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Abstract

Traditional portfolio optimization has been often criticized since it does not account for estimation risk. Theoretical considerations indicate that estimation risk is mainly driven by the parameter uncertainty regarding the expected asset returns rather than their variances and covariances. This is also demonstrated by several numerical studies. The global minimum variance portfolio has been advocated by many authors as an appropriate alternative to the traditional Markowitz approach since there are no expected asset returns which have to be estimated and thus the impact of estimation errors can be substantially reduced. But in many practical situations an investor is not willing to choose the global minimum variance portfolio, especially in the context of top down portfolio optimization. In that case the investor has to minimize the variance of the portfolio return by satisfying some specific constraints for the portfolio weights. Such a portfolio will be called 'local minimum variance portfolio'. Some finite sample hypothesis tests for global and local minimum variance portfolios are presented as well as the unconditional finite sample distribution of the estimated portfolio weights and the first two moments of the estimated expected portfolio returns.

Keywords: Estimation risk, linear regression theory, Markowitz portfolio, minimum variance portfolio, portfolio optimization, top down investment.

AMS Subject Classification: Primary 62H10, Secondary 91B28.

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LINEAR STATISTICAL INFERENCE FOR GLOBAL AND LOCAL MINIMUM VARIANCE PORTFOLIOS

GABRIEL FRAHM

ABSTRACT. Traditional portfolio optimization has been often criticized since it does not account for estimation risk. Theoretical considerations indicate that estimation risk is mainly driven by the parameter uncertainty regarding the expected asset returns rather than their variances and covariances. This is also demonstrated by several numerical studies. The global minimum variance portfolio has been advocated by many authors as an appropriate alternative to the traditional Markowitz approach since there are no expected asset returns which have to be estimated and thus the impact of estimation errors can be substantially reduced. But in many practical situations an investor is not willing to choose the global minimum variance portfolio, especially in the context of top down portfolio optimization. In that case the investor has to minimize the variance of the portfolio return by satisfying some specific constraints for the portfolio weights. Such a portfolio will be called 'local minimum variance portfolio'. Some finite sample hypothesis tests for global and local minimum variance portfolios are presented as well as the unconditional finite sample distribution of the estimated portfolio weights and the first two moments of the estimated expected portfolio returns.

1. MOTIVATION

Consider a d-dimensional random vector $R = (R_1, \ldots, R_d)$ of asset returns at the end of a certain investment horizon. It is assumed that the random vector R is multivariate normally distributed, viz $R \sim \mathcal{N}_d(\mu, \Sigma)$, where μ $(d \times 1)$ is an unknown vector of expected asset returns and Σ $(d \times d)$ is an unknown positive definite matrix containing the variances and covariances of the asset returns. The so-called 'global minimum variance portfolio' (GMVP) is defined as

$$\underset{(d\times 1)}{w} = (w_1, \ldots, w_d) := \arg\min_v \mathbb{V}\mathrm{ar}\left(R'v\right)$$

under the budget constraint $1'v = 1$, where v $(d \times 1)$ denotes a vector of portfolio weights and 1 symbolizes a vector of ones or the one scalar, respectively. Any other portfolio which minimizes the variance of the portfolio return $R'v$ but satisfies some *additional* constraints will be called 'local minimum variance portfolio' (LMVP). In this work only linear constraints will be considered.

During the past decades traditional portfolio optimization has been often criticized since it does not account for estimation risk (Klein and Bawa, 1976, Michaud, 1989). At the beginning of modern portfolio optimization it was generally agreed that the parameters of interest, i.e. the mean and (co-)variances of asset returns can be estimated accurately so that estimation errors are negligible. Although this conjecture might be true for the variances and covariances if the sample size is large enough compared to the number of assets, it is not an appropriate simplification for the expected asset returns in most practical situations (Chopra and Ziemba, 1993, Kempf and Memmel, 2002, Merton, 1980).

Nowadays many procedures for taking the parameter uncertainty into account can be found in the literature. A rather simple alternative to the traditional approach is given by the GMVP. It has been advocated by many authors (e.g. Jagannathan and Ma, 2003, Ledoit and Wolf, 2003) and indeed the GMVP seems to be a convenient choice since on the one hand it follows the basic idea of Markowitz (1952), i.e. searching for an *efficient* portfolio by diversification, but on the other hand there are no expected asset returns which have to be estimated for calculating the GMVP and thus the impact of estimation errors can be substantially reduced.

However, in many practical situations an investor cannot or, say, is not willing to choose the GMVP. For example, portfolio managers of mutual funds often have to observe certain limits regarding the choice of portfolio weights. This is a typical situation in top down portfolio management. That means the set of available assets is divided into some subsets of assets, each subset is divided into some further subsets, etc. These subsets are generally referred to as 'asset classes' usually according to some industry sector, rating, or regional classification. Now, *top down investment* means that the amount of capital is allocated to the top level partition at first. Given the portfolio weights for that partition, somebody has to choose some optimal portfolio weights for the subsequent asset classes, etc., so that each of the succeeding decisions are limited by the preceding allocations. Hence, there are a number of linear restrictions which have to be satisfied when searching for a minimum variance portfolio and thus we may be interested in testing linear hypotheses concerning the LMVP rather than the GMVP. In this work I will present some finite sample hypothesis tests for global and local minimum variance portfolios as well as the unconditional finite sample distribution of the estimated portfolio weights and the first two moments of the estimated expected portfolio returns.

2. Hypothesis Tests for the Global Minimum Variance Portfolio

2.1. **Theoretical Foundation.** Kempf and Memmel (2006) showed that minimizing the global variance of the portfolio return can be viewed as a linear regression problem. Note that the return of the GMVP can be written as

$$(1 - w_2 - \ldots - w_d)R_1 + w_2R_2 + \ldots + w_dR_d = \eta + \varepsilon\,, \tag{2.1.1}$$

where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. Now we define $\beta_1 := \eta$, $\beta_j := w_j$, $\Delta R_j := R_1 - R_j$ for $j = 2, \ldots, d$, and $u := \varepsilon$ so that Eq. 2.1.1 becomes equivalent to

$$R_1 = \beta_1 + \beta_2\Delta R_2 + \ldots + \beta_d\Delta R_d + u\,. \tag{2.1.2}$$

The following proposition is a standard result of linear regression theory. It is crucial for understanding the basic idea of the subsequent derivations and thus it will be recalled for convenience.

Proposition 2.1.1. *Let $Z = (Z_1, \ldots, Z_d)$ be a d-dimensional random vector with positive definite covariance matrix. Consider the vector*

$$\underset{(d\times 1)}{\beta} = (\beta_1, \ldots, \beta_d) := \arg\min_b \mathbb{E}\left((Z_1 - b_1 - b_2Z_2 - \ldots - b_dZ_d)^2\right),$$

where $b = (b_1, \ldots, b_d)$ and define

$$u := Z_1 - \beta_1 - \beta_2Z_2 - \ldots - \beta_dZ_d\,.$$

The vector β exists and is uniquely defined. More precisely, the subvector $\beta^{\mathrm{s}} := (\beta_2, \ldots, \beta_d)$ is given by

$$\beta^{\mathrm{s}} = \mathbb{V}\mathrm{ar}\left(Z^{\mathrm{s}}\right)^{-1} \mathbb{C}\mathrm{ov}\left(Z_1, Z^{\mathrm{s}}\right),$$

where $Z^{\mathrm{s}} := (Z_2, \ldots, Z_d)$, $\mathbb{V}\mathrm{ar}(Z^{\mathrm{s}})$ $((d-1) \times (d-1))$ is the covariance matrix of Z^{s}, and $\mathbb{C}\mathrm{ov}\left(Z_1, Z^{\mathrm{s}}\right)$ is the $(d-1) \times 1$ vector of covariances between Z_1 and Z_j $(j = 2, \ldots, d)$. Moreover, the parameter β_1 is given by

$$\beta_1 = \mathbb{E}\left(Z_1\right) - \mathbb{E}\left(Z^{\mathrm{s}}\right)' \beta^{\mathrm{s}}$$

and it holds that $\mathbb{E}(u) = 0$ as well as $\mathbb{C}\mathrm{ov}(X_j, u) = 0$ for $j = 2, \ldots, d$.

Proof. Since

$$\begin{aligned} \mathbb{E}\left(\left(Z_1 - b_1 - Z^{\mathrm{s}\prime} b^{\mathrm{s}}\right)^2\right) &= \mathbb{V}\mathrm{ar}\left(Z_1 - b_1 - Z^{\mathrm{s}\prime} b^{\mathrm{s}}\right) + \left(\mathbb{E}\left(Z_1 - b_1 - Z^{\mathrm{s}\prime} b^{\mathrm{s}}\right)\right)^2 \\ &= \mathbb{V}\mathrm{ar}\left(Z_1 - Z^{\mathrm{s}\prime} b^{\mathrm{s}}\right) + \left(\mathbb{E}\left(Z_1\right) - b_1 - \mathbb{E}\left(Z^{\mathrm{s}}\right)' b^{\mathrm{s}}\right)^2, \end{aligned}$$

where $b^{\mathrm{s}} := (b_2, \ldots, b_d)$, it is clear that

$$\beta_1 = \mathbb{E}\left(Z_1\right) - \mathbb{E}\left(Z^{\mathrm{s}}\right)' \beta^{\mathrm{s}}$$

and thus $\mathbb{E}(u) = 0$. That means we can solve the minimization problem equivalently by minimizing

$$\mathbb{E}\left(\left(Z_1^* - b_2 Z_2^* - \ldots - b_d Z_d^*\right)^2\right), \tag{2.1.3}$$

where $Z_j^* := Z_j - \mathbb{E}(Z_j)$ for $j = 1, \ldots, d$. Now we define $Z^{*\mathrm{s}} := (Z_2^*, \ldots, Z_d^*)$ so that (2.1.3) corresponds to

$$\mathbb{E}\left(\left(Z_1^* - Z^{*\mathrm{s}\prime} b^{\mathrm{s}}\right)^2\right) = \mathbb{V}\mathrm{ar}\left(Z_1\right) - 2\mathbb{C}\mathrm{ov}\left(Z_1, Z^{\mathrm{s}}\right)' b^{\mathrm{s}} + b^{\mathrm{s}\prime} \mathbb{V}\mathrm{ar}\left(Z^{\mathrm{s}}\right) b^{\mathrm{s}}.$$

