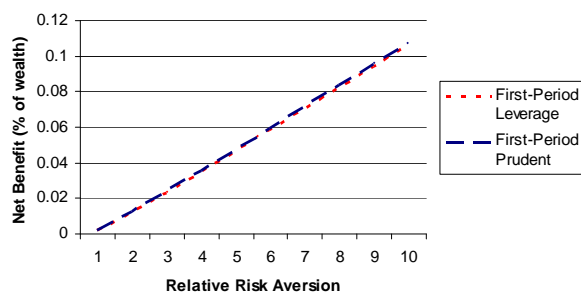
A   Proofs of Propositions

Proposition 1. *If the investor deviates from the optimal investment strategy d^* by choosing a suboptimal investment strategy $\hat{d}$ during a small proportion α of the time, her expected utility loss is bounded above.*

Proof. We need to show that the expected utility loss ΔEU satisfies $\Delta EU \leq K$, for some $K < \infty$. First, let us denote the suboptimal wealth level $\hat{W}(\hat{d})$ Now note that the expected utility loss is the difference between optimal utility with probability one and with probability $1 - \alpha$. Thus $\Delta EU \equiv U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)]$, where we drop the argument in $W()$ for simplicity. Computing the expected utility loss, we obtain

$$
\begin{aligned}
\Delta EU &\equiv U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)] \qquad\qquad (9)\\
&= \alpha[U(W^*) - U(\hat{W})].
\end{aligned}
$$

By boundedness of the utility function, the quantity in (9) is finite and bounded above, for example, by $\alpha U(W^*)$. Thus, for $K = \alpha U(W^*)$, we have that ΔEU satisfies $\Delta EU \leq K$, as was to be shown. $\quad\square$

Corollary 1. *If there are high enough costs to learning whether she is behaving suboptimally a small proportion α of the time, the investor will rationally choose to continue behaving suboptimally.*

Proof. From Proposition 1, we know that the investor loses at most K from investing suboptimally for a small portion of the time. If we set costs to K, it follows that the investor is better off using the suboptimal strategy. $\quad\square$

Proposition 2. *The cost to investors of suboptimal behavior during extremes is bounded above by a constant K, which is proportional to squared, standardized excess returns $\left(\frac{\hat{\mu}}{\sigma}\right)^2$.*

Proof. We need to show that the utility loss ΔEU from investing a proportion d^L instead of d^E during extreme periods is of the form $\Delta EU \leq K$, where $K = \theta\hat{\mu}^2$ for some positive, finite θ. In order to compute the utility loss, we just calculate the investor's objective function (3) in both cases.

<u>Optimal:</u> The optimal strategy is to invest d^E, yielding an objective function

$$
\begin{aligned}
U(W(d^E)) &= d^E\hat{\mu} + \frac{1}{2}d^E(1 - d^E)\frac{\sigma^2}{\alpha} + \frac{1}{2}(1 - \gamma)(d^E)^2\frac{\sigma^2}{\alpha} \qquad\qquad (10)\\
&= \left(\alpha d^L + \frac{1 - \alpha}{2\gamma}\right)\hat{\mu} + \frac{1}{2}\left(\alpha d^L + \frac{1 - \alpha}{2\gamma}\right)\left(1 - \alpha d^L - \frac{1 - \alpha}{2\gamma}\right)\frac{\sigma^2}{\alpha}\\
&\quad + \frac{1}{2}(1 - \gamma)\left(\alpha d^L + \frac{1 - \alpha}{2\gamma}\right)^2\frac{\sigma^2}{\alpha},
\end{aligned}
$$

where the second line uses the fact that $d^E = \alpha d^L + \frac{1-\alpha}{2\gamma}$, from expression (6).

Suboptimal: In similar fashion, the suboptimal payoff can be calculated as

$$U(W(d^L)) = d^L \hat{\mu} + \frac{1}{2} d^L (1 - d^t) \frac{\sigma^2}{\alpha} + \frac{1}{2} (1 - \gamma)(d^L)^2 \frac{\sigma^2}{\alpha}. \tag{11}$$

Now the expected utility loss from suboptimal investment is just the difference between (10) and (11):

$$
\begin{aligned}
\Delta EU &= \hat{\mu}\left(\alpha d^L + \frac{1-\alpha}{2\gamma} - d^L\right) + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\left(\alpha d^L + \frac{1-\alpha}{2\gamma}\right)\left(1 - \alpha d^L - \frac{1-\alpha}{2\gamma}\right) - d^L(1-d^L)\right] \\
&\quad + \frac{1}{2}(1-\gamma)\frac{\sigma^2}{\alpha}\left[\left(\alpha d^L + \frac{1-\alpha}{2\gamma}\right)^2 - (d^L)^2\right] \\
&= \hat{\mu}\left[\frac{1-\alpha}{2\gamma} - (1-\alpha)d^L\right] \\
&\quad + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\alpha d^L - \alpha^2(d^L)^2 - \frac{\alpha d^L(1-\alpha)}{2\gamma} + \frac{(1-\alpha)}{2\gamma} - \frac{\alpha d^L(1-\alpha)}{2\gamma} - \left(\frac{1-\alpha}{2\gamma}\right)^2 - d^L + (d^L)^2\right] \\
&\quad + \frac{1}{2}(1-\gamma)\frac{\sigma^2}{\alpha}\left[\alpha^2(d^t)^2 + \frac{2\alpha d^L(1-\alpha)}{2\gamma} + \left(\frac{1-\alpha}{2\gamma}\right)^2 - (d^L)^2\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[d^L(\alpha-1) + (d^L)^2(1-\alpha^2) - \frac{2\alpha d^L(1-\alpha)}{2\gamma} + \frac{(1-\alpha)}{2\gamma} - \left(\frac{1-\alpha}{2\gamma}\right)^2\right] \\
&\quad + \frac{1}{2}(1-\gamma)\frac{\sigma^2}{\alpha}\left[(d^L)^2(\alpha^2-1) + \frac{\alpha d^L(1-\alpha)}{\gamma} + \left(\frac{1-\alpha}{2\gamma}\right)^2\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[d^L(\alpha-1) + (d^L)^2(1-\alpha^2 + (1-\gamma)(\alpha^2-1))\right] \\
&\quad + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\frac{(1-\gamma)\alpha d^L(1-\alpha) - \alpha d^L(1-\alpha)}{\gamma} + \frac{(1-\alpha)}{2\gamma} + \left(\frac{1-\alpha}{2\gamma}\right)^2(1-\gamma-1)\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] \\
&\quad + \frac{1}{2}\frac{\sigma^2}{\alpha}[d^L(\alpha-1) + (d^L)^2(1-\alpha^2+\alpha^2-1-\gamma(\alpha^2-1))\frac{\gamma\alpha d^L(1-\alpha)}{\gamma} + \frac{(1-\alpha)}{2\gamma} - \gamma\left(\frac{1-\alpha}{2\gamma}\right)^2] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\frac{(1-\alpha)}{2\gamma} - (1-\alpha)d^L - \gamma(\alpha^2-1)(d^L)^2 - \alpha d^L(1-\alpha) - \frac{(1-\alpha)^2}{4\gamma}\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\frac{2(1-\alpha) - (1-\alpha)^2}{4\gamma} - d^L(1-\alpha)(1+\alpha) - (d^L)^2\gamma(\alpha^2-1)\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\frac{2 - 2\alpha - 1 + 2\alpha - \alpha^2}{4\gamma} - d^L(1+\alpha-\alpha-\alpha^2) + (d^L)^2\gamma(1-\alpha^2)\right] \\
&= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[\frac{1-\alpha^2}{4\gamma} - d^L(1-\alpha^2) + (d^L)^2\gamma(1-\alpha^2)\right]
\end{aligned}
$$

