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Abstract

The knowledge constraints and transactions costs imposed by geographical distance, network connections and time conspire to justify local behavior as a good approximation for global rationality. We consider a class of games to illustrate this relationship and raise some questions as to what constitutes a satisfactory solution concept.

Keywords: Local games, Network games, Advertising

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1 On Rational Optimizing Behavior

Much of economics and game theory has been based on models of rational optimizing individuals with well-defined preferences, considerable ability to compute and common knowledge. Attempts to escape from this structure have posed many difficulties. Keynes, like Alexander, cut the Gordian knot. He directly and brazenly assumed habitual behavior such as "propensity to consume." Much of macroeconomics, since then has avoided the "rationality trap" of the theory of consumer choice and of general equilibrium theory. No macro-economist worthy of her salt worries too much about individual utility functions being optimized, but is more concerned with whether one can characterize consumer behavior in a manner that provides at least a reasonable fit with short term data.

The game theorists doing evolutionary game theory can argue reasonably that the genes or the birds or bugs being discussed act as local optimal response mechanisms and do not worry heavily about how rational the agents are. Some of us believe that a better case can be made for the use of Nash noncooperative equilibrium theory in evolutionary games than in many situations involving intelligent humans. There are some situations where even a unique noncooperative equilibrium appears to be unsatisfactory (the best example was provided by Shapley, see Shubik, 1996).

Herbert Simon (1984) coined the phrase "satisficing" to indicate the situation that many practitioners in finance and economics recognize clearly. The more or less rational optimizing human being chasing after profits usually settles for "good enough." Once a reasonable goal has been achieved the hunter relaxes.

A skilled mathematical game theorist with a sense of reality knows full well that for many situations many of us operate without common knowledge. This leads to an elegant and difficult branch of game theory where we study how much the rational intelligent players without common knowledge can infer from the information provided by the playing of the game (see Aumann, 1976). John Harsanyi (1967, 1968a, b), before the development of this work devised a theory of incomplete information concerning the player types. The initial subjective probabilities needed to get his construct off the ground were assumed to be given.

In spite of Simon's clarion call to model limited or capacity constrained or habit constrained rational behavior this has proved to be far more difficult than was originally expected. A separate but parallel development in artificial intelligence (see Minsky, 1986) has shown that it is far more difficult to build machines to perform simple but even mundane non-routine tasks.

A practicing applied micro-economist knows that he is dealing with approximations. For many applied problems almost always logical niceties such as precision in the specification of the rules of the game and checking for completeness and consistency are not important. Implicitly "good enough" is implied in most modeling and analysis. It is not the evil forces of capitalism or socialism at work which makes it possible to find two sets of respectable practitioners willing to give expert witness testimony on both sides of almost any question. It is the unfortunate fact that human individual and group behavior is horrendously complex. The questions we can answer are only answered up to some degree of accuracy and are context constrained. The art of modeling is heavily devoted to providing the appropriate abstractions of the context most useful in answering the question being posed. Part of the art of modeling is to know what to accept as the primitive concepts and when to state that the subjective probability distribution being used to jump start an elaborate analysis is the author's best guess based on whatever research or learning assumptions she chooses to supply.

Many mathematical economists and game theorists are far more deeply concerned than are applied macro- or micro-economists with clarity, logical consistency and completeness of their models before they apply the heavy artillery of the mathematical analysis. We support this view because many of the questions being addressed are posed at a higher level of generality and abstraction than most of the ad hoc questions considered by applied social scientists. Questions that are both deep and difficult to tackle include: Do there exist invariant principles of efficiency and fair division; can we provide an abstract definition of power, are there basic abstractions of individual and group preferences? These are specific questions related to the more general question concerning what constitutes a satisfactory solution to an n-person non-constant sum game.

In order to approach these questions many game theorists immediately take as implicit assumptions the full definitions of the rules of the game. The pure abstraction of the rational decision-maker that forms the backbone of much economic theory and game theory provides a valuable abstraction of considerable use both in teaching and analysis. Yet it is easy to point out its limitations and to show many aspects of human behavior where this rationalistic model is a poor descriptor of human behavior. It is far more difficult to replace this model and the theories based on it with general behavioral models and with better theories based on them.

The approach adopted here is to consider how far we can go with models of strategic situations with context-constrained agents, i.e. are there conditions which result in agents who share common knowledge and know all the rules of the game nevertheless behave as though they were local optimizers?

2 Context Constrained Optimizing Behavior

There are three sets of literature and two approaches which appear to be of importance to future developments. They are models in which proximity in measurable space and time both play a direct role; models in which the key feature is the presence of network connections where distance has little influence, and the third is models based on social analogies to statistical mechanics (Durlauf, 1997). We concentrate on the first two. When even the most rationalistic game theorist pauses to reflect on his or her behavior, several items can be noted. In much of our activity we appear to be more or less rational, but to a great extent our nonconscious selves supply considerable limitations to the context of what we consider the game to be. In particular we are constrained by:

1. Space
2. Time and
5. Net

2.1 The Constraints of Space

We are a species living on a two-dimensional surface. In spite of private jet planes, large airlines, railroads and highway systems much of our day is spent in walking short distances around the house and office. Most of us do not walk at a speed of more than three to four miles an hour. Distance still matters in many aspects of daily existence. Since the work of Hotelling (1929) and Chamberlin (1933) the possibility that space confers local monopoly power has been considered. In our every day lives commuting time and distance matter; how far we will travel to go out to dinner and where the show is playing are all part and parcel of daily existence. When an individual in a Boston suburb goes out for a casual dinner she does not even allow her knowledge of Los Angeles restaurants to reach her consciousness, except possibly to make some comparisons.

The ruling out of consideration features which might be expected to produce at most second and third order effects is part and parcel of context constrained rational behavior.

2.2 The Constraints of Time

Both cardinal and ordinal time are of importance in different ways in the modeling and understanding of human behavior. The ordering of events is critical to the treatment of information in the consideration of strategic behavior. Possibly one of the major contributions of game theory to our understanding of decision-making is how information enlarges individual strategic choice.

When dealing with process there are several natural questions we must ask. If the situation under study is embedded in time, how much does history matter? Do reputations of individuals matter? Does the situation we are studying have an explicit termination date? Or is it that the termination is event driven?

Much of micro-economics and game theory has been devoted to equilibrium conditions. In Marshallian micro-economics and in oligopoly theory comparative static analysis is often discussed. Two time slices of a part of the economy are considered and the changes in the state of the system are examined. Except for careful ad hoc studies of specific economic institutions a believable full dynamics complete with equations of motion is a rarity in economics. At best even the most accurate of economic forecasts is local. Equations are updated and a prediction is made concerning motion for a limited time, after which a re-estimation is made and a new prediction is based on the re-estimation.

A key and, in our opinion unanswerable question, in general, in economics and in non-cooperative game theory is how a system out of equilibrium attains an equilibrium state. This is a far different question from if a system already in equilibrium is given a small displacement does it return to equilibrium? The Nash equilibrium analysis and allied concepts in economics such as rational expectations offer, in essence, no indication of the nature of the dynamics.

When we contemplate actual human behavior and try to construct games to illustrate what is going on, several natural questions arise. Suppose that we construct a game where we initialize all conditions such that the individuals begin with expectations supplied by the game briefing; should the players believe the referee's briefing? Where do the priors come from? If one explains the rules of a game such as a simultaneous double auction market without giving the individuals any briefing about previous prices and volume of trade the players have little on which to base their expectations. As the double auction market is a peculiarly efficient structure experimental experience from many experimenters (see e.g., Smith, 1965; Sundar and Bosch, 2000) indicate that after a few plays, at least a competitive price emerges. But in many other situations the convergence speed to an equilibrium is not necessarily fast.