Due to the positive definiteness of $\mathbb{V}\mathrm{ar}(Z)$ also $\mathbb{V}\mathrm{ar}(Z^{\mathrm{s}})$ is positive definite. Hence, we have a simple quadratic minimization problem and its unique solution is given by

$$\beta^{\mathrm{s}} = \mathbb{V}\mathrm{ar}\left(Z^{\mathrm{s}}\right)^{-1} \mathbb{C}\mathrm{ov}\left(Z_1, Z^{\mathrm{s}}\right).$$

Now we can calculate the $(d-1) \times 1$ vector of covariances between Z_j $(j = 2, \ldots, d)$ and u, i.e.

$$\begin{aligned} \mathbb{C}\mathrm{ov}\left(Z^{\mathrm{s}}, u\right) &= \mathbb{C}\mathrm{ov}\left(Z^{\mathrm{s}}, Z_1 - \beta_1 - Z^{\mathrm{s}\prime} \beta^{\mathrm{s}}\right) \\ &= \mathbb{C}\mathrm{ov}\left(Z_1, Z^{\mathrm{s}}\right) - \mathbb{V}\mathrm{ar}\left(Z^{\mathrm{s}}\right) \beta^{\mathrm{s}} = 0. \end{aligned}$$

□

The parameters $\beta_1, \ldots, \beta_d$ in Eq. 2.1.2 are chosen in such a way that $\mathbb{E}(u) = 0$ and $\mathbb{V}\mathrm{ar}(u) = E(u^2)$ is minimal, i.e. $\mathbb{C}\mathrm{ov}(\Delta R_j, u) = 0$ $(j = 2, \ldots, d)$. That means Eq. 2.1.2 indeed is a proper linear regression equation satisfying the standard assumptions of linear regression theory, especially the strict exogeneity assumption (see e.g. Hayashi, 2000, p. 7). For that reason it is possible to develop several exact hypothesis tests for the GMVP by standard methods of econometrics (cf. Kempf and Memmel, 2006).

The next corollary states that the converse of Proposition 2.1.1 is true. As we will see later on, this result implies that the standard test statistics for the GMVP generally must not be applied for testing a LMVP.

Corollary 2.1.2. *Let $Z = (Z_1, \ldots, Z_d)$ be a d-dimensional random vector with positive definite covariance matrix. Search for some numbers $b_1, \ldots, b_d$ such that $\mathbb{E}(u^*) = 0$ and $\mathbb{C}\text{ov}(Z_j, u^*) = 0$ for $j = 2, \ldots, d$, where*

$$u^* := Z_1 - b_1 - b_2 Z_2 - \ldots - b_d Z_d \,.$$

The vector $b = (b_1, \ldots, b_d)$ exists and is uniquely defined by $b = \beta$ where β is given by Proposition 2.1.1.

Proof. The proof follows immediately from the proof of Proposition 2.1.1 and noting that the linear equation

$$0 = \mathbb{C}\text{ov}\left(Z^{\text{s}}, u^*\right) = \mathbb{C}\text{ov}\left(Z_1, Z^{\text{s}}\right) - \mathbb{V}\text{ar}\left(Z^{\text{s}}\right) b^{\text{s}}$$

has a unique solution due to the positive definiteness of $\mathbb{V}\text{ar}(Z^{\text{s}})$. □

2.2. **Statistical Inference.** Of course, in practice the weights of the GMVP are unknown, i.e. they have to be estimated from historical data. Let

$$\underset{(n\times d)}{\mathbf{R}} := \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1d} \\ R_{21} & R_{22} & \cdots & R_{2d} \\ \vdots & \vdots & & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nd} \end{bmatrix}$$

be a sample of $n > d$ independent copies of R. Now we define

$$\underset{(n\times d)}{\mathbf{X}} := \begin{bmatrix} 1 & X_{12} & \cdots & X_{1d} \\ 1 & X_{22} & \cdots & X_{2d} \\ \vdots & \vdots & & \vdots \\ 1 & X_{n2} & \cdots & X_{nd} \end{bmatrix}, \tag{2.2.1}$$

where $X_{ij} := R_{i1} - R_{ij}$ $(i = 1, \ldots, n,\ j = 2, \ldots, d)$ and

$$\underset{(n\times 1)}{\mathbf{Y}} := \begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{n1} \end{bmatrix}, \tag{2.2.2}$$

where $Y_{i1} \equiv R_{i1}$ $(i = 1, \ldots, n)$. Similarly, we will also write $X := (1, X_2, \ldots, X_d)$ $(d \times 1)$, $X^{\text{s}} := (X_2, \ldots, X_d)$ $((d-1) \times 1)$, and $Y \equiv R_1$ (1×1).

According to the standard notation of linear regression theory the sample version of the linear model represented by Eq. 2.1.2 is given by

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{u}\,, \tag{2.2.3}$$

where $\beta = (\beta_1, \ldots, \beta_d)$ $(d \times 1)$ contains the weights $\beta_2, \ldots, \beta_d$ as well as the expected return β_1 of the GMVP, whereas $\mathbf{u} := (u_1, \ldots, u_n)$ is an $n \times 1$ vector of unobservable residuals. Hence, the *ordinary least squares* (OLS) estimator for β can be calculated by

$$\hat\beta_{\text{OLS}} = \left(\hat\beta_{\text{OLS},1}, \ldots, \hat\beta_{\text{OLS},d}\right) = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\mathbf{Y}\,. \tag{2.2.4}$$

In fact the weights of the GMVP – except for the first one – are estimated by

$$\hat\beta^{\text{s}}_{\text{OLS}} := \left(\hat\beta_{\text{OLS},2}, \ldots, \hat\beta_{\text{OLS},d}\right) = \widehat\Omega^{-1}\hat\omega\,,$$

where $\widehat{\Omega}$ is the sample covariance matrix of X^{s} and $\hat{\omega}$ is the $(d-1)\times 1$ vector of the sample covariances between Y and X_j $(j=2,\ldots,d)$. That means by the Gauss-Markov theorem

$$\hat{w} := \left(1 - 1'\hat{\beta}^{\mathrm{s}}_{\mathrm{OLS}}, \hat{\beta}^{\mathrm{s}}_{\mathrm{OLS}}\right)$$

is the best linear unbiased estimator for the GMVP in the context of normally distributed asset returns (Kempf and Memmel, 2006).

Now consider the fundamental least squares problem

$$(\mathbf{Y} - \mathbf{X}b)'(\mathbf{Y} - \mathbf{X}b) \to \min_b ! \tag{2.2.5}$$

under the additional constraint $Hb = h$ where H $(q \times d)$ is a matrix with $\mathrm{rk}(H) = q \leq d$ and h $(q\times 1)$ is an arbitrary vector. From linear regression theory (see e.g. Greene, 2003, p. 100) we know that the solution of this minimization problem is given by the *restricted least squares* (RLS) estimator

$$\hat{\beta}_{\mathrm{RLS}} := \hat{\beta}_{\mathrm{OLS}} - \left(\mathbf{X}'\mathbf{X}\right)^{-1} H' \left(H\left(\mathbf{X}'\mathbf{X}\right)^{-1} H'\right)^{-1}\left(H\hat{\beta}_{\mathrm{OLS}} - h\right). \tag{2.2.6}$$

The relation between the OLS estimator $\hat{\beta}_{\mathrm{OLS}}$ and the empirical residual $\hat{\mathbf{u}}$ $(n\times 1)$ can be represented by

$$\mathbf{Y} = \mathbf{X}\hat{\beta}_{\mathrm{OLS}} + \hat{\mathbf{u}}. \tag{2.2.7}$$

In contrast, we may write

$$\mathbf{Y} = \mathbf{X}\hat{\beta}_{\mathrm{RLS}} + \hat{\mathbf{u}}^* \tag{2.2.8}$$

to indicate that $\hat{\mathbf{u}}^*$ $(n\times 1)$ is the empirical residual with respect to $\hat{\beta}_{\mathrm{RLS}}$ and not to $\hat{\beta}_{\mathrm{OLS}}$.

We consider only inhomogeneous regressions and thus both $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}^*$ possess zero sample means. That is to say (2.2.5) indeed leads to the local minimum *variance* portfolio satisfying the given restriction $Hb = h$. But note that – in contrast to the unrestricted case – each column of $\mathbf{X}$ is correlated with $\hat{\mathbf{u}}^*$ in general. More precisely, due to the fact that $\mathbf{X}'\hat{\mathbf{u}} = 0$ and

$$\hat{\mathbf{u}}^* = \hat{\mathbf{u}} + \mathbf{X}\left(\hat{\beta}_{\mathrm{OLS}} - \hat{\beta}_{\mathrm{RLS}}\right)$$

(Greene, 2003, p. 101) we obtain

$$\mathbf{X}'\hat{\mathbf{u}}^* = H'\left(H\left(\mathbf{X}'\mathbf{X}\right)^{-1}H'\right)^{-1}\left(H\hat{\beta}_{\mathrm{OLS}} - h\right).$$

So the RLS estimator corresponds to

$$\hat{\beta}_{\mathrm{RLS}} = \hat{\beta}_{\mathrm{OLS}} - \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\hat{\mathbf{u}}^*,$$

where $\mathbf{X}'\hat{\mathbf{u}}^* \neq 0$ if the linear restriction is binding. This is an empirical consequence of Corollary 2.1.2.