$$= \hat{\mu}\left[(1-\alpha)(\frac{1}{2\gamma} - d^L)\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[(1-\alpha^2)\left(\frac{1}{4\gamma} - d^L + \gamma(d^L)^2\right)\right].$$

We can now substitute the expression for d^L from equation (6), to obtain

$$\Delta EU = \hat{\mu}\left[(1-\alpha)\left(\frac{1}{2\gamma} - \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2}\right)\right] \tag{12}$$

$$+\frac{1}{2}\frac{\sigma^2}{\alpha}\left[(1-\alpha^2)\left(\frac{1}{4\gamma} - \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma\sigma^2} + \gamma\frac{\left(\hat{\mu} + \frac{\sigma^2}{2}\right)^2}{\gamma^2\sigma^4}\right)\right]$$

$$= \hat{\mu}\left[(1-\alpha)\frac{\sigma^2 - 2\hat{\mu} - \sigma^2}{2\gamma\sigma^2}\right] + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[(1-\alpha^2)\frac{\sigma^4 - 4\sigma^2\hat{\mu} - 2\sigma^4 + 4\hat{\mu}^2 + 4\sigma^2\hat{\mu} + \sigma^4}{4\gamma\sigma^4}\right]$$

$$= -\frac{(1-\alpha)\hat{\mu}^2}{\gamma\sigma^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}\left[(1-\alpha^2)\frac{4\hat{\mu}^2}{4\gamma\sigma^4}\right]$$

$$= -\frac{(1-\alpha)\hat{\mu}^2}{\gamma\sigma^2} + \frac{1}{2}\frac{\sigma^2}{\alpha}\frac{(1-\alpha^2)\hat{\mu}^2}{\gamma\sigma^4}$$

$$= \frac{(1-\alpha^2)\hat{\mu}^2}{2\alpha\gamma\sigma^2} - \frac{(1-\alpha)\hat{\mu}^2}{\gamma\sigma^2}$$

$$= \frac{(1-\alpha^2)\hat{\mu}^2 - 2\alpha(1-\alpha)\hat{\mu}^2}{2\alpha\gamma\sigma^2}$$

$$= \frac{\hat{\mu}^2[1 - \alpha^2 - 2\alpha + 2\alpha^2]}{2\alpha\gamma\sigma^2}$$

$$= \frac{\hat{\mu}^2[1 - 2\alpha + \alpha^2]}{2\alpha\gamma\sigma^2}$$

$$= \frac{\hat{\mu}^2(1-\alpha)^2}{2\alpha\gamma\sigma^2}.$$

The expression in (12) is of the form $K = \theta\left(\frac{\hat{\mu}}{\sigma}\right)^2$, where $\theta = \frac{(1-\alpha)^2}{2\alpha\gamma}$, as was to be shown. $\square$

Corollary 2. *If the costs of learning about extreme events are above a threshold, the investor will prefer to over-invest during extreme periods.*

Proof. From the previous proposition, it follows that if costs are above K, the investor will be better off by over-investing. $\square$

Proposition 3. *There is a net benefit of leverage for investors who know that the environment features regime-switching in extreme events. This benefit may be expressed as a polynomial in $\hat{\mu}$ and σ^2.*

Proof. We need to show that for some parameter values, the investor's objective function from leveraged investment, P^L, exceeds that from prudent investment[36], P^P. That is, we must show that sometimes $P^L - P^P > 0$. In order to do this, we calculate the investor's expected payoff from choosing prudent and leveraged investment levels. We denote EU as the expected utility, from the objective function in equation (3). The environment is that the first period is always a low-volatility regime with variance σ^2. In the first period the investor decides whether to borrow and invest d^L, or else invest the prudent amount $d^P = 1$. In the second period, the investor will choose optimally for that period: either d^E ($d_{2\alpha}^E$) if it is extreme (endogenous), or else d^P for normal economic climates. First we compute the payoff P^P as follows[37]

$$P^P = EU(d^P|\sigma^2) + \beta\left[\alpha EU(d^E|\frac{\sigma^2}{\alpha}) + (1-\alpha)EU(d^P|\sigma^2)\right]. \tag{13}$$

Then we compute the payoff from leverage, P^L, as follows. In this case, the investor has to repay borrowing $r(d^L - 1)W_0$ in the second period, where we normalize $W_0 = 1$ to obtain $r(d^L - 1)$. Hence the payoff is

$$
\begin{aligned}
P^L &= EU(d^L|\sigma^2) + \beta\left[2\alpha\left(EU(d_{2\alpha}^E|\frac{\sigma^2}{2\alpha}) - EU(r(d^L-1)|\frac{\sigma^2}{2\alpha})\right)\right] \\
&\quad + \beta\left[(1-2\alpha)\left(EU(d^P|\sigma^2) - EU(r(d^L-1)|\sigma^2)\right)\right]
\end{aligned}
\tag{14}
$$