In the study of human behavior a reasonable question to ask is how many times, or for how long does a game have to be played until we believe that it has a reasonable chance to approach an equilibrium? We suggest that the answer is ad hoc. It depends on the situation and its context. In particular many of the most important decisions faced by the individual such as what profession to select, who to marry, where to live, when to change jobs are made over a life time with low frequencies. They provide at best extremely small samples to utilize to adjust behavior. Social network information and pressures provide a considerable reinforcement for these low frequency events. But this indicates that the gap between the equilibrium analysis of a game repeated infrequently and the explanation of adjustments towards equilibrium where the sample size may be only one, two, three or four offers us little on which to base the updating towards equilibrium. The dedicated Bayesian can reply: One takes the initial priors as given and then does a Bayesian updating as more information comes in. The logic is impeccable provided that one accepts that the priors are appropriate and that the context for the updating is given. Even if this were true the number of iterations may be far too small to achieve an equilibrium. Unfortunately in general, in important decision-making the problem is far worse. The key to good decision-making may be the realization that the individual has the appropriate model and the relevant variables, The key to good forecasting (see Ascher 1978) is less the formal methodology, but the selection of the appropriate variables.

2.3 The Constraints of Net

Along with the developments in our understanding of biology, of computer science and along with the vast changes in the speed and scope of low cost communication and calculation, for many, but not all purposes, space has been replaced by net. If A knows B well and one is in Melbourne, while the other is in New York they can be in e-mail communication with each other as easily as if they lived next door. The world of web connections deals more with connectivity than distance. Proximity is in terms of communication rather than transportation.

3 The Extended Advertising or Promotion Model

We consider the structure of some games where the operational aspects of operating at a geographical or network distance is such that what appears to be locally optimizing behavior is globally justified. Out of sight may operationally mean out of mind.

In particular, we start with a game whose simplest version has been in the competitive advertising literature for many years (Mills, 1961, Gupta and Krishnan, 1966). There are n players in the game, each of whom has to allocate his resources among various markets. The players win "market share" according to how much they spend in a market compared with other players. The payoff for a player within a market is just his market share multiplied by the market size, and the total payoff for a player is the sum of his payoffs over all the markets.

In this paper, we consider a "localized version" of the above game in which players are given geographical locations. Each player is allowed to compete only in his "home region" (where he has a certain competitive advantage) and in the home regions of his "neighbors." The idea here is to model a "global game" whose outcome is completely determined by local interactions.

In the next sections we formally define this model, solve it (showing that there is a unique Nash equilibrium), and perform some sensitivity analysis. Then we consider an extension of the model where players may also compete in the home regions of their neighbors' neighbors. Finally, we show how the Nash equilibrium existence result outlined above fits into the literature on general existence results from noncooperative theory.

Finally, we remark that certain assumptions of ours allow us to solve our game by considering each region on its own and solving it "separately." The aggregated result solves the entire model. One might argue that this amount of simplification renders the results from Sections 5 and 6 less meaningful. However, the separability *does* allow us to compute closed form solutions to the model. This allows us to generate sensitivity analysis which would carry over to "nearby" nonseparable cases.

4 The Model

Let $P = \{1, ..., p\}$ be the set of players in a game. Each player $i \in P$ is based in his own "home region," which we denote as "region i." Certain pairs of regions are designated as "neighbors," and it is this "neighbor" relation which helps define the local interactions of the game. This is because players are only allowed to act in their home region and/or in some neighboring region to their home region. Let N_i denote the set of neighbors of (region) i, with $n_i = |N_i|$. We assume $n_i \geq 1$ for all i, i.e., every player has at least one neighbor.[1] We see that $j \in N_i$ iff $i \in N_j$ for any$i$, j. Finally, we remark that i is not considered a neighbor of himself, i.e., $i \notin N_i$ for all $i \in P$.

The relation described above naturally defines a graph H, in which the vertex set is P and edge ij exists if and only if region i and region j are neighbors. We note that the relation of neighborness is not necessarily geographically defined; hence it is permitted for H to be nonplanar.

Each player i is endowed with a large finite amount M_i of investable resource (money), which she divides up into investment into her own region, investments into each of her neighboring regions, and money-not-spent (savings). Hence her strategies (in a game theoretic sense) are vectors $x \in X^i \equiv \{(x_{ii}, \{x_{ij}\}_{j \in N_i}) : x_{ii} + \sum_{j \in N_i} x_{ij} \leq M_i \text{ and } x_{ij} \geq 0 \ \forall j \in N_i \cup \{i\}\}$. The interpretation is that x_{ij} is the amount i spends in region j. Let $X = \mathcal{X}_{i \in P} X^i$. Then X represents the strategy space for all players. We note that the dimension of X will be $|P| + \sum_{i \in P} n_i$, and that the set X is compact.

At this point we should remark that our assumption that M_i is large[2] essentially means that i is not hampered by financial constraints as he searches for ways to optimize his net payoff.[3] This has some important ramifications. First, it means that i always has the option of transferring money spent in a region (either her own region or a neighbor's) to the "unspent" category, where it obtains a marginal revenue of

[1] In the case where some player has no neighbor, that player's actions are trivial and he may be removed from the game without changing the analysis for the rest of the players.

[2] In Section 4 we give a bound on just how high the M_i's need to be.

[3] In this respect our assumption is similar to the standard one made in transferable-utility cooperative games — that coalitions always have enough money on hand to effect any desired redistribution of payoff. We believe that the results with M_i of any size are qualitatively similar, but the boundary solutions would be computationally messy to obtain.

1.[4] Also, there are savings always available which may be transferred to investments.

Given a strategy vector $x \in X$, i's revenue $r_{ij}(x)$ from investing in region j depends on three factors – the "size" of region j, the relative size of her investment compared with that of other players' investments in region, and whether or not region j is her home region. Specifically, we have

$$r_{ij}(x) = \frac{m_j \alpha x_{ij}}{x_{jj} + \alpha \sum_{q \in N_j} x_{qj}} \text{ if } j \neq i. \tag{4.1}$$

$$r_{ii}(x) = \frac{m_i k_i x_{ii}}{x_{ii} + \alpha \sum_{q \in N_i} x_{qi}}. \tag{4.2}$$

Here α, $\{m_j\}_{j \in N_i \cup \{i\}}$, and k_i are parameters described below.

Let us now explain formulas (4.1) and (4.2). If $j \neq i$, the quantity $\alpha x_{ij}/(x_{jj} + \alpha \sum_{q \in N_j} x_{qj})$ represents the *proportion of influence* that i has in region j. This is just i's *proportion of the total monetary investment* in region j, except with her contribution (and the contributions of all the other neighbors of j) "damped" by a factor of α. Hence $\alpha \in [0, 1]$ is a measure of how much "power" players have in neighboring regions: if $\alpha = 0$, a player has no power in a neighboring region and hence has no incentive to invest there, while if $\alpha = 1$ the parameter makes no distinction for a player between a neighboring region and his home region. At any rate, multiplying the above proportion of influence by the "size" m_j of region j ($m_j \geq 0$) gives the return r_{ij}.[5]

If $j = i$, the term $x_{ii}/(x_{ii} + \alpha \sum_{q \in N_i} x_{qi})$ again describes i's proportion of influence (note that there is no "damping" if one is operating in one's own region). We again multiply by m_i to get a return, but then further by k_i to get the actual revenue. The k_i represents the multiplier effect of getting a return in ones own region over getting a return in other regions — we assume $k_i \geq 1$ for all i.