An exact or, say, finite sample hypothesis test against $H_0\colon H\beta = h$ is given by the next proposition. For some applications of that F test to financial data see Kempf and Memmel (2006).

Proposition 2.2.1. *Consider Eq. 2.2.3, Eq. 2.2.7, and Eq. 2.2.8, a $q\times d$ matrix H with $\mathrm{rk}(H) = q \leq d$, and an arbitrary $q\times 1$ vector h. If $H\beta = h$ then*

$$\frac{\hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^* - \hat{\mathbf{u}}'\hat{\mathbf{u}}}{q\hat{\sigma}^2} \sim F_{q,n-d}, \tag{2.2.9}$$

where $\hat{\sigma}^2 := \hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-d)$.

Proof. Due to Proposition 2.1.1 and the fact that we consider a sample of independent identically normally distributed random vectors the residual $\mathbf{u}$ does not depend on the regressor matrix $\mathbf{X}$. That means $\mathbb{E}(\mathbf{u}\,|\,\mathbf{X}) = \mathbb{E}(\mathbf{u}) = 0$ as well as $\mathbb{V}\mathrm{ar}(\mathbf{u}\,|\,\mathbf{X}) = \mathbb{V}\mathrm{ar}(\mathbf{u}) \propto I_n$. Hence, the standard assumptions of linear regression theory are satisfied and since $\mathbf{u}\,|\,\mathbf{X} \sim \mathbf{u}$ is normally distributed we can apply the F statistic given by the proposition. □

The F statistic (2.2.9) is not convenient if we do not want to calculate the LMVP and its empirical residual explicitly. Nevertheless, for testing against H_0 we do not need to calculate the LMVP at all. This is confirmed by the next theorem.

Theorem 2.2.2. *Consider Eq. 2.2.3, Eq. 2.2.4, and Eq. 2.2.7, a $q \times d$ matrix H with $\mathrm{rk}(H) = q \leq d$, and an arbitrary $q \times 1$ vector h. If $H\beta = h$ then*

$$\frac{(H\hat{\beta}_{\mathrm{OLS}} - h)'(H\,(\mathbf{X}'\mathbf{X})^{-1}\,H')^{-1}(H\hat{\beta}_{\mathrm{OLS}} - h)}{q\hat{\sigma}^2} \sim F_{q,n-d}\,, \tag{2.2.10}$$

where $\hat{\sigma}^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-d)$.

Proof. The F statistic given by the theorem corresponds to the F statistic of Proposition 2.2.1 (see e.g. Greene, 2003, p. 102). □

Another important test is given by $H_0\colon \sigma^2 \geq \sigma_0^2$ (for some $\sigma_0^2 > 0$) which can be tested by the next theorem.

Theorem 2.2.3. *Let $\hat{\sigma}^2$ be the OLS estimator given by Theorem 2.2.2 for the variance $\sigma^2 > 0$ of the GMVP return. It holds that*

$$\frac{n-d}{\sigma^2} \cdot \hat{\sigma}^2 \sim \chi^2_{n-d}\,.$$

Proof. This is a standard result from linear regression theory (see e.g. Greene, 2003, p. 50) if we recognize the linear regression model given by Eq. 2.2.3. □

Eventually the investor not only wants to know if the variance of the GMVP is bounded by some number σ_0^2 but also to test against $H_0 : \eta \leq \eta_0$ where η represents the true expected return of the GMVP. This can be done by applying the next theorem.

Theorem 2.2.4. *Consider Eq. 2.2.4 and note that $\hat{\beta}_{\mathrm{OLS},1}$ is the OLS estimator for the expected return η of the GMVP. It holds that*

$$\frac{\hat{\beta}_{\mathrm{OLS},1} - \eta}{\sqrt{\hat{\sigma}^2 \cdot (\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}} + 1)/n}} \sim t\,(n-d)\,,$$

where $t(n-d)$ denotes Student's t-distribution with $n-d$ degrees of freedom, $\hat{\sigma}^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-d)$, $\overline{\mathbf{x}}$ $((d-1)\times 1)$ is the sample mean vector, and $\widehat{\Omega}$ $((d-1)\times(d-1))$ is the sample covariance matrix of X^{s}.

Proof. From linear regression theory (see e.g. Greene, 2003, p. 51) we know that

$$\frac{\hat{\beta}_{\mathrm{OLS},1} - \eta}{\sqrt{\hat{\sigma}^2 \cdot [(\mathbf{X}'\mathbf{X})^{-1}]_{11}}} \sim t\,(n-d)\,,$$

where $\left[(\mathbf{X}'\mathbf{X})^{-1}\right]_{11}$ denotes the upper left component of $(\mathbf{X}'\mathbf{X})^{-1}$. Note that

$$\left[(\mathbf{X}'\mathbf{X})^{-1}\right]_{11} = \left(n - n\overline{\mathbf{x}}'(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}})^{-1}n\overline{\mathbf{x}}\right)^{-1} = \frac{1}{n} \cdot \left(1 - n\overline{\mathbf{x}}'(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}})^{-1}\overline{\mathbf{x}}\right)^{-1},$$

where $\mathbf{X}^{\mathrm{s}}$ $(n \times (d-1))$ symbolizes the regressor matrix $\mathbf{X}$ without the column of ones. Since

$$\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}} = n \cdot \left(\widehat{\Omega} + \overline{\mathbf{x}}\,\overline{\mathbf{x}}'\right),$$

due to the binomial inverse theorem (Press, 2005, p. 23) we conclude that

$$n \cdot \left(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}}\right)^{-1} = \left(\widehat{\Omega} + \overline{\mathbf{x}}\,\overline{\mathbf{x}}'\right)^{-1} = \widehat{\Omega}^{-1} - \frac{\widehat{\Omega}^{-1}\overline{\mathbf{x}}\,\overline{\mathbf{x}}'\widehat{\Omega}^{-1}}{1 + \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}}.$$

That is

$$1 - n\overline{\mathbf{x}}'(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}})^{-1}\overline{\mathbf{x}} = 1 - \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}} + \frac{(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}})^2}{1 + \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}} = \frac{1}{1 + \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}}$$

and thus

$$\left[(\mathbf{X}'\mathbf{X})^{-1}\right]_{11} = \frac{1}{n} \cdot \left(1 - n\overline{\mathbf{x}}'(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}})^{-1}\overline{\mathbf{x}}\right)^{-1} = \frac{\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}} + 1}{n}.$$

□

3. Hypothesis Tests for Local Minimum Variance Portfolios

3.1. Theoretical Foundation.

Consider the LMVP

$$\underset{(d\times 1)}{w^*} = (w_1^*, \ldots, w_d^*) := \arg\min_v \mathbb{V}\mathrm{ar}\left(R'v\right), \tag{3.1.1}$$

where $1'v = 1$ and some additional linear constraints are fulfilled. Using the definitions above this can be formulated as a least squares problem, i.e.

$$\underset{(d\times 1)}{\beta^*} := \arg\min_b \mathbb{E}\left(\left(Y - X'b\right)^2\right) \tag{3.1.2}$$

under a linear constraint $Hb = h$ where only the parameters $b_2, \ldots, b_d$ are affected. Note that the unrestricted linear model given by Eq. 2.1.2 can be equivalently written as

$$Y = X'\beta + u = \beta_1 + X^{\mathrm{s}\prime}\beta^{\mathrm{s}} + u\,, \tag{3.1.3}$$

whereas the optimal solution (3.1.2) of the restricted minimization problem is indicated by the linear equation

$$Y = X'\beta^* + u^* = \beta_1^* + X^{\mathrm{s}\prime}\beta^{*\mathrm{s}} + u^*\,, \tag{3.1.4}$$

where $\beta^* = (\beta_1^*, \ldots, \beta_d^*)$ and $\beta^{*\mathrm{s}} := (\beta_2^*, \ldots, \beta_d^*)$. Eq. 3.1.4 is *not* a proper linear regression equation since – due to Corollary 2.1.2 – the residual u^* generally depends on the components of X. Thus we cannot apply the standard F statistics given by Eq. 2.2.9 or Eq. 2.2.10 to test against a linear hypothesis for a LMVP. However, Eq. 3.1.4 can be rewritten so as to obtain a proper linear regression equation and the standard F tests become applicable.

Since we are primarily interested in restricting the portfolio weights $b_2, \ldots, b_d$ rather than the expected portfolio return b_1 in (3.1.2) we may substitute the linear restriction $Hb = h$ by $H^{\mathrm{s}}b^{\mathrm{s}} = h$, where H^{s} is a $q \times (d-1)$ matrix with $\mathrm{rk}(H^{\mathrm{s}}) = q \leq d-1$. More precisely, the $q \times d$ matrix H is decomposed into

$$H = \begin{bmatrix} 0 & H^{\mathrm{s}} \end{bmatrix},$$

where 0 is a $q \times 1$ vector of zeros and H^{s} is the residual part of H.