Now to see conditions under which it is optimal to have excessive leverage (excess in the sense that it raises the likelihood of extremes), we compute $P^L - P^P$ from (13) and (14) and see when it is positive:

$$
\begin{aligned}
P^L - P^P &= EU(d^L|\sigma^2) - EU(d^P|\sigma^2)[1 + \beta(1-\alpha) - \beta(1-2\alpha)] \\
&\quad + \alpha\beta\left[2EU(d_{2\alpha}^E|\frac{\sigma^2}{2\alpha}) - EU(d^E|\frac{\sigma^2}{\alpha})\right] \\
&\quad - \beta\left[2\alpha EU(r(d^L-1)|\frac{\sigma^2}{2\alpha}) + (1-2\alpha)EU(r(d^L-1)|\sigma^2)\right] \\
&= EU(d^L|\sigma^2) - (1+\alpha\beta)EU(d^P|\sigma^2) + \alpha\beta\hat{\mu}\left[\frac{3\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right] \\
&\quad - \beta\left[r(d^L-1)\hat{\mu} + (2-2\alpha)\frac{\sigma^2}{2}r(d^L-1)[(1-r(d^L-1)) + (1-\gamma)r(d^L-1)]\right],
\end{aligned}
\tag{15}
$$

---

[36] Alternatively, we could phrase it in terms of whether expected borrowing costs B are beneath a certain threshold. Then we need to show that the optimal choice is a cutoff

$$
\begin{aligned}
d &= d^P, \text{ if } B < \bar{B} \\
&= d^L, \text{ if } B > \bar{B}.
\end{aligned}
$$

[37] We show in (7) that d^L involves leverage. So in the second period, since it is the end of economic activity, the agents cannot borrow, they just invest as much as they can, $d^P = 1$.

where the second and third terms come from substitution into the EU definition (3) in the following way. Let us denote the second term S (for second) and the third term T (for third). To see the second term, use equation (3) to obtain

$$
\begin{aligned}
S &= \alpha\beta\left[2EU(d_{2\alpha}^E|\frac{\sigma^2}{2\alpha}) - EU(d^E|\frac{\sigma^2}{\alpha})\right] \\
&= \alpha\beta\left[2d_{2\alpha}^E\hat{\mu} + d_{2\alpha}^E(1-d_{2\alpha}^E)\frac{\sigma^2}{2\alpha} + (1-\gamma)(d_{2\alpha}^E)^2\frac{\sigma^2}{2\alpha}\right] \\
&\quad -\alpha\beta\left[d^E\hat{\mu} + \frac{1}{2}d^E(1-d^E)\frac{\sigma^2}{\alpha} + \frac{1}{2}(1-\gamma)(d^E)^2\frac{\sigma^2}{\alpha}\right] \\
&= \beta\left[2\alpha d_{2\alpha}^E\hat{\mu} + d_{2\alpha}^E(1-d_{2\alpha}^E)\frac{\sigma^2}{2} + (1-\gamma)(d_{2\alpha}^E)^2\frac{\sigma^2}{2}\right] \\
&\quad -\beta\left[\alpha d^E\hat{\mu} + d^E(1-d^E)\frac{\sigma^2}{2} + (1-\gamma)(d^E)^2\frac{\sigma^2}{2}\right] \\
&= \beta\left[\alpha\hat{\mu}(2d_{2\alpha}^E) + \frac{\sigma^2}{2}\left(d_{2\alpha}^E - \gamma(d_{2\alpha}^E)^2\right)\right] - \beta\left[\alpha\hat{\mu}(d^E) + \frac{\sigma^2}{2}\left(d^E - \gamma(d^E)^2\right)\right] \\
&= \beta\left[\alpha\hat{\mu}(2d_{2\alpha}^E - d^E) + \frac{\sigma^2}{2}\left(d_{2\alpha}^E - d^E - \gamma(d_{2\alpha}^E)^2 + \gamma(d^E)^2\right)\right].
\end{aligned}
$$

Now we can substitute in the definitions of d^E and $d_{2\alpha}^E$ from (6) and (8), to obtain

$$
\begin{aligned}
S &= \beta\alpha\hat{\mu}\left(\frac{8\alpha\hat{\mu} + 2\sigma^2 - 2\alpha\hat{\mu} - \sigma^2}{2\gamma\sigma^2}\right) + \beta\frac{\sigma^2}{2}\left(\frac{4\alpha\hat{\mu} + \sigma^2 - 2\alpha\hat{\mu} - \sigma^2}{2\gamma\sigma^2}\right) \\
&\quad +\beta\frac{\sigma^2}{2}\left(-\gamma\frac{(16\alpha^2\hat{\mu}^2 + 8\alpha\hat{\mu}\sigma^2 + \sigma^4)}{4\gamma^2\sigma^4} + \gamma\frac{4\alpha^2\hat{\mu}^2 + 4\alpha\hat{\mu}\sigma^2 + \sigma^4}{4\gamma^2\sigma^4}\right) \\
&= \beta\left[\alpha\hat{\mu}\left(\frac{6\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right) + \frac{\sigma^2}{2}\left(\frac{2\alpha\hat{\mu}}{2\gamma\sigma^2} - \frac{12\alpha^2\hat{\mu}^2 + 4\alpha\hat{\mu}\sigma^2}{4\gamma\sigma^4}\right)\right] \\
&= \beta\left[\alpha\hat{\mu}\left(\frac{6\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right) + \frac{\alpha\hat{\mu}}{2\gamma} - \frac{12\alpha^2\hat{\mu}^2 + 4\alpha\hat{\mu}\sigma^2}{8\gamma\sigma^2}\right] \\
&= \beta\left[\frac{6\alpha^2\hat{\mu}^2 + \alpha\hat{\mu}\sigma^2 + \alpha\hat{\mu}\sigma^2}{2\gamma\sigma^2} - \frac{4(3\alpha^2\hat{\mu}^2 + \alpha\hat{\mu}\sigma^2)}{4(2\gamma\sigma^2)}\right] \\
&= \beta\left[\frac{6\alpha^2\hat{\mu}^2 + \alpha\hat{\mu}\sigma^2 + \alpha\hat{\mu}\sigma^2 - 3\alpha^2\hat{\mu}^2 - \alpha\hat{\mu}\sigma^2}{2\gamma\sigma^2}\right] \\
&= \beta\left[\frac{3\alpha^2\hat{\mu}^2 + \alpha\hat{\mu}\sigma^2}{2\gamma\sigma^2}\right] = \alpha\beta\hat{\mu}\left[\frac{3\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right].
\end{aligned}
$$