Finally, in the degenerate case where formulas (4.1) or (4.2) give "0/0," we interpret the fraction to be 0. This reflects the natural assumption that if no one invests in a region, then the revenue to everyone from that region is zero.[6]

To figure i's total net payoffs resulting from a vector $x \in X$, we need to add the quantities r_{ij} over all $j \in N_i \cup \{i\}$ and then subtract off the amount she spends, i.e.

$$u^i(x) = \frac{m_i k_i x_{ii}}{x_{ii} + \alpha \sum_{q \in N_i} x_{qi}} + \sum_{j \in N_i} \frac{m_j \alpha x_{ij}}{x_{jj} + \alpha \sum_{q \in N_j} x_{qj}} - x_{ii} - \sum_{j \in N_i} x_{ij}. \tag{4.3}$$

The notations above are sufficient to define a *local advertising game* (LAG). Formally, a LAG is given by a sextuple $G = (P, \{N_i\}_{i \in P}, \alpha, \{k_i\}_{i \in P}, \{m_i\}_{i \in P}, X)$, where

P = player set = $\{1, ..., p\}$.

[4] The marginal revenue is 1 because of the last two terms of the payoff function (2.3), below, imply that the marginal cost of *spent* money is 1.

[5] The reader will note that we can model a player with no home region simply by assigning him a "dummy region" of size 0.

[6] However, we should remark here that our payoff functions ($\{u_i\}_{i \in P}$, defined below) are not continuous at x's in which there is a region where noone invests. This causes problems when we try to prove existence in generalizations of our model. See Section 9.

N_i = set of neighbors for player i.
α = measure of players' power in neighboring regions ($\alpha \in [0, 1]$).
k_i = multiplier effect of i's evaluation of revenue in her home region ($k_i \geq 1$).
m_i = size of region i ($m_i \geq 0$).
X = strategy space for the players (see above).

Let us also remind the reader of two other notations we have made above:
$r_{ij}(x)$ = revenue function for player i in Region j.
$u^i(x)$ = payoff function for player i.

To close the section, we compare our model to those of Harker (1986) and Miller–Tobin–Friesz (1991). We may think of our model as corresponding to a case of theirs in which there are no transportation costs, the "production function" is linear, and the "inverse demand functions" have the specific functional form $\theta(D) = 1/D$ (their notation). However, their models cannot accomodate our parameters α and $\{k_i\}$, and the function $1/D$ does not satisfy the continuity and concavity conditions which ensure existence/uniqueness of Nash equilibria in their papers.

5 Nash Equilibrium

Our aim now is to "solve" the game, by finding a Nash equilibrium for it. Recall that in a noncooperative game with player set $P = \{1, ..., p\}$, strategy set $X = (X^1, ..., X^p)$ and payoff function $u = (u^1, ..., u^p)$, a *Nash equilibrium* (NE) is defined as a set of strategies $\hat{x} \in X$ for which $u^i(\hat{x}) \geq u^i(\hat{x}^1, ..., \hat{x}^{i-1}, x^i, \hat{x}^{i+1}, ..., \hat{x}^p)$ for all i and all $x^i \in X^i$. In words, an NE is a set of strategies, one for each player, with the property that each player is simulataneously playing a best response against what the others are doing.

In our game, we can find a closed form solution for an NE. Specifically, we have

Theorem 5.1: *Let $G = (P, \{N_i\}_{i\in P}, \alpha, \{k_i\}_{i\in P}, \{m_i\}_{i\in P}, X)$ be a LAG, with $\alpha > 0$.*[7] *Then an NE for G is given by:*

$$\hat{x}_{ij} = \frac{\alpha m_j n_j k_j}{(k_j n_j + \alpha)^2}, \ \forall i, j \in P : i \neq j, \tag{5.1}$$

$$\hat{x}_{ii} = \frac{\alpha k_i m_i n_i (k_i n_i - \alpha(n_i - 1))}{(k_i n_i + \alpha)^2}, \ \forall i \in P. \tag{5.2}$$

Proof: Suppose $i \in P$, and let us assume that all other players (except for i) are playing according to (5.1) and (5.2). We need to show that (5.1) and (5.2) give a

[7] {The reader will note that the formulas (5.1)-(5.2) do not constitute an NE in the case where $\alpha = 0$. Indeed, the proof below involves many algebraic cancellations, and so does not work when $\alpha = 0$ because in this case many of these cancellations are "0/0"'s. In fact, we shall show in Remark 5.9 (at the end of this section) that when $\alpha = 0$ there are no NEs! It thus may be useful to look at the cases where α *approaches* zero, as these will not have the strange behavior of the $\alpha = 0$ case.}

best response for i. This means that the vector $(\hat{x}_{ii}, \{\hat{x}_{ij}\}_{j\in N_i})$ (as defined in (5.1) and (5.2)) is the optimal solution to the nonlinear program (NLP)

$$\max_{(x_{ii},\{x_{ij}\}_{j\in N_i})} \frac{k_i m_i x_{ii}}{x_{ii} + \alpha \sum_{j\in N_i} \hat{x}_{ji}} + \sum_{j\in N_i} \frac{\alpha m_j x_{ij}}{\hat{x}_{jj} + \alpha x_{ij} + \alpha \sum_{q\in N_j : q\neq i} \hat{x}_{qj}} - x_{ii} - \sum_{j\in N_i} x_{ij}$$

$$\text{s.t. } x_{ii} + \sum_{j\in N_i} x_{ij} \leq M_i \tag{5.3}$$

$$x_{ii} \geq 0, x_{ij} \geq 0 \ \forall j \in N_i. \tag{5.4}$$

We note two items. First, since we assume M_i is large, $\hat{x}^i$ (given by (5.1)–(5.2)) satisfies (5.3). Second, it is clear that $\hat{x}^i$ is also nonnegative. Hence, since the objective function is concave in its variables, all we need to do to prove the theorem is show that $\hat{x}^i$ satisfies the first-order-conditions for the objective function f of (NLP). The first such condition is:

$$\frac{\partial f}{\partial x_{ii}} = 0 \Rightarrow \frac{(x_{ii} + \alpha \sum_{j\in N_i} \hat{x}_{ji}) k_i m_i - k_i m_i x_{ii}}{(x_{ii} + \alpha \sum_{j\in N_i} \hat{x}_{ji})^2} - 1 = 0$$

$$\Rightarrow \alpha k_i m_i \sum_{j\in N_i} \hat{x}_{ji} = (x_{ii} + \alpha \sum_{j\in N_i} \hat{x}_{ji})^2.$$

We see from (5.1) that $\sum_{j\in N_i} \hat{x}_{ji}$ is equal to $n_i[\alpha m_i n_i k_i/(k_i n_i + \alpha)^2]$. Hence our first first-order-condition becomes

$$\frac{\alpha^2 k_i^2 m_i^2 n_i^2}{(k_i n_i + \alpha)^2} = \left(x_{ii} + \frac{\alpha^2 k_i m_i n_i^2}{(k_i n_i + \alpha)^2} \right)^2.$$

One may readily verify (using a bit of algebra!) that the expression for $x_{ii} = \hat{x}_{ii}$ does indeed satisfy this equality.

Our second (set of) first-order-conditions is for each $j \neq i$,

$$\frac{\partial f}{\partial x_{ij}} = 0 \Rightarrow \frac{(\hat{x}_{jj} + \alpha x_{ij} + \alpha \sum_{q\in N_j : q\neq i} \hat{x}_{qj}) \alpha m_j - \alpha^2 m_j x_{ij}}{(\hat{x}_{jj} + \alpha x_{ij} + \alpha \sum_{q\in N_j : q\neq i} \hat{x}_{qj})^2} - 1 = 0$$

$$\Rightarrow \alpha m_j \hat{x}_{jj} + \alpha^2 m_j \sum_{q\in N_j : q\neq i} \hat{x}_{qj} = \left(\hat{x}_{jj} + \alpha x_{ij} + \alpha \sum_{q\in N_j : q\neq i} \hat{x}_{qj} \right)^2.$$