According to Rao (1965, p. 189) any general solution of the linear equation $H^{\mathrm{s}}b^{\mathrm{s}} = h$ can be written as $b^{\mathrm{s}} = b_0^{\mathrm{s}} + Ta^{\mathrm{s}}$ where $b_0^{\mathrm{s}} := (b_{02}, \ldots, b_{0d})$ is some *particular* solution of the linear equation, T $((d-1) \times (d-q-1))$ is a matrix with $\mathrm{rk}(T) = d-q-1$ such that $H^{\mathrm{s}}T = 0$, and $a^{\mathrm{s}} := (a_2, \ldots, a_{d-q})$ is an *arbitrary* $(d-q-1)\times 1$ vector. That means the minimization problem given by (3.1.2) can be reformulated as

$$\underset{((d-q)\times 1)}{\alpha} := \arg\min_a \mathbb{E}\left(\left(Y - a_1 - X^{s\prime}b_0^s - X^{s\prime}Ta^s\right)^2\right), \tag{3.1.5}$$

without any restriction for $a := (a_1,\ldots,a_{d-q})$ $((d-q)\times 1)$. Now, if we define $Y^* := Y - X^{s\prime}b_0^s$ (1×1) and $X^{*s} := (X_2^*,\ldots,X_{d-q}^*) = T'X^s$ $((d-q-1)\times 1)$, Eq. 3.1.5 becomes equivalent to

$$\alpha = \arg\min_a \mathbb{E}\left(\left(Y^* - a_1 - X^{*s\prime}a^s\right)^2\right) = \arg\min_a \mathbb{E}\left(\left(Y^* - X^{*\prime}a\right)^2\right)$$

with $X^* := (1, X_2^*,\ldots,X_{d-q}^*)$ $((d-q)\times 1)$. Thus we obtain the modified linear model

$$Y^* = X^{*\prime}\alpha + u^* = \alpha_1 + X^{*s\prime}\alpha^s + u^*\,, \tag{3.1.6}$$

where $\alpha_1 = \beta_1^*$ and $\alpha^s := (\alpha_2,\ldots,\alpha_{d-q})$, which is quite similar to (3.1.4). However, the vector α is chosen *without any restriction* from $\mathbb{R}^{d-q}$ so that $\mathbb{V}\mathrm{ar}(u^*)$ becomes minimal and it is guaranteed that the condition $H^s\beta^{*s} = h$ is always satisfied after the reparameterization

$$\beta^{*s} := b_0^s + T\alpha^s\,.$$

Since there is no restriction for α, Eq. 3.1.6 represents a proper linear regression equation, i.e. $\mathbb{E}(u^*) = 0$ and $\mathbb{C}\mathrm{ov}(X_j^*, u^*) = 0$ for $j = 2,\ldots,d-q$. Recall that $\beta_2^*,\ldots,\beta_d^*$ correspond to the weights of the LMVP except for the first one and $b_{02},\ldots,b_{0d}$ are some weights of an arbitrary and thus possibly *inefficient* portfolio satisfying the linear restriction. Hence the LMVP is attained by translating the inefficient portfolio $w_0 := (1 - 1'b_0^s, b_0^s)$ into the efficient portfolio $w^* = (1 - 1'\beta^{*s}, \beta^{*s})$ under the given linear constraints, i.e.

$$w^* = w_0 + \begin{bmatrix} -1'T \\ T \end{bmatrix} \alpha^s\,.$$

So the return of the LMVP amounts to

$$R'w^* = R'w_0 + R'\begin{bmatrix} -1'T \\ T \end{bmatrix} \alpha^s\,, \tag{3.1.7}$$

where $R'w_0$ is the return of the inefficient portfolio. The other term on the right hand side of Eq. 3.1.7 can be also interpreted as a portfolio return. Note that each column of the $d\times(d-q-1)$ matrix

$$\begin{bmatrix} -1'T \\ T \end{bmatrix}$$

sums up to zero. Here we can think of $d-q-1$ *self-financing* rebalancing strategies which on the one hand are necessary to translate the inefficient portfolio into the LMVP and on the other hand sufficient to satisfy the given linear restrictions. Now, similar to the approach of Kempf and Memmel (2006) we can find the LMVP by solving the linear regression equation

$$R'w_0 = \alpha_1 + R'\begin{bmatrix} 1'T \\ -T \end{bmatrix} \alpha^s + u^*\,. \tag{3.1.8}$$

In fact, since

$$\begin{bmatrix} 1'T \\ -T \end{bmatrix}' R = T'1R_1 - T' \begin{bmatrix} R_2 \\ R_3 \\ \vdots \\ R_d \end{bmatrix} = T' \begin{bmatrix} R_1 - R_2 \\ R_1 - R_3 \\ \vdots \\ R_1 - R_d \end{bmatrix} = T'X^{\mathrm{s}} = X^{*\mathrm{s}}$$

and

$$R'w_0 = R_1 \cdot \left(1 - 1'b_0^{\mathrm{s}}\right) + \begin{bmatrix} R_2 \\ R_3 \\ \vdots \\ R_d \end{bmatrix}' b_0^{\mathrm{s}} = R_1 - \begin{bmatrix} R_1 - R_2 \\ R_1 - R_3 \\ \vdots \\ R_1 - R_d \end{bmatrix}' b_0^{\mathrm{s}} = Y - X^{\mathrm{s}\prime} b_0^{\mathrm{s}} = Y^*,$$

Eq. 3.1.8 is equivalent to Eq. 3.1.6.

An important question is how to derive the needed quantities b_0^{s} and T. First of all we assume that the matrix H^{s} is structured in such a way that it can be decomposed into

$$H^{\mathrm{s}} = \begin{bmatrix} H_2 & H_3 \end{bmatrix}, \tag{3.1.9}$$

where H_2 is a $q \times q$ matrix with $\mathrm{rk}(H_2) = q$ and H_3 is a $q \times (d - q - 1)$ matrix. A structure like this can be always found by a permutation of the columns of H^{s} since $\mathrm{rk}(H^{\mathrm{s}}) = q$. Now let $b_0^{\mathrm{s}} = (\gamma, 0)$ where γ is a $q \times 1$ vector and 0 is a $d - q - 1$ vector of zeros. That means we are searching for a vector γ such that $H^{\mathrm{s}} b_0^{\mathrm{s}} = h$ or, equivalently, $\gamma = H_2^{-1} h$, i.e.

$$b_0^{\mathrm{s}} := \begin{bmatrix} H_2^{-1} h \\ 0 \end{bmatrix}. \tag{3.1.10}$$

Furthermore, consider the matrix

$$T = \begin{bmatrix} \Gamma \\ I_{d-q-1} \end{bmatrix},$$

where Γ is a $q \times (d - q - 1)$ matrix. Now we are searching for Γ such that

$$H^{\mathrm{s}} T = H_2 \Gamma + H_3 = 0\,.$$

This leads to the solution

$$\Gamma = -H_2^{-1} H_3\,,$$

which means that

$$T := \begin{bmatrix} -H_2^{-1} H_3 \\ I_{d-q-1} \end{bmatrix}. \tag{3.1.11}$$

3.2. **Statistical Inference.** Due to the preceding theoretical arguments we see that the parameter vector α can be readily estimated by the OLS estimator

$$\hat{\alpha}_{\mathrm{OLS}} := \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} \mathbf{X}^{*\prime}\mathbf{Y}^*\,, \tag{3.2.1}$$

where

$$\underset{(n \times (d-q))}{\mathbf{X}^*} := \begin{bmatrix} 1 & X_{12}^* & \cdots & X_{1,d-q}^* \\ 1 & X_{22}^* & \cdots & X_{2,d-q}^* \\ \vdots & \vdots & & \vdots \\ 1 & X_{n2}^* & \cdots & X_{n,d-q}^* \end{bmatrix} \tag{3.2.2}$$

and

$$\underset{(n\times 1)}{\mathbf{Y}^*} := \begin{bmatrix} Y_1^* \\ Y_2^* \\ \vdots \\ Y_n^* \end{bmatrix}.$$

The relationship between the empirical residual $\hat{\mathbf{u}}^*$ $(n\times 1)$ and the OLS estimator $\hat{\alpha}_{\text{OLS}}$ can be represented by

$$\mathbf{Y}^* = \mathbf{X}^* \hat{\alpha}_{\text{OLS}} + \hat{\mathbf{u}}^*\,, \tag{3.2.3}$$

whereas for the RLS estimator

$$\hat{\beta}_{\text{RLS}} = \left(\hat{\beta}_{\text{RLS},1}, \ldots, \hat{\beta}_{\text{RLS},d}\right) \tag{3.2.4}$$

we possess Eq. 2.2.8. The RLS estimator for $\beta^{*\text{s}}$ corresponds to

$$\hat{\beta}^{\text{s}}_{\text{RLS}} := \left(\hat{\beta}_{\text{RLS},2}, \ldots, \hat{\beta}_{\text{RLS},d}\right) = b_0^{\text{s}} + T\hat{\alpha}^{\text{s}}_{\text{OLS}} = b_0^{\text{s}} + T\widehat{\Omega}^{*-1}\hat{\omega}^*\,, \tag{3.2.5}$$

where $\hat{\alpha}^{\text{s}}_{\text{OLS}} := (\hat{\alpha}_{\text{OLS},2}, \ldots, \hat{\alpha}_{\text{OLS},d-q})$, $\widehat{\Omega}^*$ is the sample covariance matrix of $X^{*\text{s}}$ and $\hat{\omega}^*$ is the $(d-q-1)\times 1$ vector of the sample covariances between Y^* and X_j^* $(j = 2, \ldots, d-q)$. Hence, by the Gauss-Markov theorem

$$\hat{w}^* := \left(1 - 1'\hat{\beta}^{\text{s}}_{\text{RLS}}, \hat{\beta}^{\text{s}}_{\text{RLS}}\right)$$

is the best linear unbiased estimator for the LMVP w^* represented by Eq. 3.1.1. However, note that $\hat{\beta}_{\text{RLS},1} = \hat{\alpha}_{\text{OLS},1}$ is the least squares estimator for the expected return of the LMVP.