Similarly, for the third term in (15), we use expression (3) to obtain

$$
\begin{aligned}
T = \ & 2\alpha EU(r(d^L - 1)|\frac{\sigma^2}{2\alpha}) + (1 - 2\alpha)EU(r(d^L - 1)|\sigma^2) \\
= \ & r(d^L - 1)\hat{\mu} + (2 - 2\alpha)\left[\frac{\sigma^2}{2}r(d^L - 1)(1 - r(d^L - 1))\right] \\
& + (2 - 2\alpha)\left[\frac{\sigma^2}{2}(1 - \gamma)(r(d^L - 1))^2\right] \\
= \ & r(d^L - 1)\hat{\mu} + (2 - 2\alpha)\frac{\sigma^2}{2}r(d^L - 1)[(1 - r(d^L - 1)) + (1 - \gamma)(r(d^L - 1))].
\end{aligned}
$$

Now we return to computing $P^L - P^P$, from expression (15), as follows: First, we use (3) with $d^P = 1$ and d^L to obtain

$$
\begin{aligned}
P^L - P^P = \ & d^L\hat{\mu} + \frac{1}{2}d^L(1 - d^L)\sigma^2 + \frac{1}{2}(1 - \gamma)(d^L)^2\sigma^2 \\
& - (1 + \alpha\beta)\left[\hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2\right] + \alpha\beta\hat{\mu}\left[\frac{3\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right] \\
& - \beta\left[r(d^L - 1)\hat{\mu} + (2 - 2\alpha)\frac{\sigma^2}{2}r(d^L - 1)[(1 - r(d^L - 1)) + (1 - \gamma)r(d^L - 1)]\right] \\
= \ & d^L\hat{\mu} + \frac{1}{2}d^L(1 - d^L)\sigma^2 + \frac{1}{2}(1 - \gamma)(d^L)^2\sigma^2 \\
& - (1 + \alpha\beta)\left[\hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2\right] + \alpha\beta\hat{\mu}\left[\frac{3\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right] \\
& - \beta\left[r(d^L - 1)\hat{\mu} + (2 - 2\alpha)\frac{\sigma^2}{2}r(d^L - 1)[1 - \gamma r(d^L - 1)]\right].
\end{aligned}
$$

Expanding this expression, then collecting terms in d^L, $(d^L)^2$, $\hat{\mu}$, and $\frac{\sigma^2}{2}$ yields

$$
\begin{aligned}
P^L - P^P = \ & d^L\hat{\mu} + \frac{1}{2}d^L\sigma^2 - \frac{1}{2}(d^L)^2\sigma^2 + \frac{1}{2}(d^L)^2\sigma^2 - \frac{1}{2}\gamma(d^L)^2\sigma^2 \\
& - (1 + \alpha\beta)\left[\hat{\mu} + \frac{1}{2}(1 - \gamma)\sigma^2\right] + \alpha\beta\hat{\mu}\left[\frac{3\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}\right] \\
& - \beta\left[d^L(r\hat{\mu}) - r\hat{\mu} + (2 - 2\alpha)\frac{\sigma^2}{2}rd^L - (2 - 2\alpha)\frac{\sigma^2}{2}r - (2 - 2\alpha)\frac{\sigma^2}{2}\gamma r^2(d^L)^2\right] \\
& - \beta\left[2(2 - 2\alpha)\frac{\sigma^2}{2}\gamma r^2 d^L - (2 - 2\alpha)\frac{\sigma^2}{2}\gamma r^2\right] \\
= \ & d^L\left[\hat{\mu} + \frac{\sigma^2}{2} - \beta r\hat{\mu} - \beta(2 - 2\alpha)\frac{\sigma^2}{2}r - 2\beta(2 - 2\alpha)\frac{\sigma^2}{2}\gamma r^2\right]
\end{aligned}
$$

$$+(d^L)^2\left[-\frac{1}{2}\gamma\sigma^2+\beta(2-2\alpha)\frac{\sigma^2}{2}\gamma r^2\right]$$

$$+\hat{\mu}\left[\alpha\beta\left(\frac{3\alpha\hat{\mu}+\sigma^2}{2\gamma\sigma^2}\right)-(1+\alpha\beta)+\beta r\right]$$

$$+\frac{\sigma^2}{2}[-(1+\alpha\beta)(1-\gamma)+\beta(2-2\alpha)r+\beta(2-2\alpha)\gamma r^2]$$

$$=\quad d^L\left[\hat{\mu}(1-\beta r)+\frac{\sigma^2}{2}[1-\beta(2-2\alpha)r(1+2\gamma r)]\right]$$

$$+(d^L)^2\left[\frac{\gamma\sigma^2}{2}[\beta(2-2\alpha)r^2-1]\right]$$

$$+\hat{\mu}\left[\alpha\beta\left(\frac{3\alpha\hat{\mu}+\sigma^2}{2\gamma\sigma^2}-1\right)-1+\beta r\right]$$

$$+\frac{\sigma^2}{2}[\beta(2-2\alpha)r(1+\gamma r)-(1+\alpha\beta)(1-\gamma)].$$