From (5.1) we have $\sum_{q\in N_j : q\neq i} \hat{x}_{qj} = (n_j - 1)[\alpha m_j n_j k_j/(k_j n_j + \alpha)^2]$ and equation (5.2) gives $\hat{x}_{jj} = [\alpha k_j m_j n_j (k_j n_j - \alpha(n_j - 1))/(k_j n_j + \alpha)^2]$. Our first-order-conditions become

$$\frac{\alpha^2 k_j m_j^2 n_j (k_j n_j - \alpha(n_j - 1))}{(k_j n_j + \alpha)^2} + \frac{\alpha^3 k_j m_j^2 n_j (n_j - 1)}{(k_j n_j + \alpha)^2}$$

$$= \left(\frac{\alpha k_j m_j n_j (k_j n_j - \alpha(n_j - 1))}{(k_j n_j + \alpha)^2} + \alpha x_{ij} + (n_j - 1) \frac{\alpha^2 m_j n_j k_j}{(k_j n_j + \alpha)^2} \right)^2.$$

Again we may verify that $x_{ij} = \hat{x}_{ij}$ satisfies this. ■

Theorem 5.2: *The NE found in* (5.1)–(5.2) *for* G *is unique.*

Proof: For the uniqueness, we shall show that if $\hat{x}$ is an NE, then $\hat{x}_{11}$ and $\{\hat{x}_{i1}\}_{i \in N_1}$ are uniquely determined; the same proof could be used to show that $\hat{x}_{jj}$ and $\{\hat{x}_{ij}\}_{i \in N_j}$ are uniquely determined for any $j \in P$.

Our proof begins with a series of small lemmas:

Lemma 5.3: *If* $m_1 = 0$*, then* $\hat{x}_{i1}$ *must be equal to zero for all* $i \in N_1 \cup \{1\}$*.*

Proof: Otherwise, $\hat{x}_{i1} > 0$ for some $i \in N_1 \cup \{1\}$. But then i has a profitable deviation by diverting investment from region 1 to savings. So $\hat{x}$ is not an NE. ■

Hence, for the rest of the proof we assume that $m_1 > 0$.

Lemma 5.4: *There must be at least two distinct players* i *and* q *for which* $\hat{x}_{i1} > 0$ *and* $\hat{x}_{q1} > 0$*.*

Proof: If there was no such player i (or q), then any player would have a profitable deviation from $\hat{x}$ by switching funds from savings to Region 1. If there was exactly one such player, then that player would have a profitable deviation from $\hat{x}$ by switching half his investment in Region 1 to savings (he would still win 100% of the Region 1 market, but only use half the expenditure to do so). Either way, since there is a profitable deviation $\hat{x}$ would not be an NE. ■

An important consequence of Lemma 5.4 is that, since there is positive total spending in region 1 from $\hat{x}$, the derivatives $(\partial/\partial x_{11})[k_1 m_1 x_{11}/(x_{11} + \alpha \sum_{q \in N_1} x_{q1})]$ and $(\partial/\partial x_{i1})[\alpha m_1 x_{i1}/(x_{11} + \alpha \sum_{q \in N_1} x_{q1})]$ (for any $i \in N_1$) must exist at $x = \hat{x}$. Also, we have

Lemma 5.5: *There exists a neighbor* i *of 1 for whom* $\hat{x}_{i1} > 0$*.*

Proof: Follows directly from Lemma 5.4. ■

Without loss of generality, let us assume that player 2 is a neighbor of 1 with $\hat{x}_{21} > 0$.

Lemma 5.6: *If either*

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{11}} \left(\frac{k_1 m_1 x_{11}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) < 1 \textit{and } \hat{x}_{11} = 0, \textit{ OR} \tag{5.5}$$

$$\exists i \in N_1 \textit{ with } |_{x=\hat{x}} \frac{\partial}{\partial x_{i1}} \left(\frac{\alpha m_1 x_{i1}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) < 1 \textit{ and } \hat{x}_{i1} = 0, \tag{5.6}$$

then

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{21}} \left(\frac{\alpha m_1 x_{21}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) < 1.$$

Proof: If (5.5) holds, we have

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{21}} \left(\frac{\alpha m_1 x_{21}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) = \frac{m_1 \alpha (\hat{x}_{11} + \alpha \sum_{q \in N_1 : q \neq 2} \hat{x}_{q1})}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2} = \frac{m_1 \alpha^2 \sum_{q \in N_1, q \neq 2} \hat{x}_{q1}}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2}$$

$$< \alpha \frac{k_1 m_1 \alpha \sum_{q \in N_1} \hat{x}_{q1}}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2} = \alpha |_{x=\hat{x}} \frac{\partial}{\partial x_{11}} \left(\frac{k_1 m_1 x_{11}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) < 1.$$

[Here the second equality follows from the assumption $\hat{x}_{11} = 0$; the first inequality follows because $k_1 \geq 1$ and $\hat{x}_{21} > 0$; and the last inequality follows from assumption (5.5).]

Similarly, if (5.6) holds for i,

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{21}} \left(\frac{\alpha m_1 x_{21}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) = \frac{m_1 \alpha (\hat{x}_{11} + \alpha \sum_{q \in N_1 : q \neq 2} \hat{x}_{q1})}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2}$$

$$= \frac{m_1 \alpha (\hat{x}_{11} + \sum_{q \in N_1, q \neq 2, q \neq i} \hat{x}_{q1})}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2}$$

$$< \frac{m_1 \alpha (\hat{x}_{11} + \sum_{q \in N_1, q \neq i} \hat{x}_{q1})}{(\hat{x}_{11} + \sum_{q \in N_1} \hat{x}_{q1})^2}$$

■

Lemma 5.7: *If $\hat{x}$ is an NE, then*

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{11}} \left(\frac{k_1 m_1 x_{11}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) = 1, \text{ and} \tag{5.7}$$

$$|_{x=\hat{x}} \frac{\partial}{\partial x_{i1}} \left(\frac{\alpha m_1 x_{i1}}{x_{11} + \alpha \sum_{q \in N_1} x_{q1}} \right) = 1 \textit{for all } i \in N_1. \tag{5.8}$$

Proof: Suppose (5.7) did not hold, with $|_{x=\hat{x}}(\partial/\partial x_{11})[k_1 m_1 x_{11}/(x_{11}+\alpha \sum_{q \in N_1} x_{q1})] > 1$. The left-hand side of this inequality is the rate-of-return (at $x = \hat{x}$) of investment for player 1 in his home region; the "1" on the right-hand-side represents the rate-of-return of saving (see footnote 2). The inequality here means that at $\hat{x}$, player 1 could take some money that he was saving and do better by investing it in his home region. Hence $\hat{x}$ could not be an NE. Next, if $|_{x=\hat{x}}(\partial/\partial x_{11})[k_1 m_1 x_{11}/(x_{11}+\alpha \sum_{i \in N_1} x_{i1})] < 1$, there are two cases. If $\hat{x}_{11} > 0$, player 1 can profitably deviate from $\hat{x}$ by diverting some money from his home region to savings. If $\hat{x}_{11} = 0$, Lemma 5.6 implies that $|_{x=\hat{x}}(\partial/\partial x_{21})[\alpha m_1 x_{21}/(x_{11} + \alpha \sum_{i \in N_1} x_{i1})] < 1$. But then player 2 can profitably deviate. Either way, $\hat{x}$ is not an NE, and we have a contradiction.