Now consider another linear hypothesis $G\beta^* = g$, where G $(p\times d)$ is a matrix with $\text{rk}(G) = p \leq d-q$ such that the stacked matrix

$$\underset{((p+q)\times d)}{S} := \begin{bmatrix} G \\ H \end{bmatrix}$$

has rank $p+q$ and g is an arbitrary $p\times 1$ vector. We can decompose G by

$$G = \begin{bmatrix} G_1 & G^{\text{s}} \end{bmatrix},$$

where G_1 $(p\times 1)$ is the first column of G and its residual part is given by G^{s} $(p\times(d-1))$. Note that $G\beta^* = g$ is equivalent to

$$G_1\alpha_1 + G^{\text{s}}T\alpha^{\text{s}} = g - G^{\text{s}}b_0^{\text{s}}\,.$$

Thus, by defining

$$\underset{(p\times(d-q))}{G^*} := \begin{bmatrix} G_1 & G^{\text{s}}T \end{bmatrix}$$

and $g^* := g - G^{\text{s}}b_0^{\text{s}}$ $(p\times 1)$ we can substitute the original restriction $G\beta^* = g$ by $G^*\alpha = g^*$. Further, we can decompose G^{s} into

$$G^{\text{s}} = \begin{bmatrix} G_2 & G_3 \end{bmatrix},$$

where G_2 $(p\times q)$ and G_3 $(p\times(d-q-1))$. Now the stacked matrix S – which contains the structure of our linear restrictions – is given by

$$\underset{((p+q)\times d)}{S} = \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} \underset{(p\times 1)}{G_1} & \underset{(p\times q)}{G_2} & \underset{(p\times(d-q-1))}{G_3} \\ \underset{(q\times 1)}{0} & \underset{(q\times q)}{H_2} & \underset{(q\times(d-q-1))}{H_3} \end{bmatrix}, \tag{3.2.6}$$

where $\mathrm{rk}(S) = p + q$. We can also define the vector

$$\underset{((p+q)\times 1)}{s} := \begin{bmatrix} \underset{(p\times 1)}{g} \\ \underset{(q\times 1)}{h} \end{bmatrix} \tag{3.2.7}$$

so as to calculate the quantities

$$G^* = \begin{bmatrix} G_1 & G_3 - G_2 H_2^{-1} H_3 \end{bmatrix} \tag{3.2.8}$$

and $g^* = g - G_2 H_2^{-1} h$ simply by s and S. This leads to the next theorem.

Theorem 3.2.1. *Consider Eq. 2.2.8 and Eq. 3.1.4, as well as the quantities given by Eq. 3.1.9, Eq. 3.2.6, Eq. 3.2.7, and Eq. 3.2.8. Let* $\beta^* = (\beta_1^*, \beta^{*\mathrm{s}})$ *be the parameter vector of the LMVP satisfying the linear restriction* $H^{\mathrm{s}}\beta^{*\mathrm{s}} = h$. *If in addition* $G\beta^* = g$ *then*

$$\frac{(G\hat{\beta}_{\mathrm{RLS}} - g)'(G^* \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} G^{*\prime})^{-1}(G\hat{\beta}_{\mathrm{RLS}} - g)}{p\hat{\sigma}^{*2}} \sim F_{p,n-d+q}\,, \tag{3.2.9}$$

where $\hat{\sigma}^{*2} := \hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^*/(n-d+q)$.

Proof. Similar to the proof of Theorem 2.2.2, from linear regression theory we know that

$$\frac{(G^*\hat{\alpha}_{\mathrm{OLS}} - g^*)'(G^* \left(\mathbf{X}^{*\prime}\mathbf{X}^*\right)^{-1} G^{*\prime})^{-1}(G^*\hat{\alpha}_{\mathrm{OLS}} - g^*)}{p\hat{\sigma}^{*2}} \sim F_{q,n-d+q}$$

and note that $G^*\hat{\alpha}_{\mathrm{OLS}} - g^* = G\hat{\beta}_{\mathrm{RLS}} - g$. □

Of course, we could have also give the F statistic in terms of the restricted and unrestricted sum of squared residuals as in Proposition 2.2.1. Then one would have to estimate not only the LMVP given by the linear restriction $H^{\mathrm{s}}\beta^{*\mathrm{s}} = h$ but also the LMVP characterized by the additional constraint $G\beta^* = g$ which seems to be rather cumbersome in practical situations.

The next two theorems are the natural counterparts of Theorem 2.2.3 and Theorem 2.2.4.

Theorem 3.2.2. *Let* $\hat{\sigma}^{*2}$ *be the RLS estimator given by Theorem 3.2.1 for the variance* $\sigma^{*2} > 0$ *of the return of a LMVP with* q *linear restrictions for the portfolio weights (without the budget constraint). It holds that*

$$\frac{n-d+q}{\sigma^{*2}} \cdot \hat{\sigma}^{*2} \sim \chi^2_{n-d+q}\,.$$

Proof. As mentioned in the proof of Theorem 2.2.3 this is a standard result of linear regression theory where we have to consider only $d - q$ instead of d dimensions. □

Theorem 3.2.3. *Consider Eq. 3.2.4 and note that* $\hat{\beta}_{\mathrm{RLS},1}$ *is the RLS estimator for the expected return* η^* *of a LMVP with* q *linear restrictions for the portfolio weights (without the budget constraint). It holds that*

$$\frac{\hat{\beta}_{\mathrm{RLS},1} - \eta^*}{\sqrt{\hat{\sigma}^{*2} \cdot (\overline{\mathbf{x}}^{*\prime}\widehat{\Omega}^{*-1}\overline{\mathbf{x}}^* + 1)/n}} \sim t\,(n-d+q)\,,$$

where $t(n-d+q)$ *denotes Student's t-distribution with* $n-d+q$ *degrees of freedom,* $\hat{\sigma}^{*2} = \hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^*/(n-d+q)$, $\overline{\mathbf{x}}^*$ $((d-q-1)\times 1)$ *is the sample mean vector, and* $\widehat{\Omega}^*$ $((d-q-1)\times(d-q-1))$ *is the sample covariance matrix of* $X^{*\mathrm{s}}$.

Proof. Note that $\hat{\beta}_{\text{RLS},1} = \hat{\alpha}_{\text{OLS},1}$ and follow the proof of Theorem 2.2.4. □

The latter two theorems can be used for testing against $H_0 \colon \sigma^{*2} \geq \sigma_0^2 > 0$ or $H_0 \colon \eta^* \leq \eta_0$, respectively.

4. Distribution of the Estimated Portfolio Weights

As it was shown before, finding a minimum variance portfolio is equivalent to the fundamental least squares problem of linear regression theory. We have seen that this is true not only for the GMVP (Kempf and Memmel, 2006) but also for any LMVP possessing some linear restrictions for the portfolio weights after some appropriate transformation of the data. Since the parameters of interest can be represented by a proper linear regression equation, linear statistical inference can be readily done by applying standard methods of econometrics. More precisely, we can derive the *conditional* distribution of $\hat{\beta}_{\text{OLS}}$ under each realization of $\mathbf{X}$ and so we are able to conduct exact hypothesis tests, calculate exact confidence intervals, etc. However, I will not go into the details of linear statistical inference given some realization of $\mathbf{X}$ since suchlike results for the GMVP can be already found in Kempf and Memmel (2006). Note that after transforming the data the same instruments can be applied for any LMVP. Instead, in the following section I will concentrate on the unconditional finite sample distribution of the estimated weights of global and local minimum variance portfolios. Although this is only loosely connected to linear statistical inference the unconditional distribution of the estimated portfolio weights might be of interest in its own right.

4.1. Preliminary Definitions.

The subsequent statements follow from linear regression theory and so they are not merely valid in the context of portfolio estimation, but rather for least squares parameter estimation in general. Nevertheless, I will refer only to the estimation of portfolio weights and drop the standard notation of linear regression. That means we turn back to the initial notation. From now on $\hat{w}$ denotes the estimator for the GMVP whereas $\hat{w}^*$ is an estimator for a LMVP. Correspondingly, w symbolizes the true GMVP and w^* is the true LMVP. The expected return of the GMVP is denoted by η whereas the expected return of the LMVP is given by η^*. Moreover, σ^2 is the variance of the GMVP return and σ^{*2} is the variance of the LMVP return. The corresponding unbiased least squares estimators are given by $\hat{\eta}$, $\hat{\eta}^*$, $\hat{\sigma}^2$, and $\hat{\sigma}^{*2}$. In the following $t_k(a, B, \nu)$ (with $t(\cdot) \equiv t_1(\cdot)$) stands for the k-variate t-distribution with $\nu > 0$ degrees of freedom, location vector a ($k \times 1$), and positive semi-definite dispersion matrix B ($k \times k$), i.e.

$$a + \frac{\zeta}{\sqrt{\chi^2_\nu / \nu}} \sim t_k(a, B, \nu),$$

where $\zeta \sim \mathcal{N}_k(0, B)$ is stochastically independent of χ^2_ν. Here we suppose that

$$\zeta \sim B^{\frac{1}{2}} \xi \,,$$

where $\xi \sim \mathcal{N}_k(0, I_k)$ and $B^{\frac{1}{2}}$ is a matrix such that $B^{\frac{1}{2}} B^{\frac{1}{2}\prime} = B$. Furthermore, the following notation will be useful for the subsequent derivations:

$$\underset{((d-1)\times d)}{\mathcal{I}} := \begin{bmatrix} 1 & -I_{d-1} \end{bmatrix}, \tag{4.1.1}$$

so that $\mathcal{I}R = X^{\mathrm{s}}$ and thus $\Omega := \mathcal{I}\Sigma\mathcal{I}'$ is the covariance matrix of X^{s}. Analogously, by the definition

$$\underset{((d-q-1)\times d)}{\mathcal{T}} := T'\mathcal{I} = \begin{bmatrix} T'1 & -T' \end{bmatrix}, \tag{4.1.2}$$

we obtain $\mathcal{T}R = X^{*\mathrm{s}}$ and $\Omega^* := \mathcal{T}\Sigma\mathcal{T}'$ is the covariance matrix of $X^{*\mathrm{s}}$.

4.2. **Global Minimum Variance Portfolio.** The next theorem provides the unconditional finite sample distribution of the OLS estimator for the GMVP. Another variant of this theorem can be found in Okhrin and Schmid (2006).

Theorem 4.2.1. *Let $w = (w_1, \ldots, w_d)$ be the GMVP of d assets and $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_d)$ the corresponding OLS estimator for a sample of asset returns with size $n \geq d$. It holds that*

$$(\hat{w}_2, \ldots, \hat{w}_d) \sim t_{d-1}\left((w_2, \ldots, w_d), \frac{\sigma^2}{n-d+1} \cdot \left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}, n-d+1\right),$$

where $\mathcal{I}$ is given by Eq. 4.1.1, $\sigma^2 = w'\Sigma w$ is the variance of the GMVP return, and Σ is the covariance matrix of R.