We now remove all terms except the basic parameters $\hat{\mu}$, α, γ, σ^2, by expressing $d^L=\frac{2\hat{\mu}+\sigma^2}{2\gamma\sigma^2}$ as in (6) to obtain

$$P^L-P^P\quad=\quad\frac{2\hat{\mu}+\sigma^2}{2\gamma\sigma^2}\left[\hat{\mu}(1-\beta r)+\frac{\sigma^2}{2}[1-\beta(2-2\alpha)r(1+2\gamma r)]\right]$$

$$+\frac{4\hat{\mu}^2+4\hat{\mu}\sigma^2+\sigma^4}{4\gamma^2\sigma^4}\left[\frac{\gamma\sigma^2}{2}[\beta(2-2\alpha)r^2-1]\right]$$

$$+\hat{\mu}\left[\alpha\beta\left(\frac{3\alpha\hat{\mu}+\sigma^2-2\gamma\sigma^2}{2\gamma\sigma^2}\right)-1+\beta r\right]$$

$$+\frac{\sigma^2}{2}[\beta(2-2\alpha)r(1+\gamma r)-(1+\alpha\beta)(1-\gamma)]$$

$$=\quad\frac{\hat{\mu}}{\gamma\sigma^2}\left[\hat{\mu}(1-\beta r)+\frac{\sigma^2}{2}[1-\beta(2-2\alpha)r(1+2\gamma r)]\right]$$

$$+\frac{1}{2\gamma}\left[\hat{\mu}(1-\beta r)+\frac{\sigma^2}{2}[1-\beta(2-2\alpha)r(1+2\gamma r)]\right]$$

$$+\frac{\hat{\mu}^2}{\gamma\sigma^2}\left[\frac{1}{2}[\beta(2-2\alpha)r^2-1]\right]+\frac{\hat{\mu}}{\gamma}\left[\frac{1}{2}[\beta(2-2\alpha)r^2-1]\right]$$

$$+\frac{1}{4\gamma}\left[\frac{\sigma^2}{2}[\beta(2-2\alpha)r^2-1]\right]+\hat{\mu}\left[\alpha\beta\left(\frac{3\alpha\hat{\mu}+\sigma^2-2\gamma\sigma^2}{2\gamma\sigma^2}\right)-1+\beta r\right]$$

$$+\frac{\sigma^2}{2}[\beta(2-2\alpha)r(1+\gamma r)-(1+\alpha\beta)(1-\gamma)]$$

$$=\quad\frac{\hat{\mu}}{\gamma\sigma^2}[\hat{\mu}(1-\beta r)]+\frac{\hat{\mu}}{2\gamma}[1-\beta(2-2\alpha)r(1+2\gamma r)]$$

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$$+\frac{\hat{\mu}}{2\gamma}[1-\beta r]+\frac{\sigma^2}{4\gamma}[1-\beta(2-2\alpha)r(1+2\gamma r)]$$

$$+\frac{\hat{\mu}^2}{2\gamma\sigma^2}[\beta(2-2\alpha)r^2-1]+\frac{\hat{\mu}}{2\gamma}[\beta(2-2\alpha)r^2-1]+\frac{\sigma^2}{8\gamma}[\beta(2-2\alpha)r^2-1]$$

$$+\hat{\mu}\left[\alpha\beta\left(\frac{3\alpha\hat{\mu}+\sigma^2-2\gamma\sigma^2}{2\gamma\sigma^2}\right)-1+\beta r\right]+\frac{\sigma^2}{2}[\beta(2-2\alpha)r(1+\gamma r)-(1+\alpha\beta)(1-\gamma)].$$

Now we collect the terms to obtain the desired polynomial in $\hat{\mu}$, $\hat{\mu}^2$, and σ^2 :

$$
\begin{aligned}
P^L - P^P &= \hat{\mu}\left[\frac{1-\beta(2-2\alpha)r(1+2\gamma r)+1-\beta r+\beta(2-2\alpha)r^2-1}{2\gamma}\right]\\
&+\hat{\mu}\left[\alpha\beta\frac{(1-2\gamma)}{2\gamma}-1+\beta r\right]\\
&+\hat{\mu}^2\left[\frac{1-\beta r}{\gamma\sigma^2}+\frac{\beta(2-2\alpha)r^2-1+3\alpha^2\beta}{2\gamma\sigma^2}\right]\\
&+\frac{\sigma^2}{2}\left[\frac{1-\beta(2-2\alpha)r(1+2\gamma r)}{2\gamma}+\frac{\beta(2-2\alpha)r^2-1}{4\gamma}\right]\\
&+\frac{\sigma^2}{2}[\beta(2-2\alpha)r(1+\gamma r)-(1+\alpha\beta)(1-\gamma)]\\
&= \hat{\mu}\left[\frac{1-\beta(2-2\alpha)r(1+2\gamma r)-\beta r+\beta(2-2\alpha)r^2}{2\gamma}\right]\\
&+\hat{\mu}\left[\frac{\alpha\beta(1-2\gamma)-2\gamma+2\beta\gamma r}{2\gamma}\right]\\
&+\hat{\mu}^2\left[\frac{2-2\beta r+\beta(2-2\alpha)r^2-1+3\alpha^2\beta}{2\gamma\sigma^2}\right]\\
&+\frac{\sigma^2}{2}\left[\frac{2-2\beta(2-2\alpha)r(1+2\gamma r)+\beta(2-2\alpha)r^2-1}{4\gamma}\right]\\
&+\frac{\sigma^2}{2}\left[\frac{4\beta\gamma(2-2\alpha)r(1+\gamma r)-4\gamma(1+\alpha\beta)(1-\gamma)}{4\gamma}\right]\\
&= \hat{\mu}\left[\frac{1+(2-2\alpha)\beta r(r-1-2\gamma r)-\beta r+\alpha\beta-2\alpha\beta\gamma-2\gamma+2\beta\gamma r}{2\gamma}\right]\\
&+\hat{\mu}^2\left[\frac{1-2\beta r+2\beta r^2-2\alpha\beta r^2+3\alpha^2\beta}{2\gamma\sigma^2}\right]\\
&+\frac{\sigma^2}{2}\left[\frac{1-(4\beta r-4\alpha\beta r)(1+2\gamma r)+2\beta r^2-2\alpha\beta r^2+(8\beta\gamma r-8\alpha\beta\gamma r)(1+\gamma r)}{4\gamma}\right]\\
&+\frac{\sigma^2}{2}\left[\frac{-4\gamma(1+\alpha\beta)+4\gamma^2(1+\alpha\beta)}{4\gamma}\right]\\
&= \hat{\mu}\left[\frac{1+2\beta r(r-1-2\gamma r)-2\alpha\beta r(r-1-2\gamma r)-\beta r+\alpha\beta-2\alpha\beta\gamma-2\gamma+2\beta\gamma r}{2\gamma}\right]
\end{aligned}
$$

$$+\hat{\mu}^2 \left[\frac{1 - 2\beta r + 2\beta r^2 - 2\alpha\beta r^2 + 3\alpha^2\beta}{2\gamma\sigma^2} \right]$$

$$+\frac{\sigma^2}{2} \left[\frac{1 - (4\beta r - 4\alpha\beta r) - (8\beta\gamma r^2 - 8\alpha\beta\gamma r^2) + 2\beta r^2 - 2\alpha\beta r^2 + 8\beta\gamma r - 8\alpha\beta\gamma r}{4\gamma} \right]$$

$$+\frac{\sigma^2}{2} \left[\frac{8\beta\gamma^2 r^2 - 8\alpha\beta\gamma^2 r^2 - 4\gamma - 4\alpha\beta\gamma + 4\gamma^2 + 4\alpha\beta\gamma^2}{4\gamma} \right].$$