A similar proof shows (5.8). ■

Lemma 5.7 implies that if $\hat{x}$ is an NE, then

$$\frac{k_1 m_1 \alpha \sum_{q\in N_1} \hat{x}_{q1}}{(\hat{x}_{11} + \alpha \sum_{q\in N_1} \hat{x}_{q1})^2} = 1, \text{ and} \tag{5.9}$$

$$\frac{m_1 \alpha(\hat{x}_{11} + \alpha \sum_{q\in N_1, q\neq i} \hat{x}_{q1})}{(\hat{x}_{11} + \alpha \sum_{q\in N_1} \hat{x}_{q1})^2} = 1 \text{ for all } i \in N_1. \tag{5.10}$$

Now let $y = \hat{x}_{11} + \alpha \sum_{q\in N_1} \hat{x}_{q1}$. It is clear that since $\hat{x}$ is an NE, that $y > 0$ (see Lemma 5.4). Thus, we can rewrite (5.9) and (5.10) as:

$$k_1 m_1 (y - \hat{x}_{11}) = y^2 \tag{5.11}$$

$$m_1 \alpha (y - \alpha \hat{x}_{i1}) = y^2 \ \forall i \in N_1. \tag{5.12}$$

$$y = \hat{x}_{11} + \alpha \sum_{q\in N_1} \hat{x}_{q1}. \tag{5.13}$$

Equations (5.12) imply that $\hat{x}_{i1}$ is constant over all $i \in N_1$, say $\hat{x}_{i1} = W \ \forall i \in N$. Then (5.13) gives $y = x_{11} + \alpha n_1 W$ or $y - x_{11} = \alpha n_1 W$. Replacing into (5.11) gives $k_1 m_1 \alpha n_1 W = y^2$, or $W = y^2 / k_1 m_1 \alpha n_1$. Finally, since $W = \hat{x}_{i1}$, replacement into (5.12) gives

$$m_1 \alpha \left(y - \frac{\alpha y^2}{k_1 m_1 \alpha n_1} \right) = y^2 \Rightarrow y = 0 \text{ or } y = \frac{k_1 m_1 \alpha n_1}{k_1 n_1 + \alpha}.$$

But $y = 0$ is impossible, so $y = k_1 m_1 \alpha n_1 / (k_1 n_1 + \alpha)$. Since y thus is uniquely defined, it is clear that $\hat{x}_{i1} = W = y^2 / k_1 m_1 \alpha n_1$ is also uniquely defined for $i \in N_1$. Finally, $\hat{x}_{11}$ is uniquely defined because of (5.13). ■

Remark 5.8: As a first remark, let us comment that the NE given by (5.1)–(5.2) is *never* Pareto optimal.[8] Indeed, if we consider the outcome $\delta \hat{x}$, where $\delta < 1$, we see that each player's proportion of influence (see section 2) in every region is unchanged, and so his revenue is unchanged; hence, since his expenditures have gone down, he is better off.

Remark 5.9: As a second remark, we now can describe what happens in the case where $\alpha = 0$. If $m_i = 0$ for all i, we are in a trivial case where the unique NE is for no player to invest any money in any region. Otherwise, suppose WLOG that $m_1 > 0$. Lemma 5.5 still holds here; i.e., in any NE there must be a neighbor of player 1 who invests a positive amount in Region 1. Yet it is clear that since $\alpha = 0$ that neighbor would have incentive to cancel that investment. Conclusion: in the case where $\alpha = 0$ there are no NEs.

[8] Except in the trivial case where $m_i = 0$ for all i.

6 Sensitivity Analysis

We now perform sensitivity analysis on the NE (5.1)–(5.2), to examine the effects of varying some of the model's parameters.

The first thing we notice is that the expressions for $\hat{x}_{ij}$ have j's and α's in them but not i's; in other words, *the size of investments are dependent on characteristics of the region into which the investments are made, and on the overall power of players in neighboring regions, but not on the identity of the investors.* [The only exception to this is that investment in a region will be larger (all other things being equal) if the region happens to be the investor's home region.] We suspect that this will cease to be the case if a) we allow the α's to become pair-specific or b) if the quantities $\{M_i\}_{i\in P}$ are lowered.

We now get some interesting results by taking derivatives of the expressions (5.1)–(5.2) with respect to some of the parameters, starting with the size parameters $\{m_i\}$:

$$\frac{\partial \hat{x}_{ij}}{\partial m_j} = \frac{\alpha n_j k_j}{(k_j n_j + \alpha)^2} \text{ if } i \neq j$$

$$\frac{\partial \hat{x}_{ii}}{\partial m_i} = \frac{\alpha k_i n_i (k_i n_i - \alpha(n_i - 1))}{(k_j n_j + \alpha)^2}.$$

Since the quantities on the right are both positive, this gives the intuitive result that as the size of a region gets bigger, players will invest more in it.

Next consider changes in α. Recall that α measures, on a scale from 0 to 1, how much power players have in neighboring regions:

$$\frac{\partial \hat{x}_{ij}}{\partial \alpha} = \frac{k_j m_j n_j (k_j n_j - \alpha)}{(k_j n_j + \alpha)^3} \text{ if } i \neq j.$$

The quantity on the right is always positive, so this gives the intuitive result that as players have more power in neighboring regions, their investments there go up.

Next we investigate what happens to "home region investments" as α changes.

$$\frac{\partial \hat{x}_{ii}}{\partial \alpha} = \frac{k_i^2 m_i n_i^2 (k_i n_i - 2\alpha n_i + \alpha)}{(k_i n_i + \alpha)^3}.$$

The quantity on the right is positive if $\alpha < k_i n_i/(2n_i - 1)$ and negative when $\alpha > k_i n_i/(2n_i - 1)$. We can explain these results by noting two "effects" as α increases. First is a "substitution effect": as α increases, neighboring regions become more attractive investments, and therefore players replace some spending in their home regions with spending in neighboring regions. Second is a "home competition effect": as α increases, it requires more spending in one's home region to compete with outsiders. We see that for values of α near 0, a player need not spend much in his home region to "win" it, but as α rises the home competition effect causes him to increase his spending. However, at a certain level $(k_i n_i/(2n_i - 1))$ the substitution effect becomes stronger, and home spending decreases thereafter. Note that in the case where $k_i > (2n_i - 1)/n_i$ the substitution effect never dominates the home competition effect, because the "dividing line" $k_i n_i/(2n_i - 1)$ is greater than 1; in this case it is so important for i to win his home region that he always responds to a rise in α by an increase in home spending.

Next, let us examine changes to a player's spending habits as the number of his neighbors changes, and as the number of his neighbor's neighbors changes:

$$\frac{\partial \hat{x}_{ij}}{\partial n_j} = \frac{\alpha m_j k_j (\alpha - k_j n_j)}{(k_j n_j + \alpha)^3}.$$

The quantity on the right is always negative. This is again intuitive, because n_j is a measure of how much competition there is in region j for market share. So, the more such competition in a region, the less each competitor (from a neighboring region) will want to spend.

$$\frac{\partial \hat{x}_{ii}}{\partial n_i} = \frac{\alpha^2 k_i m_i (k_i n_i - 2\alpha n_i + \alpha)}{(k_i n_i + \alpha)^3}$$

This partial derivative is positive if $k_i > 2\alpha - \alpha/n_i$ and negative if $k_i < 2\alpha - \alpha/n_i$. To interpret this, we begin by pointing out that n_i is a measure of outside competition in i's home region. The condition $k_i > 2\alpha - \alpha/n_i$ means that i values revenue in his home region relatively highly and has relatively low power in neighboring regions. Hence he reacts to an increase in the number of competitors in his home region by raising spending there. If $k_i < 2\alpha - \alpha/n_i$ he replaces some home spending with spending in newly available regions.

Finally, we note how far one may lower the M_i's without changing the validity of formulas (5.1)–(5.2). This is equivalent to calculating upper bounds on the amounts $\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij}$. We do this now in the "symmetric case," where $k_i = K$, and $m_i = m$ for all i, and where the associated graph H is regular (i.e., $n_i = N$ for all i):

$$\begin{aligned}\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij} &= \frac{\alpha K m N (KN - \alpha(N-1))}{(KN + \alpha)^2} + N \frac{\alpha K m N}{(KN + \alpha)^2} \\ &= \frac{\alpha K^2 m N^2 - \alpha^2 K m N^2 + \alpha^2 K m N + \alpha K m N^2}{(KN + \alpha)^2}.\end{aligned}$$

We note two things — first, that the quantity on the right is less than m if $\alpha = 1$.[9] Second, the derivative $(\partial/\partial\alpha)(\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij})$ is positive for all $\alpha \in (0, 1)$.[10] Hence $\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij}$ is less than m for all α.

