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IMPLICATIONS**

By Beatrice Pataracchia

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Ambiguity and Volatility: Asset Pricing Implications

Beatrice Pataracchia*

Tilburg University

Abstract

Using a simple dynamic consumption-based asset pricing model, this paper explores the implications of a representative investor with smooth ambiguity averse preferences [Klibanoff, Marinacci and Mukerji, *Econometrica* (2005)] and provides a comparative analysis of risk aversion and ambiguity aversion. The perception of ambiguity is described by a hidden Markovian consumption growth process. The hidden states differ both for the mean and the volatility. We show that the ambiguity-averse investor downweights high-mean states in favor of low-mean ones. However, such distortion appears much stronger in low-volatility regimes: high volatility attenuates the distortion due to ambiguity concerns. It follows that (i) ambiguity aversion always implies higher equity premia but sustained levels of ambiguity aversion do not help explaining the high volatility of the equity premium observed in the data (volatility puzzle); (ii) our calibrated model can match the moments of the equity premium and risk free rate and can generate asset-price stylized facts like a procyclical price-dividend ratio and countercyclical conditional equity premia; however, (iii) high levels of ambiguity aversion, necessary to explain high equity returns, produce counterfactual price-dividend ratio time series across volatility states.

Keywords: Ambiguity aversion, volatility, asset pricing puzzles, robustness.

JEL: D81, E44, G12

1 Introduction

In this paper we examine a standard dynamics asset pricing model in which the endowment law of motion is modelled as a hidden regime switching process and the representative agent learns the probabilities of each state via Bayes rule. We depart from rational expectations assuming that the investor feels *ambiguous* about the probabilities of the realization of each state, which differ both for the mean and the volatility values. Her preferences are represented

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by a generalized recursive smooth ambiguity model proposed in Klibanoff et al. (2005, 2008), which nests the standard Epstein and Zin preferences framework and the maxmin expected utility framework of Gilboa and Schmeidler (1989) as two special cases, respectively with no and maximal ambiguity aversion. Smooth ambiguity aversion is manifested through a pessimistic distortion of the pricing kernel in the sense that the agent attaches more weight to the states with lower continuation values. In this respect, this approach is more flexible than the maxmin approach according to which the decision maker conditions her decisions exclusively on the worst case model. The pessimistic distortion implied by the ambiguity-averse preferences suggests that such formalization can potentially contribute to the resolution of the equity premium puzzle (Mehra and Prescott, 1985). Knight (1921) has been the first to distinguish among *risk* and *uncertainty*. In this sense, the theory of subjective probabilities nullifies this distinction by reducing all uncertainty to risk through the use of beliefs expressible as probabilities. In our case, ambiguity aversion on the plausible probability distributions posterior implies that the posterior of the hidden state and the conditional distribution of the consumption process cannot be reduced to a compound predictive distribution, unlike the standard Bayesian framework. The experimental motivation of such preferences modeling is provided by the well known Ellsberg paradox (Ellsberg, 1961), showing that decision makers prefer the risky urn to the ambiguous urn, in a way that violates the expected utility framework.

The goal of our work is understanding and assessing the extents to which ambiguity averse preferences can help explaining the main asset pricing puzzles: the risk free rate puzzle (Weil, 1989), the volatility puzzle (Shiller, 1981) together with the equity premium puzzle. Furthermore, we also assess the capacity of the ambiguity-averse preferences to imply procyclical price-dividend ratios and countercyclical conditional expected equity premia documented in the literature (e.g. Campbell and Shiller, 1988). Finally, we compare the time series of the price-dividend ratios and the equity premia implied by the model with the observed one. Our attention is focused on the ability of the model to capture the striking increase in prices observed during the great moderation period. Figure 1 and figure 2 depict, respectively, the post-war time series of the consumption growth rate and the CRSP Value-Weighted price-dividend ratios. They suggest a quite evident negative correlation between consumption volatility and the price-dividend ratios, documented also in Lettau et al. (2008) and Bansal and Yaron (2004). We do not ask ourselves the causes of the decline of volatility, rather we want to investigate its effects on prices and the role of ambiguity aversion in this respect. In order to capture the switching in volatility, we model the consumption growth rate as a hidden Markov switching process with two hidden Markov chains, each of them determining, respectively, the switching between mean regimes and volatility regimes.