Proof. From linear regression theory we know that

$$(\hat{\eta}, \hat{w}_2, \ldots, \hat{w}_d) \,|\, \mathbf{X} \sim \mathcal{N}_d\left((\eta, w_2, \ldots, w_d), \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1}\right).$$

Define $\hat{w}^{\mathrm{s}} := (\hat{w}_2, \ldots, \hat{w}_d)$ and consider the partition

$$\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \begin{bmatrix} a & b' \\ b & C \end{bmatrix},$$

where a (1×1), b $((d-1) \times 1)$, and C $((d-1) \times (d-1))$. Then

$$\begin{aligned} \sigma^2 C &= \sigma^2 \cdot \left(\mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}} - \frac{1}{n} \cdot \mathbf{X}^{\mathrm{s}\prime}11'\mathbf{X}^{\mathrm{s}}\right)^{-1} \\ &= \frac{\sigma^2}{n} \cdot \left(\frac{1}{n} \cdot \mathbf{X}^{\mathrm{s}\prime}\mathbf{X}^{\mathrm{s}} - \overline{\mathbf{x}}\,\overline{\mathbf{x}}'\right)^{-1} = \frac{\sigma^2}{n} \cdot \widehat{\Omega}^{-1} \end{aligned}$$

is the covariance matrix of $\hat{w}^{\mathrm{s}} \,|\, \mathbf{X}$. Hence,

$$\hat{w}^{\mathrm{s}} \,|\, \mathbf{X} \equiv \hat{w}^{\mathrm{s}} \,|\, \widehat{\Omega}^{-1} \sim \mathcal{N}_{d-1}\left((w_2, \ldots, w_d), \frac{\sigma^2}{n} \cdot \widehat{\Omega}^{-1}\right)$$

and note that $\widehat{\Omega}^{-1}$ is inverse Wishart distributed. More precisely, it has a density function of the form

$$p\left(\widehat{\Omega}^{-1}\right) \propto |\widehat{\Omega}^{-1}|^{-\frac{n+d-1}{2}} \cdot \exp\left(-\frac{1}{2} \cdot \operatorname{tr} \widehat{\Omega}\, (\Omega/n)^{-1}\right)$$

(Press, 2005, p. 117). So the joint density function of $\hat{w}^{\mathrm{s}}$ and $\widehat{\Omega}^{-1}$ is given by

$$\begin{aligned} p\left(\hat{w}^{\mathrm{s}}, \widehat{\Omega}^{-1}\right) &= p\left(\hat{w}^{\mathrm{s}} \,|\, \widehat{\Omega}^{-1}\right) \cdot p\left(\widehat{\Omega}^{-1}\right) \\ &\propto |\widehat{\Omega}^{-1}|^{-\frac{n+d}{2}} \times \\ &\quad \exp\left(-\frac{1}{2} \cdot \operatorname{tr} \widehat{\Omega}\left((\Omega/n)^{-1} + n/\sigma^2 \cdot (\hat{w}^{\mathrm{s}} - w^{\mathrm{s}})(\hat{w}^{\mathrm{s}} - w^{\mathrm{s}})'\right)\right), \end{aligned}$$

where $w^{\mathrm{s}} := (w_2, \ldots, w_d)$. Integrating the joint density function with respect to $\widehat{\Omega}^{-1}$ leads to

$$p\left(\hat{w}^{\mathrm{s}}\right) \propto \frac{1}{\left(1 + (\hat{w}^{\mathrm{s}} - w^{\mathrm{s}})'(\Omega/\sigma^2)(\hat{w}^{\mathrm{s}} - w^{\mathrm{s}})\right)^{\frac{n}{2}}}$$

(Press, 2005, p. 186), which is equivalent to

$$p\left(\hat{w}^{\mathrm{s}}\right) \propto \frac{1}{\left(1+\frac{(\hat{w}^{\mathrm{s}}-w^{\mathrm{s}})'(\sigma^2/(n-d+1)\cdot\Omega^{-1})^{-1}(\hat{w}^{\mathrm{s}}-w^{\mathrm{s}})}{n-d+1}\right)^{\frac{(d-1)+(n-d+1)}{2}}}.$$

After substituting Ω by $\mathcal{I}\Sigma\mathcal{I}'$ this corresponds to the density function of the multivariate t-distribution (Press, 2005, p. 136) given by the theorem. □

An unbiased estimator for the covariance matrix of $(\hat{w}_2,\ldots,\hat{w}_d)$ is provided by the next corollary.

Corollary 4.2.2. *Consider a sample of asset returns with size $n \geq d+2$ and let $(\hat{w}_2,\ldots,\hat{w}_d)$ be the OLS estimator for the GMVP except for the first portfolio weight. Then the matrix*

$$\widehat{\mathbb{V}\mathrm{ar}}\left((\hat{w}_2,\ldots,\hat{w}_d)\right) := \frac{\hat{\sigma}^2}{n}\cdot\left(\mathcal{I}\widehat{\Sigma}\mathcal{I}'\right)^{-1}$$

is an unbiased estimator for the covariance matrix of $(\hat{w}_2,\ldots,\hat{w}_d)$, where $\mathcal{I}$ is given by Eq. 4.1.1, $\hat{\sigma}^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-d)$ is an unbiased estimator for the variance of the GMVP, and $\widehat{\Sigma}$ denotes the sample covariance matrix of R.

Proof. The preceding theorem implies that the covariance matrix of $(\hat{w}_2,\ldots,\hat{w}_d)$ is given by

$$\mathbb{V}\mathrm{ar}\left((\hat{w}_2,\ldots,\hat{w}_d)\right) = \frac{\sigma^2}{n-d-1}\cdot\Omega^{-1}.$$

From Wishart theory we know that $\widehat{\Omega}^{-1} \sim W_{d-1}^{-1}((\Omega/n)^{-1}, n+d-1)$ (Press, 2005, p. 117). Hence we obtain

$$\mathbb{E}\left(\widehat{\Omega}^{-1}\right) = \frac{(\Omega/n)^{-1}}{(n+d-1)-2\cdot(d-1)-2} = \frac{n}{n-d-1}\cdot\Omega^{-1}$$

(Press, 2005, p. 119). Moreover, from linear regression theory we know that $\hat{\sigma}^2$ is a conditionally unbiased estimator for σ^2. That means

$$\begin{aligned}\mathbb{E}\left(\frac{\hat{\sigma}^2}{n}\cdot\widehat{\Omega}^{-1}\right) &= \mathbb{E}\left(\mathbb{E}\left(\frac{\hat{\sigma}^2}{n}\cdot\widehat{\Omega}^{-1}\,|\,\widehat{\Omega}^{-1}\right)\right) = \mathbb{E}\left(\frac{\sigma^2}{n}\cdot\widehat{\Omega}^{-1}\right)\\ &= \frac{\sigma^2}{n-d-1}\cdot\Omega^{-1} = \mathbb{V}\mathrm{ar}\left((\hat{w}_2,\ldots,\hat{w}_d)\right)\end{aligned}$$

and note that $\widehat{\Omega} = \mathcal{I}\widehat{\Sigma}\mathcal{I}'$. □

The next theorem complements Theorem 4.2.1 since it provides the unconditional distribution of the first (or, after an appropriate rearrangement, any other) portfolio weight estimate.

Theorem 4.2.3. *Let $w = (w_1,\ldots,w_d)$ be the GMVP of d assets and $\hat{w} = (\hat{w}_1,\ldots,\hat{w}_d)$ the corresponding OLS estimator for a sample of asset returns with size $n \geq d$. Then $\hat{w}_1 = 1-\hat{w}_2-\ldots-\hat{w}_d$ and it holds that*

$$\hat{w}_1 \sim t\left(w_1, \frac{\sigma^2}{n-d+1}\cdot 1'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}1, n-d+1\right).$$

Proof. Note that $\hat{w}_1 = 1 - 1'\hat{w}^{\mathrm{s}}$ and from Theorem 4.2.1 we conclude that

$$1'\hat{w}^{\mathrm{s}} \sim t\left(1'w^{\mathrm{s}}, \frac{\sigma^2}{n-d+1}\cdot 1'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}1, n-d+1\right),$$

so that $\hat{w}_1$ possesses the distribution given by the theorem. □

Corollary 4.2.2 immediately leads to the next one.

Corollary 4.2.4. *Consider a sample of asset returns with size $n \geq d+2$ and let $\hat{w}_1$ be the OLS estimator for the first weight of the GMVP. Then*

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{w}_1\right) := \frac{\hat{\sigma}^2}{n} \cdot 1' \left(\mathcal{I}\widehat{\Sigma}\mathcal{I}'\right)^{-1} 1$$

with $\hat{\sigma}^2 = \hat{\mathbf{u}}'\hat{\mathbf{u}}/(n-d)$ is an unbiased estimator for the variance of $\hat{w}_1$.

Proof. The variance of $\hat{w}_1$ is given by

$$\mathbb{V}\mathrm{ar}\left(\hat{w}_1\right) = \mathbb{V}\mathrm{ar}\left(1 - 1'\hat{w}^{\mathrm{s}}\right) = \mathbb{V}\mathrm{ar}\left(1'\hat{w}^{\mathrm{s}}\right) = 1'\mathbb{V}\mathrm{ar}\left(\left(\hat{w}_2, \ldots, \hat{w}_d\right)\right) 1$$

and due to Corollary 4.2.2 the presented estimator is unbiased for $\mathbb{V}\mathrm{ar}(\hat{w}_1)$. □

Principally, we can find a stochastic representation for $\hat{\eta}$, i.e. the OLS estimator for the expected return of the GMVP. However, this is not very useful for econometrical purposes. Instead, the next theorem provides the first two moments of the distribution of $\hat{\eta}$. Note that

$$\hat{\eta} = \hat{\beta}_{\mathrm{OLS},1} = \overline{\mathbf{r}}'\hat{w}\,,$$

where $\overline{\mathbf{r}} := \mathbf{R}'1/n$ $(d \times 1)$ is the sample mean vector of R.