However, the above bound on spending does not hold for all LAGs. In particular, if there is an agent who has incentive to spend money in neighboring regions, she will frequently spend much more than m. For instance, consider the case where the network H is a star, i.e with one central player adjacent to each of N "satellite" players. [None of the satellite players are adjacent to one another.] The central player thus has incentive to spend in the satellite regions, because each such region

[9] In fact it is equal to $m[K^2N^2 + KN]/[K^2N^2 + 2KN + 1]$.

[10] In fact it works out to $[K^2N^3(K - 2\alpha + 1) + \alpha K^2 N^2 + 3\alpha K N^2]/[(KN + \alpha)^3 m$ ${K^2N^3(K-2\alpha, which is clearly positive because $K > \alpha$ and $1 > \alpha$.

only has one neighbor. In addition, suppose $K = 1$ and $\alpha = 1$. Then it turns out that $\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij}$ is equal to $N(1/4 + 1/(N+1)^2)m$, which is quite large for large values of N (or m).

To get a crude upper bound on the quantity $\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij}$ for *any* player i in *any* LAG, suppose M be the maximal size of any region and suppose N is the maximal number of neighbors that any region can have. From our sensitivity analysis above we see that $\hat{x}_{ij}$ (for any $j \in N_i$) is maximized when $m_j = M$, $k_j = 1$ and when $n_j = 1$. We also see that $\hat{x}_{ii}$ is maximized when $m_i = M$. So,

$$\hat{x}_{ii} + \sum_{j \in N_i} \hat{x}_{ij} \leq \frac{\alpha k_i M n_i (k_i n_i - \alpha(n_i - 1)}{(k_i n_i + \alpha)^2} + n_i \frac{\alpha M}{(1+\alpha)^2},$$

which is less than $M(1 + N)$.

7 Beyond Immediate Neighbors

An intriguing extension to our model is to allow players to operate further away from their home regions. In particular, what happens if the players are also allowed to invest in regions which are neighbors of their neighbors? Will players invest much in such "second-degree neighbor" regions?

To answer this question, we consider a symmetric case in which each player i has a set N_{i1} of "first degree" neighbors and a set N_{i2} of "second-degree" neighbors, with $|N_{i1}| = n_1$ and $|N_{i2}| = n_2$ for all i.[11] We also assume that for simplicity, $k_i = 1$ and $m_i = 1$ for all i. Finally, the damping factor α remains in effect for first-degree neighbors, but becomes α^2 for second-degree neighbors.

Player i's utility function is

$$u^i(x) = \frac{x_{ii}}{x_{ii} + \alpha \sum_{q \in N_{i1}} x_{qi} + \alpha^2 \sum_{q \in N_{i2}} x_{qi}} + \sum_{j \in N_{i1}} \frac{\alpha x_{ij}}{x_{jj} + \alpha \sum_{q \in N_{j1}} x_{qj} + \alpha^2 \sum_{q \in N_{j2}} x_{qj}}$$
$$+ \sum_{\ell \in N_{i2}} \frac{\alpha^2 x_{i\ell}}{x_{\ell\ell} + \alpha \sum_{q \in N_{\ell 1}} x_{q\ell} + \alpha^2 \sum_{q \in N_{\ell 2}} x_{q\ell}} - x_{ii} - \sum_{j \in N_{i1}} x_{ij} - \sum_{l \in N_{i2}} x_{i\ell}.$$

To analyze the model above, we again write down player i's maximization problem:

$$\max_{x_{ii}, \{x_{ij}\}_{j \in N_{i1}}, \{x_{il}\}_{\ell \in N_{i2}}} u^i(x)$$
$$\text{s.t. } x_{ii} + \sum_{j \in N_{i1}} x_{ij} + \sum_{l \in N_{i2}} x_{i\ell} \leq M_i$$
$$x_{ii} \geq 0, \; x_{ij} \geq 0 \; \forall j \in N_{i1}, \; x_{i\ell} \geq 0 \; \forall \ell \in N_{i2}.$$

Again assume the constraint is vacuous (i.e., M_i is large), and, for now, ignore the nonnegativity constraints. Since u^i is concave in its variables, we see that if $\hat{x}$ is

[11] Necessarily $n_2 \leq n_1(n_1 - 1)$.

nonnegative and satisfies

$$\frac{\partial u^i}{\partial x_{iq}} = 0 \text{ for all } i, q \in N_{i1} \cup N_{i2} \cup \{i\}, \tag{7.1}$$

then $\hat{x}$ is an NE.

To find such an $\hat{x}$, we take advantage of the model's symmetry and assume that a) each player spends the same amount A in his home region; b) each player spends the same amount B in each of his first-degree-neighbor regions; and c) each player spends the same amount C in each of his second-degree-neighbor regions. In this case, the functions $\{u^i\}_{i\in P}$ become

$$\begin{aligned} u^i(x) \;=\; & \frac{x_{ii}}{x_{ii} + \alpha n_1 B + \alpha^2 n_2 C} + \sum_{j\in N_{i1}} \frac{\alpha x_{ij}}{A + \alpha x_{ij} + \alpha(n_1 - 1)B + \alpha^2 n_2 C} \\ & + \sum_{\ell \in N_{i2}} \frac{\alpha^2 x_{i\ell}}{A + \alpha n_1 B + \alpha^2 x_{i\ell} + \alpha^2 (n_2 - 1)C} - x_{ii} - \sum_{j\in N_{i1}} x_{ij} - \sum_{l\in N_{i2}} x_{i\ell}. \end{aligned}$$

The first-order-conditions (7.1) become

$$\frac{\alpha B n_1 + \alpha^2 C N_2}{(A + \alpha B n_1 + \alpha^2 C n_2)^2} - 1 = 0 \tag{7.2}$$

$$\frac{\alpha A + \alpha^2 B(n_1 - 1) + \alpha^3 C n_2}{(A + \alpha B n_1 + \alpha^2 C n_2)^2} - 1 = 0 \tag{7.3}$$

$$\frac{\alpha^2 A + \alpha^3 B n_1 + \alpha^4 C(n_2 - 1)}{(A + \alpha B n_1 + \alpha^2 C n_2)^2} - 1 = 0 \tag{7.4}$$

This solves (again, after a lot of algebra) with

$$A = \frac{\alpha^6 (n_1 + n_2)(n_2 + \alpha n_1 + \alpha^2 - \alpha^2 n_1 - \alpha^2 n_2)}{(\alpha^4 n_1 - \alpha^4 - 2\alpha^3 n_1 - \alpha^2 n_2 - \alpha n_1 n_2 + 2\alpha^2 n_1 n_2 - \alpha^3 n_1 n_2)^2} \tag{7.5}$$

$$B = \frac{\alpha^5 (n_1 + n_2)(n_2 - \alpha n_2 + \alpha^2)}{(\alpha^4 n_1 - \alpha^4 - 2\alpha^3 n_1 - \alpha^2 n_2 - \alpha n_1 n_2 + 2\alpha^2 n_1 n_2 - \alpha^3 n_1 n_2)^2} \tag{7.6}$$

$$C = \frac{\alpha^4 (n_1 + n_2)(\alpha^2 + \alpha n_1 - n_1)}{(\alpha^4 n_1 - \alpha^4 - 2\alpha^3 n_1 - \alpha^2 n_2 - \alpha n_1 n_2 + 2\alpha^2 n_1 n_2 - \alpha^3 n_1 n_2)^2} \tag{7.7}$$

We see that in the case where $\alpha = 1$ we have $A = B = C$, and all three quantities are equal to the positive quantity $(n_1 + n_2)/(1 + n_1 + n_2)^2$. Essentially this is because $\alpha^2 = \alpha$, so we can think of this as a symmetric case of our original game in which each player has $n_1 + n_2$ neighbors.