This expression can be further simplified for calibration purposes, as

$$
\begin{aligned}
P^L - P^P &= \hat{\mu} \left[\frac{1 + 2\beta r^2 - 2\beta r - 4\beta\gamma r^2 - 2\alpha\beta r^2 + 2\alpha\beta r + 4\alpha\beta\gamma r^2 - \beta r + \alpha\beta - 2\alpha\beta\gamma - 2\gamma + 2\beta\gamma r}{2\gamma} \right] \\
&+\hat{\mu}^2 \left[\frac{1 - 2\beta r + 2\beta r^2 - 2\alpha\beta r^2 + 3\alpha^2\beta}{2\gamma\sigma^2} \right] \\
&+\frac{\sigma^2}{2} \left[\frac{1 - 4\beta r + 4\alpha\beta r - 8\beta\gamma r^2 + 8\alpha\beta\gamma r^2 + 2\beta r^2 - 2\alpha\beta r^2 + 8\beta\gamma r - 8\alpha\beta\gamma r}{4\gamma} \right] \\
&+\frac{\sigma^2}{2} \left[\frac{8\beta\gamma^2 r^2 - 8\alpha\beta\gamma^2 r^2 - 4\gamma - 4\alpha\beta\gamma + 4\gamma^2 + 4\alpha\beta\gamma^2}{4\gamma} \right] \\
&= \hat{\mu} \left[\frac{1 + 2\beta r^2 - 2\alpha\beta r^2 - 3\beta r + 2\alpha\beta r + \alpha\beta - 2\gamma + 2\beta\gamma r - 4\beta\gamma r^2 + 4\alpha\beta\gamma r^2 - 2\alpha\beta\gamma}{2\gamma} \right] \\
&+\hat{\mu}^2 \left[\frac{1 - 2\beta r + 2\beta r^2 - 2\alpha\beta r^2 + 3\alpha^2\beta}{2\gamma\sigma^2} \right] \\
&+\frac{\sigma^2}{2} \left[\frac{1 - 4\beta r(1 - \alpha) - 8\beta\gamma r^2(1 - \alpha) + 2\beta r^2(1 - \alpha) + 8\beta\gamma r(1 - \alpha) + 8\beta\gamma^2 r^2(1 - \alpha)}{4\gamma} \right] \\
&+\frac{\sigma^2}{2} \left[\frac{-4\gamma - 4\alpha\beta\gamma + 4\gamma^2 + 4\alpha\beta\gamma^2}{4\gamma} \right].
\end{aligned}
$$

Finally, we can factor the above expression further in terms of β everywhere to obtain

$$
\begin{aligned}
P^L - P^P &= \hat{\mu} \left[\frac{1 - 2\gamma + \beta[\alpha + 2r^2(1 - \alpha) + 2\alpha r - 3r + 2\gamma r - 4\gamma r^2(1 - \alpha) - 2\alpha\gamma]}{2\gamma} \right] \qquad (16)\\
&+\hat{\mu}^2 \left[\frac{1 + \beta[3\alpha^2 - 2r + 2r^2(1 - \alpha)]}{2\gamma\sigma^2} \right] \\
&+\frac{\sigma^2}{2} \left[\frac{1 + 4\gamma(\gamma - 1) + 2\beta[-2r(1 - \alpha) - 4\gamma r^2(1 - \alpha) + (4\gamma r + 4\gamma^2 r^2 + r^2)(1 - \alpha) + 2\alpha\gamma(\gamma - 1)]}{4\gamma} \right] \\
&= \hat{\mu} \left[\frac{\beta[\alpha(1 + 2r) + (1 - \alpha)(2r^2 - 4\gamma r^2) - 3r + 2\gamma r - 2\alpha\gamma] + (1 - 2\gamma)}{2\gamma} \right]
\end{aligned}
$$

$$+\hat{\mu}^2 \left[\frac{1 + \beta[3\alpha^2 - 2r + 2r^2(1 - \alpha)]}{2\gamma\sigma^2} \right]$$

$$+\frac{\sigma^2}{2} \left[\frac{1 + 4\gamma(\gamma - 1) + 2\beta[(1 - \alpha)(4\gamma r + 4\gamma^2 r^2 + r^2 - 2r - 4\gamma r^2) + 2\alpha\gamma(\gamma - 1)]}{4\gamma} \right].$$

Equation (16) is the desired polynomial in $\hat{\mu}$ and σ^2, which represents the net utility gain from following the leveraged versus the prudent strategy. Upon inspection this quantity can be confirmed as bounded.

$\square$

Proposition 4. *The utility loss from following a suboptimal strategy is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.*

Proof. We have to show that $\Delta EU \leq K$, for some positive constant K. To do this, we compute the difference between payoffs to the optimal strategies (P^L, P^P) and their suboptimal counterparts $(\hat{P}^L, \hat{P}^P)$. That is, we compute $P^L - \hat{P}^L$ and $P^L - \hat{P}^L$. Below we first compute the optimal, then suboptimal payoffs.