Theorem 4.2.5. *Let $w = (w_1, \ldots, w_d)$ be the GMVP of d assets and $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_d)$ the corresponding OLS estimator for a sample of asset returns with size $n \geq d+2$. Further, let $\hat{\eta} = \hat{\beta}_{\mathrm{OLS},1}$ be the OLS estimator for the expected return η of the GMVP. Then*

$$\mathbb{E}\left(\hat{\eta}\right) = \eta$$

and

$$\mathbb{V}\mathrm{ar}\left(\hat{\eta}\right) = \frac{\sigma^2}{n-d-1} \cdot \left(\mu'\mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}\mathcal{I}\mu + 1\right) - 2 \cdot \frac{\sigma^2/n}{n-d-1}\,.$$

Proof. We know from linear regression theory that $\hat{\beta}_{\mathrm{OLS},1}$ is conditionally unbiased, i.e. $\mathbb{E}(\hat{\eta} \,|\, \mathbf{X}) = \eta$ so that

$$\mathbb{E}\left(\hat{\eta}\right) = \mathbb{E}\left(\mathbb{E}\left(\hat{\eta} \,|\, \mathbf{X}\right)\right) = \eta\,.$$

The variance of $\hat{\eta}$ is given by

$$\mathbb{V}\mathrm{ar}\left(\hat{\eta}\right) = \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(\hat{\eta} \,|\, \mathbf{X}\right)\right) + \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left(\hat{\eta} \,|\, \mathbf{X}\right)\right) = \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(\hat{\eta} \,|\, \mathbf{X}\right)\right),$$

where

$$\hat{\eta} \,|\, \mathbf{X} \sim \mathcal{N}\left(\eta, \sigma^2 \cdot \left(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}} + 1\right)/n\right)$$

(Kempf and Memmel, 2006). Note that $\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}$ essentially follows a noncentral F-distribution (Muirhead, 1982, p. 24) since $(n-1) \cdot \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}$ corresponds to Hotelling's T^2 statistic (Press, 2005, p. 132) and thus

$$\frac{n-d+1}{d-1} \cdot \overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}} \sim F_{d-1,n-d+1}\left(\lambda\right)$$

with noncentrality parameter

$$\lambda = n \cdot \mathbb{E}(X^{\mathrm{s}})'\Omega^{-1}\mathbb{E}(X^{\mathrm{s}}) = n\mu'\mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}\mathcal{I}\mu\,. \tag{4.2.1}$$

Hence, using the expected value of a noncentral F-distribution (Muirhead, 1982, p. 25) we obtain

$$\begin{aligned}\mathbb{E}\left(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}\right) &= \mathbb{E}\left(F_{d-1,n-d+1}(\lambda)\right)\cdot\frac{d-1}{n-d+1}\\ &= \frac{(n-d+1)\cdot(d-1+\lambda)}{(d-1)\cdot(n-d-1)}\cdot\frac{d-1}{n-d+1}=\frac{d-1+\lambda}{n-d-1}\end{aligned}$$

and thus $\mathbb{V}\mathrm{ar}(\hat{\eta})$ corresponds to

$$\mathbb{E}\left(\frac{\sigma^2}{n}\cdot\left(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}+1\right)\right)=\frac{\sigma^2}{n}\cdot\frac{n-2+\lambda}{n-d-1}.$$

That leads to the desired formula if we substitute λ using the expression given by Eq. 4.2.1. □

Corollary 4.2.6. *Consider a sample of asset returns with size $n \geq d+2$ and let $\hat{\eta}=\hat{\beta}_{\mathrm{OLS},1}$ be the OLS estimator for the expected return η of the GMVP. Then*

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{\eta}\right):=\frac{\hat{\sigma}^2}{n}\cdot\left(\overline{\mathbf{r}}'\mathcal{I}'\left(\mathcal{I}\widehat{\Sigma}\mathcal{I}'\right)^{-1}\mathcal{I}\,\overline{\mathbf{r}}+1\right),$$

is an unbiased estimator for $\mathbb{V}\mathrm{ar}(\hat{\eta})$, where $\overline{\mathbf{r}}$ $(d\times 1)$ is the sample mean vector and $\widehat{\Sigma}$ $(d\times d)$ is the sample covariance matrix of R.

Proof. Note that

$$\overline{\mathbf{r}}'\mathcal{I}'\left(\mathcal{I}\widehat{\Sigma}\mathcal{I}'\right)^{-1}\mathcal{I}\,\overline{\mathbf{r}}=\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}$$

and, since $\hat{\sigma}^2$ is conditionally unbiased, the expected value of $\widehat{\mathbb{V}\mathrm{ar}}(\hat{\eta})$ is given by

$$\mathbb{E}\left(\mathbb{E}\left(\frac{\hat{\sigma}^2}{n}\cdot\left(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}+1\right)\mid\mathbf{X}\right)\right)=\mathbb{E}\left(\frac{\sigma^2}{n}\cdot\left(\overline{\mathbf{x}}'\widehat{\Omega}^{-1}\overline{\mathbf{x}}+1\right)\right).$$

By the proof of Theorem 4.2.5 this corresponds to $\mathbb{V}\mathrm{ar}(\hat{\eta})$. □

Note that

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{\eta}\right)=\overline{\mathbf{r}}'\mathcal{I}'\,\widehat{\mathbb{V}\mathrm{ar}}\left((\hat{w}_2,\ldots,\hat{w}_d)\right)\mathcal{I}\,\overline{\mathbf{r}}+\frac{\hat{\sigma}^2}{n},$$

where $\widehat{\mathbb{V}\mathrm{ar}}((\hat{w}_2,\ldots,\hat{w}_d))$ is given by Corollary 4.2.2. Since

$$\mathcal{I}'\left(\hat{w}_2,\ldots,\hat{w}_d\right)=\left(\sum_{i=2}^{d}w_i,-w_2,\ldots,-w_d\right)$$

we can see that

$$\mathcal{I}'\,\widehat{\mathbb{V}\mathrm{ar}}\left((\hat{w}_2,\ldots,\hat{w}_d)\right)\mathcal{I}=\widehat{\mathbb{V}\mathrm{ar}}\left((\hat{w}_1,\ldots,\hat{w}_d)\right)$$

corresponds to the positive semi-definite covariance matrix of $\hat{w}$. Thus we obtain the nice representation

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{\eta}\right)=\overline{\mathbf{r}}'\,\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{w}\right)\overline{\mathbf{r}}+\frac{\hat{\sigma}^2}{n}$$

for the unbiased estimator. That means the estimation risk concerning the expected GMVP return can be decomposed into a part quantifying the estimation risk of the portfolio weights and another part carrying the variance of the GMVP return.

The next theorem provides a similar result for the *out-of-sample* variance of the return of the estimated GMVP. That is the variance of the portfolio return where the *estimated* GMVP weights are taken and combined with some *future* asset returns $R_1, \ldots, R_d$. This is a typical situation of an investor who estimates the GMVP from historical data and takes these portfolio weights for a future investment.

Theorem 4.2.7. *Let $w = (w_1, \ldots, w_d)$ be the GMVP of d assets and $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_d)$ the corresponding OLS estimator for a sample of asset returns with size $n \geq d+2$. For the out-of-sample portfolio return $R'\hat{w}$ it holds that*

$$\mathbb{E}\left(R'\hat{w}\right) = \eta$$

and

$$\mathbb{V}\mathrm{ar}\left(R'\hat{w}\right) = \mu'\mathbb{V}\mathrm{ar}\left(\hat{w}\right)\mu + \sigma^2 \cdot \frac{n-2}{n-d-1},$$

where

$$\mathbb{V}\mathrm{ar}\left(\hat{w}\right) = \frac{\sigma^2}{n-d-1} \cdot \mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}\mathcal{I}.$$

Proof. The expected out-of-sample portfolio return $R'\hat{w}$ corresponds to

$$\mathbb{E}\left(R'\hat{w}\right) = \mathbb{E}\left(\mathbb{E}\left(R'\hat{w} \mid \hat{w}\right)\right) = \mathbb{E}\left(\mu'\hat{w}\right) = \mu'w = \eta.$$

For the out-of-sample variance we obtain

$$\begin{aligned} \mathbb{V}\mathrm{ar}\left(R'\hat{w}\right) &= \mathbb{E}\left(\mathbb{V}\mathrm{ar}\left(R'\hat{w} \mid \hat{w}\right)\right) + \mathbb{V}\mathrm{ar}\left(\mathbb{E}\left(R'\hat{w} \mid \hat{w}\right)\right) \\ &= \mathbb{E}\left(\hat{w}'\Sigma\hat{w}\right) + \mu'\mathbb{V}\mathrm{ar}\left(\hat{w}\right)\mu. \end{aligned}$$

Note that

$$\hat{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \mathcal{I}'\left(\hat{w}_2, \ldots, \hat{w}_d\right),$$

so that

$$\mathbb{V}\mathrm{ar}\left(\hat{w}\right) = \mathcal{I}'\mathbb{V}\mathrm{ar}\left(\left(\hat{w}_2, \ldots, \hat{w}_d\right)\right)\mathcal{I} = \frac{\sigma^2}{n-d-1} \cdot \mathcal{I}'\Omega^{-1}\mathcal{I}$$

(see the proof of Corollary 4.2.2), and since $\Omega = \mathcal{I}\Sigma\mathcal{I}'$ we obtain

$$\mathbb{V}\mathrm{ar}\left(\hat{w}\right) = \frac{\sigma^2}{n-d-1} \cdot \mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}\mathcal{I}.$$