As we lower α from 1 we see that NE investments in first- and second-degree neighbor regions to both fall, with those in second-degree regions falling faster. In fact, A and B are nonnegative for any value of α, but that C is nonnegative only if $\alpha^2 + \alpha n_1 - n_1 \geq 0$, i.e., $\alpha \geq (\sqrt{n_1^2 + 4n_1} - n_1)/2$. The quantity $(\sqrt{n_1^2 + 4n_1} - n_1)/2$ is equal to $\sqrt{(N+1)^2 - 1} - N$ (where $N = n_1/2$), which approaches 1 as n_1 gets large.

In conclusion, we see that while first-degree investments stay positive for any positive α, second-degree investments hit zero at a value of α very close to 1. This suggests an interesting rationale for the theory that human behavior is based only on extremely local conditions.

8 Extending the Player Set

We remark here that all of the results from Sections 5, 6, and 7 hold if we allow the player set P to be countably infinite, i.e. $P = \{1, 2, 3, ...\}$.[12] However, we would still require that the number of neighbors for any player would be finite.

9 Generalizing the Model

In this section we generalize the model from Section 4 by replacing the specific functional forms for given in (4.1)–(4.2) with arbitrary revenue functions. We then investigate in what sense we still have NE existence results.

We begin by presenting the generalized model. Let $P = \{1, ..., p\}$ be the player set, and let $\mathbb{R}^P$ be p-dimensional Euclidean space in which the coordinate axes are associated with the players in P. Suppose $i \in P$. Then, as in Section 4, let N_i the set of neighbors of Player i and M_i the monetary endowment of Player i.

Again let $i \in P$. We define $X^i = \{x \in \mathbb{R}^P : \sum_{j \in P} x_{ij} \leq M_i,\ x_{ij} \geq 0\ \forall j \in P$, and $x_{ij} = 0$ if $j \not\in N_i \cup \{i\}\}$. This definition of i's strategy set is technically different than the one proposed in Section 4: now i's strategies are no longer $1 + N_i$-vectors, but instead are p-vectors with all but $1 + N_i$ components set to 0. Also, this time we define the strategy space X to be an *arbitrary compact subset* of $\mathcal{X}_{i \in P} X^i$. Hence it will be possible to forbid certain joint strategies for the players.

If $x \in X$ is a vector of joint strategies for the players, the revenue for player i from Region j is given by an arbitrary function $r_{ij}(x)$. We assume the following concerning these functions $\{r_{ij}\}$:

1) $r_{ij}(x) = 0$ if $x_{ij} = 0$ for any $i \in P$, $j \in P$.

The first assumption states that if a player invests zero into a region, he gets a revenue of zero out of that region.

For the second assumption, we need some notation. Let x be a joint strategy vector. Then $x^{\sim ij=y}$ is the same vector as x, except with the ijth component changed to y.

2) $r_{ij}(x^{\sim kl=y}) = r_{ij}(x)$ for any $x, y, i, j, k, l : l \neq j$.

This assumption says that if a player changes his investment in a single region, it does not affect anyone's revenue from any *other* region.

3) $r_{ij}(x)$ is an increasing concave function of x_{ij}, for all i, j.

Player i's revenues in any region are an increasing concave function of his investment there.

4) $r_{ij}(x)$ is a decreasing function of x_{qj}, for any i and for $q \neq i$.

[12] We note that the strategy set X is no longer compact in this case.

The last assumption says that Player i's revenue from any region is a decreasing function of what other players invest there.

Under this notation, Player i's utility function is simply his total revenues (added up over all regions) minus his total expenditures, or $u^i(x) = \sum_{j=1}^{p} r_{ij}(x) - \sum_{j=1}^{p} x_{ij}$. We can think of the functions $\{u^i\}_{i\in P}$ as payoff functions in a noncooperative game, in which the player set is P and the joint strategy set is X. Formally, we say that the quantities $G = (P, \{N_i\}_{i\in P}, \{r_{ij}\}_{i,j\in P}, X)$ define a *generalized* local advertising game (GLAG).

We note that the model described in Section 4 is an example of a GLAG, in which the functions $\{r_{ij}\}$ are given by $r_{ij}(x) = \alpha m_j x_{ij}/(x_{jj} + \sum_{q:q\in N_j} x_{qj})$ if $i \neq j$ and $x_{jj}+\sum_{q:q\in N_j} x_{qj} > 0$, $r_{ii}(x) = k_i m_i x_{ii}/(x_{ii}+\sum_{j:j\in N_i} x_{ij})$ if $x_{ii}+\sum_{j:j\in N_i} x_{ij} > 0$, and $r_{iq}(x) = 0$ if $x_{qq} + \sum_{l\in N_q} x_{lq} = 0$.[13]

Theorem 9.1: *Let $G = (P, \{N_i\}_{i\in P}, X, \{r_{ij}\}_{i,j\in P}, X)$ be a GLAG. If the functions $\{r_{ij}(x)\}$ are continuous on X, then there exists an NE for G.*[14]

Proof: Follows directly from Theorem 1 of Rosen (1965). ■

In Theorem 5.1 we proved the existence of NE in a model in which the functions $\{r_{ij}\}$ are *not* continuous on all of X. Hence Theorem 9.1 is not completely satisfactory because it does not imply (the existence result part of) Theorem 5.1. We now rectify this problem by doing the following:

1) Given a LAG G, we define the "snipped game" G^S, which is the same as G except we "snip off" the corners of the strategy set X on which the r_{ij}'s are discontinuous. We shall prove that G^S is "equivalent" to G in that both games must have the exact same (unique) NE.

2) The snipped game *does* satisfy the hypotheses of Theorem 9.1, and so that theorem does imply the existence of an NE in G^S, and hence in G.

Let $G = (P, \{N_i\}_{i\in P}, \alpha, \{k_i\}_{i\in P}, \{m_i\}_{i\in P}, X)$ be a LAG. Define the *snipped game* G^S by $G^S = (P, \{N_i\}_{i\in P}, \alpha, \{k_i\}_{i\in P}, \{m_i\}_{i\in P}, X^S)$, where

$$X^S = \left\{ x \in X : x_{jj} + \alpha \sum_{i\in N_j} x_{ij} \geq \frac{\alpha k_j m_j n_j}{2(\alpha + k_j n_j)} \ \forall j \in P \right\}.$$

Our aim is to show that the set of G's NEs is exactly the same as the set of G^S's NEs.

Lemma 9.2: *Let G be a LAG, and let j any region for which $m_j > 0$. If x^* is an NE of game G or of G^S, then*

$$x^*_{jj} + \alpha \sum_{i\in N_j} x^*_{ij} \geq \frac{\alpha k_j m_j n_j}{\alpha + k_j n_j}.$$

[13] Note that even if $m_j > 0$ and $x_{qj} = 0$ for all $q \neq i$, the function $r_{ij}(x)$ is a concave function of x_{ij}, although it is not a continuous function of x_{ij}.

[14] We note that the Theorem holds even if we drop the assumption of "largeness" for the M_i's.

Proof: First, we remark that the quantity $x^*_{jj} + \alpha \sum_{q\in N_j} x^*_{qj}$ cannot be zero. In game G^S, this is by virtue of the definition of X^S;[15] in game G this is because of Lemma 5.6.[16] Either way, we note that $|_{x=x^*(\partial r_{ij}/\partial x_{ij})}$ exists for all $i, j \in P$.