Optimal Payoffs (P^L, P^P). These are the same as above, in equations (14) and (13):

$$P^P = EU(d^P|\sigma^2) + \beta \left[\alpha EU(d^E|\frac{\sigma^2}{\alpha}) + (1 - \alpha)EU(d^P|\sigma^2) \right]$$

and

$$P^L = EU(d^L|\sigma^2) + \beta \left[2\alpha \left(EU(d^E_{2\alpha}|\frac{\sigma^2}{2\alpha}) - EU(r(d^L - 1)|\frac{\sigma^2}{2\alpha}) \right) \right]$$
$$+ \beta \left[(1 - 2\alpha) \left(EU(d^P|\sigma^2) - EU(r(d^L - 1)|\sigma^2) \right) \right].$$

Suboptimal Payoffs $(\hat{P}^L, \hat{P}^P)$. The strategy here involves demanding either d^L or d^P in period 1 (again depending on parameter values). Then in period 2 the investor repays any borrowing, and since she mistakenly believes the world is always in the typical regime, she always demands the most she can, $d^P = 1$, regardless of whether the realized regime is extreme or typical. To compute the results, we proceed as follows. If she over-invests by choosing d^L in the first period, the likelihood of extremes raises from α to 2α, and her payoff $\hat{P}^L$ is

$$\hat{P}^L = EU(d^L|\sigma^2) + \beta \left[2\alpha \left(EU(d^P|\frac{\sigma^2}{2\alpha}) - EU(r(d^L - 1)|\frac{\sigma^2}{2\alpha}) \right) \right] \qquad (17)$$
$$+ \beta \left[(1 - 2\alpha) \left(EU(d^P|\sigma^2) - EU(r(d^L - 1)|\sigma^2) \right) \right].$$

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Conversely, if she chooses d^P in the first period, the likelihood of extremes remains unchanged at α and her payoff $\hat{P}^P$ is

$$\hat{P}^P = EU(d^P|\sigma^2) + \beta\left[\alpha EU(d^P|\frac{\sigma^2}{\alpha}) + (1-\alpha)EU(d^P|\sigma^2)\right]. \tag{18}$$

Utility Differentials $P^L - \hat{P}^L$ and $P^P - \hat{P}^P$. First we compute $P^L - \hat{P}^L$. Using equations (14) and (17), we obtain

$$P^L - \hat{P}^L = 2\alpha\beta\left[\left(EU(d_{2\alpha}^E|\frac{\sigma^2}{2\alpha}) - EU(d^P|\frac{\sigma^2}{2\alpha})\right)\right],$$

which from equations (3) and (6) yields

$$
\begin{aligned}
P^L - \hat{P}^L &= 2\alpha\beta\left[d_{2\alpha}^E\hat{\mu} + \frac{1}{2}d_{2\alpha}^E(1 - d_{2\alpha}^E)\frac{\sigma^2}{2\alpha} + \frac{1}{2}(1-\gamma)(d_{2\alpha}^E)^2\frac{\sigma^2}{2\alpha}\right] \\
&\quad -2\alpha\beta\left[d^P\hat{\mu} + \frac{1}{2}d^P(1 - d^P)\frac{\sigma^2}{2\alpha} + \frac{1}{2}(1-\gamma)(d^P)^2\frac{\sigma^2}{2\alpha}\right] \\
&= 2\alpha\beta\left[d_{2\alpha}^E\hat{\mu} + \frac{1}{2}d_{2\alpha}^E(1 - d_{2\alpha}^E)\frac{\sigma^2}{2\alpha} + \frac{1}{2}(1-\gamma)(d_{2\alpha}^E)^2\frac{\sigma^2}{2\alpha}\right] - 2\alpha\beta\left[\hat{\mu} + \frac{1}{2}(1-\gamma)\frac{\sigma^2}{2\alpha}\right].
\end{aligned}
$$

We now factor this expression into terms involving $\hat{\mu}$ and $\frac{\sigma^2}{2}$, to obtain

$$
\begin{aligned}
P^L - \hat{P}^L &= \beta\frac{\sigma^2}{2}[d_{2\alpha}^E(1 - d_{2\alpha}^E) + (1-\gamma)(d_{2\alpha}^E)^2 - (1-\gamma)] - 2\alpha\beta\hat{\mu}\left[1 - d_{2\alpha}^E\right] \\
&= \beta\frac{\sigma^2}{2}[d_{2\alpha}^E - (d_{2\alpha}^E)^2 + (d_{2\alpha}^E)^2 - \gamma(d_{2\alpha}^E)^2 - (1-\gamma)] - 2\alpha\beta\hat{\mu}\left[1 - d_{2\alpha}^E\right] \\
&= \beta\frac{\sigma^2}{2}[d_{2\alpha}^E - 1 - \gamma(d_{2\alpha}^E)^2 + \gamma] - 2\alpha\beta\hat{\mu}\left[1 - d_{2\alpha}^E\right] \\
&= \beta\frac{\sigma^2}{2}[d_{2\alpha}^E - 1 + \gamma(1 - (d_{2\alpha}^E)^2)] - 2\alpha\beta\hat{\mu}\left[1 - d_{2\alpha}^E\right] \\
&= \beta\gamma\frac{\sigma^2}{2}[1 - (d_{2\alpha}^E)^2] - \beta(1 - d_{2\alpha}^E)[2\alpha\hat{\mu} + \frac{\sigma^2}{2}].
\end{aligned}
$$

Now, substituting $d_{2\alpha}^E = \frac{4\alpha\hat{\mu}+\sigma^2}{2\gamma\sigma^2}$ from (8) yields