Further, note that due to Theorem 4.2.1

$$\hat{w} \sim t_d\left(w, \frac{\sigma^2}{n-d+1} \cdot \mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-1}\mathcal{I}, n-d+1\right),$$

so that we can write $\hat{w} \sim w + \varepsilon$, where

$$\varepsilon \sim \frac{\sigma}{\sqrt{n-d+1}} \cdot \mathcal{I}'\left(\mathcal{I}\Sigma\mathcal{I}'\right)^{-\frac{1}{2}} \cdot \frac{\xi}{\sqrt{\chi^2_{n-d+1}/(n-d+1)}}$$

and $\xi \sim \mathcal{N}_{d-1}(0, I_{d-1})$ is stochastically independent of χ^2_{n-d+1}. That means

$$\varepsilon'\Sigma\varepsilon \sim \frac{\sigma^2}{n-d+1} \cdot \frac{\xi^2}{\chi^2_{n-d+1}/(n-d+1)} \sim \sigma^2 \cdot \frac{d-1}{n-d+1} \cdot F_{d-1,n-d+1}$$

and thus

$$\mathbb{E}\left(\hat{w}'\Sigma\hat{w}\right) = w'\Sigma w + \mathbb{E}\left(\varepsilon'\Sigma\varepsilon\right) = \sigma^2 + \sigma^2 \cdot \frac{d-1}{n-d-1} = \sigma^2 \cdot \frac{n-2}{n-d-1}.$$

□

4.3. **Local Minimum Variance Portfolios.** From the previous discussion we know that we can search for any LMVP in the same manner as for the GMVP by transforming the sample $\mathbf{X}$ into the sample $\mathbf{X}^*$. That means we take a linear transformation $T'X^{\mathrm{s}}$ of $X^{\mathrm{s}} = (\Delta R_2, \ldots, \Delta R_d)$ or, more precisely, $\mathcal{T}R$ where the operator $\mathcal{T}$ defined by (4.1.2) is determined by the specific linear restrictions of the LMVP.

According to the definitions of b_0^{s} and T given by Eq. 3.1.10 and Eq. 3.1.11 the RLS estimator $\hat{w}^* = (\hat{w}_1^*, \ldots, \hat{w}_d^*)$ of a LMVP can be represented by

$$\left(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*\right) = \hat{\alpha}_{\mathrm{OLS}}^{\mathrm{s}} ,$$

$$\left(\hat{w}_2^*, \ldots, \hat{w}_{q+1}^*\right) = H_2^{-1}\left(h - H_3\hat{\alpha}_{\mathrm{OLS}}^{\mathrm{s}}\right),$$

and $\hat{w}_1^* = 1 - \hat{w}_2^* - \ldots - \hat{w}_d^*$. For notational convenience consider once again Eq. 3.2.5, which implies that

$$\left(\hat{w}_2^*, \ldots, \hat{w}_d^*\right) = b_0^{\mathrm{s}} + T\left(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*\right)$$

and thus

$$\hat{w}^* = \begin{bmatrix}1\\0\end{bmatrix} - \mathcal{I}'\left(\hat{w}_2^*, \ldots, \hat{w}_d^*\right) = \begin{bmatrix}1\\0\end{bmatrix} - \mathcal{I}'b_0^{\mathrm{s}} - \mathcal{T}'\left(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*\right). \tag{4.3.1}$$

Since the entire weight estimator is a linear function of the last $d-q-1$ weight estimates, the distribution of $\hat{w}^*$ is concentrated on a $(d-q-1)$-dimensional linear subspace of $\mathbb{R}^d$. For that reason the following theorem is restricted to the distribution of $(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*)$.

Theorem 4.3.1. *Let $w^* = (w_1^*, \ldots, w_d^*)$ be a LMVP of d assets obeying $q < d-1$ linear restrictions for the portfolio weights (without the budget constraint) and $\hat{w}^* = (\hat{w}_1^*, \ldots, \hat{w}_d^*)$ be the corresponding RLS estimator. For a sample of asset returns with size $n \geq d-q$ it holds that*

$$\left(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*\right) \sim t_{d-q-1}\left(\left(w_{q+2}^*, \ldots, w_d^*\right), \frac{\sigma^{*2}}{n-d+q+1}\cdot\left(\mathcal{T}\Sigma\mathcal{T}'\right)^{-1}, \nu\right),$$

where $\nu = n-d+q+1$, $\sigma^{*2} = w^{*\prime}\Sigma w^*$ is the variance of the LMVP return and Σ is the covariance matrix of R.

Proof. Note that $(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*)$ corresponds to the OLS estimator for α^{s} given by the linear regression equation 3.1.6. Since this is an unrestricted least squares estimator we can argue as in the proof of Theorem 4.2.1 but we have to consider $d-q$ dimensions, the covariance matrix of the transformed data $\mathcal{T}R$, i.e. $\mathcal{T}\Sigma\mathcal{T}'$, and the variance σ^{*2} of the LMVP. □

Similarly, the remaining assertions follow from the theorems and corollaries already derived for the GMVP simply by substituting d by $d-q$, η (or $\hat{\eta}$) by η^* (or $\hat{\eta}^*$), σ^2 (or $\hat{\sigma}^2$) by σ^{*2} (or $\hat{\sigma}^{*2}$), and $\mathcal{I}$ by $\mathcal{T}$. For example, following Corollary 4.2.2 we may conclude that

$$\widehat{\mathrm{Var}}\left(\left(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*\right)\right) = \frac{\hat{\sigma}^{*2}}{n}\cdot\left(\mathcal{T}\hat{\Sigma}\mathcal{T}'\right)^{-1}$$

is an unbiased estimator for the covariance matrix of $(\hat{w}_{q+2}^*, \ldots, \hat{w}_d^*)$, where

$$\hat{\sigma}^{*2} = \frac{\hat{\mathbf{u}}^{*\prime}\hat{\mathbf{u}}^*}{n-d+q}$$

is an unbiased estimator for σ^{*2}, i.e. the variance of the LMVP return, and $\widehat{\Sigma}$ is the sample covariance matrix of R. Here we have only to assume that $n \geq d - q + 2$. Due to Eq. 4.3.1 the covariance matrix of $\hat{w}^*$ is given by

$$\mathbb{V}\mathrm{ar}\left(\hat{w}^*\right) = \mathcal{T}'\mathbb{V}\mathrm{ar}\left(\left(\hat{w}^*_{q+2}, \ldots, \hat{w}^*_d\right)\right)\mathcal{T} = \frac{\sigma^{*2}}{n - d + q - 1} \cdot \mathcal{T}'\left(\mathcal{T}\Sigma\mathcal{T}'\right)^{-1}\mathcal{T}$$

and so

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{w}^*\right) = \frac{\hat{\sigma}^{*2}}{n} \cdot \mathcal{T}'\left(\mathcal{T}\widehat{\Sigma}\mathcal{T}'\right)^{-1}\mathcal{T}$$

is an unbiased estimator for $\mathbb{V}\mathrm{ar}(\hat{w}^*)$. From Eq. 4.1.2 and Eq. 4.3.1 we see that

$$\hat{w}^*_1 = 1 - 1'b^{\mathrm{s}}_0 - 1'T\left(\hat{w}^*_{q+2}, \ldots, \hat{w}^*_d\right)$$

and thus – in the line of Theorem 4.2.3 – it holds that

$$\hat{w}^*_1 \sim t\left(w^*_1, \frac{\sigma^{*2}}{n - d + q + 1} \cdot 1'T\left(\mathcal{T}\Sigma\mathcal{T}'\right)^{-1}T'1, n - d + q + 1\right)$$

and – similar to Corollary 4.2.4 – an unbiased estimator for $\mathbb{V}\mathrm{ar}(\hat{w}^*_1)$ is given by

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{w}^*_1\right) = \frac{\hat{\sigma}^{*2}}{n} \cdot 1'T\left(\mathcal{T}\widehat{\Sigma}\mathcal{T}'\right)^{-1}T'1\,.$$

Moreover, due to Theorem 4.2.5 we know that

$$\mathbb{E}\left(\hat{\eta}^*\right) = \eta^*$$

and

$$\mathbb{V}\mathrm{ar}\left(\hat{\eta}^*\right) = \frac{\sigma^{*2}}{n - d + q - 1} \cdot \left(\mu'\mathcal{T}'\left(\mathcal{T}\Sigma\mathcal{T}'\right)^{-1}\mathcal{T}\mu + 1\right) - 2 \cdot \frac{\sigma^{*2}/n}{n - d + q - 1}\,,$$

where $\hat{\eta}^*$ denotes the RLS estimator for the expected LMVP return η. According to Corollary 4.2.6,

$$\widehat{\mathbb{V}\mathrm{ar}}\left(\hat{\eta}^*\right) := \frac{\hat{\sigma}^{*2}}{n} \cdot \left(\bar{\mathbf{r}}'\mathcal{T}'\left(\mathcal{T}\widehat{\Sigma}\mathcal{T}'\right)^{-1}\mathcal{T}\,\bar{\mathbf{r}} + 1\right)$$

is an unbiased estimator for $\mathbb{V}\mathrm{ar}(\hat{\eta}^*)$. Finally, for the out-of-sample return of a LMVP we obtain

$$\mathbb{E}\left(R'\hat{w}^*\right) = \eta^*$$

and following the proof of Theorem 4.2.7 we can find that

$$\mathbb{V}\mathrm{ar}\left(R'\hat{w}^*\right) = \mu'\mathbb{V}\mathrm{ar}\left(\hat{w}^*\right)\mu + \sigma^{*2} \cdot \frac{n - 2}{n - d + q - 1}\,,$$

where

$$\mathbb{V}\mathrm{ar}\left(\hat{w}^*\right) = \frac{\sigma^{*2}}{n - d + q - 1} \cdot \mathcal{T}'\left(\mathcal{T}\Sigma\mathcal{T}'\right)^{-1}\mathcal{T}\,.$$

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