Now we prove the Lemma by assuming it is false. Thus, suppose there exists a region j and a positive $\varepsilon < \alpha k_j m_j n_j/(\alpha + k_j n_j)$ with $x^*_{jj} + \alpha \sum_{i\in N_j} x^*_{ij} = \varepsilon$.[17] Now suppose $i \neq j$. Since x^* is an NE, it is true that i cannot profitably switch resources from savings to Region j. Hence $|_{x=x^*(\partial r_{ij}/\partial x_{ij})} \leq 1$, i.e. $(\alpha m_j x^*_{jj} + \alpha^2 m_j \sum_{q\in N_j : q\neq i} x^*_{qj})/(x^*_{jj} + \alpha \sum_{q\in N_j} x^*_{qj})^2 \leq 1$, or

$$\alpha m_j x^*_{jj} + \alpha^2 m_j \sum_{q\in N_j : q\neq i} x^*_{qj} \leq \varepsilon^2. \tag{9.1}$$

Adding up inequalities (9.1) over all $i \in N_j$ gives

$$\alpha m_j n_j x^*_{jj} + \alpha^2 m_j (n_j - 1) \sum_{q\in N_j} x^*_{qj} \leq \varepsilon^2 n_j. \tag{9.2}$$

We also know that j cannot profitably switch resources from savings to his home region. Hence $|_{x=x^*} (\partial r_{jj}/\partial x_{jj}) \leq 1$, i.e., $\alpha k_j m_j \sum_{q\in N_j} x^*_{qj}/(x^*_{jj} + \alpha \sum_{q\in N_j} x^*_{qj})^2 \leq 1$, or

$$\alpha^2 m_j \sum_{q\in N_j} x^*_{qj} \leq \frac{\alpha\varepsilon^2}{k_j}. \tag{9.3}$$

Adding (9.2) and (9.3) gives

$$\alpha m_j n_j x^*_{jj} + \alpha^2 m_j n_j \sum_{q\in N_j} x^*_{qj} \leq \varepsilon^2 \left(n_j + \frac{\alpha}{k_j} \right)$$

The last inequality above gives a contradiction. ■

Theorem 9.3: *Suppose G is a LAG. Then x^* is an NE of G iff x^* is an NE of G^S.*

Proof: First let us suppose x^* is an NE of G. We see that if $m_j = 0$, we trivially have $x^*_{jj} + \alpha \sum_{i\in N_j} x^*_{ij} \geq [\alpha k_j m_j n_j / 2(\alpha + k_j n_j)]$; and, if $m_j > 0$, Lemma 9.2 ensures that $x^*_{jj} + \alpha \sum_{i\in N_j} x^*_{ij} \geq \alpha k_j m_j n_j/[2(\alpha + k_j n_j)]$. Hence, $x^* \in X^S$. But $X^S \subseteq X$, so this implies that x^* is an NE of G^S as well.

For the converse, let us suppose x^* is an NE of G^S but *not* an NE of G. Hence, there is some player i who has a deviation from x^* to some $\tilde{x}$ which makes him better off in G, i.e., $u^i(\tilde{x}) > u^i(x^*)$. Since x^* is an NE in G^S, the deviation $\tilde{x}$ must lie in X but

[15] The assumption $m_j > 0$ is crucial here.

[16] See in particular the sentence immediately following the proof of Lemma 5.6.

[17] The assumption that $m_j > 0$ is necessary here to ensure that ε is positive.

not in X^S. Hence the set $J = \{j \in P : \tilde{x}_{jj} + \alpha \sum_{q \in N_j} \tilde{x}_{qj} < \alpha k_j m_j n_j / [2(\alpha + k_j n_j)]\}$ must be nonempty. Note that necessarily $m_j > 0$ for all $j \in J$.

By virtue of Lemma 9.2 and the fact that x^* is an NE of G^S, we have

$$x^*_{jj} + \alpha \sum_{q \in N_j} x^*_{qj} \geq \frac{\alpha k_j m_j n_j}{\alpha + k_j n_j} \ \forall j \in J.$$

Now consider the line segment connecting $\tilde{x}$ and x^*. There must be a point y on this segment for which $y \neq x^*$ but still $y_{jj} + \alpha \sum_{q \in N_j} y_{qj} \geq \alpha k_j m_j n_j / [2(\alpha + k_j n_j)]$ for all $j \in P$, i.e., $y \in X^S$.

But then

$$u^i(y) = u^i(\lambda x^* + (1 - \lambda)\tilde{x}) \text{ for some } \lambda \in (0, 1)$$

Hence y is also a deviation which makes i better off.

But, since $y \in X^S$, we have a contradiction of the fact that x^* is an NE of G^S.■

10 Concluding Remarks

In Section 2 the constraints of space, time and net were noted. In Sections 3–4 a model was specified. In section 5 we carried out an analysis, which by the appropriate selection of parameters could be interpreted either in terms of space or net. Time and information conditions were left out in the sense that we considered the noncooperative game as a one period game in strategic form and solved for its NE. We did not perform a sensitivity analysis of the solution concept.

We close with a simple example, which serves to illustrate some of the basic differences in the treatment of what is considered to be a solution to a game as we vary the complexity of available strategies. We consider a four player game where each player has two neighbors:

Player 1 has 2 and 3 as neighbors
Player 2 has 1 and 4 as neighbors
Player 3 has 1 and 4 as neighbors
Player 4 has 2 and 3 as neighbors

Each pair of neighbors plays either a prisoner's dilemma or coordination game, as follows. First, in the game of 2 vs. 4, and the game of 1 vs. 3, the agents play the prisoner's dilemma:

$$\begin{bmatrix} (3,3) & (-1,4) \\ (4,-1) & (0,0) \end{bmatrix}$$

Meanwhile, the games 1 vs. 2 and 3 vs. 4 play a coordination game:

$$\begin{bmatrix} (3,3) & (0,0) \\ (0,0) & (-1,-1) \end{bmatrix}$$

Hence each player plays two others, one in one game and the other in a different game.

We may use this four person game to illustrate four solution concepts of differing levels of complexity ranging from zero intelligence and perceptions to a strategy using recent memory.

Solution 1 A fixed strategy

Solution 2 Select either A or B for all games

Solution 3 Select A or B for each game separately

Solution 4 Select A or B for each game separately, based on previous history

Suppose that survival requires a positive payoff. The first solution is a no intelligence, no learning solution. If by chance all agents are hard-wired to play A, each will earn 6 each period and survive well. If there is a mixture of agents playing A and B the A agents will be wiped out in their encounters with the B agents and then the B agents will wipe each other out.

In the second solution, if the agents do not use any history, but are locally optimal, but are constrained (like many mass anonymous agencies) to treat all agents in the same manner, then each will select strategy B and wipe each other out. [See Morris (2000) for an analysis of this type of game.]

The third solution is still purely local in time, but an agent is able to distinguish between neighbors. In this instance each agent will play B in the prisoner's dilemma game and A in the coordination game. This permits all to survive and make 2 each period.

The fourth solution extends locally into time. It permits strategies to be selected contingently based on the history in the previous period. Thus the strategy is based on AA, AB, BA, or BB. In the prisoner's dilemma game the agent can resort to a "tit-for-tat" strategy, as was selected by Rapoport. In the coordination game it is not clear that the "tit-for-tat" strategy has the nice intuitive "eye-for-eye" interpretation that it has in the prisoner's dilemma.

In order to well define the game with history, initial conditions must be supplied. For example if all agents are informed the their neighbors played A in both games, then the solution where all play A in both games is a noncooperative equilibrium enforced by strategies using one period of history. If the initialization is, say BA then the agents will cycle on the tit-for-tat strategy without further learning. Humans tend to learn from more than immediate history. Furthermore, unlike in Axelrod's (1984), simulation, encounters with neighbors are not like new random encounters with strangers.

In the analysis part of this paper we have investigated the NE solution based on considerations of neighborhood in space, and net. Further solutions differ based on the flexibility and information processing abilities we attribute to the individuals embedded in time. The simple four person game example presented here illustrates a sensitivity analysis on solution concepts. The simple treat all-alike behavior appears to be better associated with models of simple animal or organism behavior than for humans; games such as "blockwatch" or "keep our town green." These depend on the facts of locality which in turn make moderately simple strategies utilizing not too many contingencies, reasonably effective, yet they do require learning and intelligence higher than offered by evolutionary game theory models.

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