$$
\begin{aligned}
P^L - \hat{P}^L &= \beta\gamma\frac{\sigma^2}{2}\left[\frac{4\gamma^2\sigma^4 - 16\alpha^2\hat{\mu}^2 - 8\alpha\hat{\mu}\sigma^2 - \sigma^4}{4\gamma^2\sigma^4}\right] \\
&\quad -\beta\left[\frac{2\gamma\sigma^2 - 4\alpha\hat{\mu} - \sigma^2}{2\gamma\sigma^2}\right]\left[\frac{4\alpha\hat{\mu} + \sigma^2}{2}\right] \\
&= \beta\left[\frac{4\gamma^2\sigma^4 - 16\alpha^2\hat{\mu}^2 - 8\alpha\hat{\mu}\sigma^2 - \sigma^4}{8\gamma\sigma^2}\right] \\
&\quad -\beta\left[\frac{8\alpha\gamma\hat{\mu}\sigma^2 + 2\gamma\sigma^4 - 16\alpha^2\hat{\mu}^2 - 4\alpha\hat{\mu}\sigma^2 - 4\alpha\hat{\mu}\sigma^2 - \sigma^4}{4\gamma\sigma^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta}{8\gamma\sigma^2}\left[4\gamma^2\sigma^4 - 16\alpha^2\hat{\mu}^2 - 8\alpha\hat{\mu}\sigma^2 - \sigma^4 - 16\alpha\gamma\hat{\mu}\sigma^2 - 4\gamma\sigma^4 + 32\alpha^2\hat{\mu}^2 + 16\alpha\hat{\mu}\sigma^2 + 2\sigma^4\right] \\
&= \frac{\beta}{8\gamma\sigma^4}[4\gamma^2\sigma^4 + 16\alpha^2\hat{\mu}^2 + 8\alpha\hat{\mu}\sigma^2 + \sigma^4 - 16\alpha\gamma\hat{\mu}\sigma^2 - 4\gamma\sigma^4] \\
&= \frac{\beta}{8\gamma\sigma^2}\left[\hat{\mu}(16\alpha^2\hat{\mu} + 8\alpha\sigma^2 - 16\alpha\gamma\sigma^2) + \sigma^4(4\gamma^2 - 4\gamma + 1)\right] \\
&= \frac{\beta}{8\gamma\sigma^2}\left[8\alpha\hat{\mu}(2\alpha\hat{\mu} + \sigma^2(1 - 2\gamma)) + \sigma^4[4\gamma(\gamma - 1) + 1]\right] \\
&= \frac{\beta}{8\gamma\sigma^2}\left[8\alpha\hat{\mu}(2\alpha\hat{\mu} - \sigma^2(2\gamma - 1)) - \sigma^4[-1 - 4\gamma(\gamma - 1)]\right]. \tag{19}
\end{aligned}
$$

The expression in (19) represents the net utility gain from following the optimal versus the suboptimal strategy, in the case of leverage. Upon inspection this quantity can be confirmed as bounded.

We now consider the differential between P^P and its suboptimal counterpart $\hat{P}^P$.

$P^P - \hat{P}^P$: From equations (13) and (18) we have

$$
\begin{aligned}
P^P - \hat{P}^P &= \alpha\beta\left[EU(d^E|\frac{\sigma^2}{\alpha}) - EU(d^P|\frac{\sigma^2}{\alpha})\right] \\
&= \alpha\beta\left[d^E\hat{\mu} + \frac{1}{2}d^E(1 - d^E)\frac{\sigma^2}{\alpha} + \frac{1}{2}(1 - \gamma)(d^E)^2\frac{\sigma^2}{\alpha}\right] \\
&\quad -\alpha\beta\left[\hat{\mu} + \frac{1}{2}(1 - \gamma)\frac{\sigma^2}{\alpha}\right] \\
&= \alpha\beta\hat{\mu}[d^E - 1] + \beta\frac{\sigma^2}{2}[d^E(1 - d^E) + (1 - \gamma)(d^E)^2 - (1 - \gamma)] \\
&= \beta\frac{\sigma^2}{2}[d^E - \gamma(d^E)^2 - 1 + \gamma] - \alpha\beta\hat{\mu}[1 - d^E] \\
&= \beta\frac{\gamma\sigma^2}{2}[1 - (d^E)^2] - \beta[1 - d^E]\left[\alpha\hat{\mu} + \frac{\sigma^2}{2}\right].
\end{aligned}
$$

Now we can substitute $d^E = \frac{2\alpha\hat{\mu} + \sigma^2}{2\gamma\sigma^2}$ from (6) to obtain

$$
\begin{aligned}
P^P - \hat{P}^P &= \beta\frac{\gamma\sigma^2}{2}\left[\frac{4\gamma^2\sigma^4 - 4\alpha^2\hat{\mu}^2 - 4\alpha\hat{\mu}\sigma^2 - \sigma^4}{4\gamma^2\sigma^4}\right] \\
&\quad -\beta\left[\frac{2\gamma\sigma^2 - 2\alpha\hat{\mu} - \sigma^2}{2\gamma\sigma^2}\right]\left[\frac{2\alpha\hat{\mu} + \sigma^2}{2}\right] \\
&= \beta\left[\frac{4\gamma^2\sigma^4 - 4\alpha^2\hat{\mu}^2 - 4\alpha\hat{\mu}\sigma^2 - \sigma^4}{8\gamma\sigma^2}\right] \\
&\quad -\beta\left[\frac{4\alpha\gamma\hat{\mu}\sigma^2 + 2\gamma\sigma^4 - 4\alpha^2\hat{\mu}^2 - 2\alpha\hat{\mu}\sigma^2 - 2\alpha\hat{\mu}\sigma^2 - \sigma^4}{4\gamma\sigma^2}\right]
\end{aligned}
$$

$$= \frac{\beta}{8\gamma\sigma^2} \left[4\gamma^2\sigma^4 - 4\alpha^2\hat{\mu}^2 - 4\alpha\hat{\mu}\sigma^2 - \sigma^4 - 8\alpha\gamma\hat{\mu}\sigma^2 - 4\gamma\sigma^4 + 8\alpha^2\hat{\mu}^2 + 8\alpha\hat{\mu}\sigma^2 + 2\sigma^4 \right]$$

$$= \frac{\beta}{8\gamma\sigma^2} \left[4\gamma^2\sigma^4 + 4\alpha^2\hat{\mu}^2 + 4\alpha\hat{\mu}\sigma^2 + \sigma^4 - 8\alpha\gamma\hat{\mu}\sigma^2 - 4\gamma\sigma^4 \right]$$

$$= \frac{\beta}{8\gamma\sigma^2} \left[\hat{\mu}(4\alpha^2\hat{\mu} + 4\alpha\sigma^2 - 8\alpha\gamma\sigma^2) + \sigma^4(4\gamma^2 + 1 - 4\gamma) \right]$$

$$= \frac{\beta}{8\gamma\sigma^2} \left[4\alpha\hat{\mu}(\alpha\hat{\mu} - \sigma^2(2\gamma - 1)) - \sigma^4(-4\gamma(\gamma - 1) - 1) \right]. \tag{20}$$

The expression in (20) represents the net utility gain from following the optimal versus the suboptimal strategy, in the case of prudent first-period investment. Upon inspection this quantity can be confirmed as bounded. $\square$