We show that the ambiguity-averse investor downweights high-mean states in favour of low-mean ones. However, such distortion appears much stronger in low-volatility regimes. Between two high-mean states, the ambiguity-averse investor removes more mass probability from the best state (high mean and

low volatility) rather than from the high-mean and high-volatility case. Similarly, between two low mean states, the distortion downweights more the highly volatile states' probabilities rather than the stable states' probabilities. The intuition is the following: when the investor perceives a high-mean observation, the worst case is to think that the quality of the observation is quite bad. Conversely, after a low-mean observation, the ambiguity averse investor will add mass probability to the low-volatility case since a good quality new is the worst case ¹. In other words, ambiguity aversion has a first order effect among mean states. The distortion among volatility states is a second order effect. It follows that (i) ambiguity aversion always implies higher equity premia but (ii) the slighter distortion which results in high volatility times implies a moderation in the price volatility and, overall, an uncertain effect of ambiguity aversion on volatility of returns and prices occurs; (iii) our calibrated model can match the moments of the equity premium and risk free rate and can generate asset-price stylized facts like a procyclical price-dividend ratio and countercyclical conditional equity premia; however, (iv) high levels of ambiguity aversion, necessary to explain high equity returns, produce counterfactual price-dividend ratio time series across volatility states.

Ambiguity concerns have recently received a huge popularity in the macroeconomic and finance literature. Contributions have been mainly focused on the maxmin expected utility (or multiple priors) model of Gilboa and Schmeidler (1989) and the robustness theory developed by Hansen and Sargent (2001, 2008). In the maxmin framework, there exist several papers which consider the effect of ambiguity concerns in asset pricing. Epstein and Schneider (2008) analyze the effects of ambiguous quality of intangible information on asset prices, Leippold et al. (2008) analyze a learning model under ambiguity in a continuous time framework. This literature is also related to contributions which impose pessimism and beliefs' distortion in a very specific ways without a formalized decision theoretic foundation (Cecchetti et al. 2000, Abel et al., 2002, and Brandt et al., 2004 among others). Our work is closely related to Ju and Miao (2007) who also considers a smooth ambiguity averse investor. However, their work is based on Cecchetti et al. (2000) data set, which ends on 1993, when the decline of consumption volatility was not appreciable yet: the authors do not impose any switching among volatility states and consequently, do not investigate the ability of such preferences to reproduce the negative observed correlation between consumption volatility and prices.

The paper is structured as follows. Section 2 describes the theoretical framework of the smooth ambiguity averse preferences. Section 3 presents the model and the theoretical asset pricing implications. Section 4 provides a comparative analysis of the effects of degree of ambiguity aversion, while Section 5 describes the quantitative implications of the model, both in terms of the analysis of the moments and the time series implications. Section 6 concludes.

¹The intuition is similar to Epstein and Schneider (2008) who consider a different framework where ambiguity is imposed on the quality of the intangible information.

2 Smooth Ambiguity Preferences

Recursive preferences have become a standard tool for studying economic behaviour in dynamic stochastic environments and for parameterizing risk aversion and intertemporal substitution. We assume that time is discrete, with dates $t = 0, 1, 2, \dots$. At each time $t > 0$, let $\{I_t : t \geq 0\}$ denote the sequence of conditioning information sets available to an investor at date t . Adapted to this sequence are consumption processes $\{C_t : t \geq 0\}$ and a corresponding sequence of continuation values $\{V_t : t \geq 0\}$ associated with this consumption process. The date t components C_t and V_t are restricted to be in the date t conditioning information set. The continuation values are determined recursively and used to rank alternative consumption processes.

As a starting point, we use Epstein and Zin (1989) recursive preferences, which separate between risk aversion and the elasticity of intertemporal substitution such that consumers are not indifferent to the timing of the resolution of uncertainty. We use the following constant elasticity of substitution - CES recursion:

$$V_t = H(C_t, [\mathcal{R}_t(V_{t+1})]) = \left[(1 - \beta)(C_t)^{1-\rho} + \beta \mathcal{R}_t(V_{t+1})^{1-\rho} \right]^{\frac{1}{1-\rho}}, \quad (1)$$

where H is a time aggregator and \mathcal{R}_t is a certainty equivalent function. The time aggregator is all that matters in deterministic settings. The rate of time preferences, $\beta \in (0, 1)$ is assumed to be built in and constant and $\frac{1}{\rho}$ determines the intertemporal elasticity of substitution (IES) for deterministic consumption paths. Following Epstein and Zin, many recent applications, particularly in dynamic asset pricing models, use the homothetic version of the utility function which combines the constant elasticity time aggregator in (1) with a linear homogenous (constant relative risk aversion) expected utility certainty equivalent:

$$\mathcal{R}_t(V_{t+1}) = \left[E_t(V_{t+1})^{1-\gamma} \right]^{\frac{1}{1-\gamma}}, \quad (2)$$

where γ is the constant Arrow Pratt coefficient for relative risk aversion. When V_t is convex (concave) in its second argument, that is when $\gamma > \rho$ ($\gamma < \rho$), the consumer exhibits preference for early (late) resolution of uncertainty.

Considering smooth ambiguity preferences implies considering an additional operator, which will define the certainty equivalent for future utilities taking into account ambiguity concerns. As in Hansen and Sargent (2006, 2007), we impose ambiguity over the transition probabilities of the hidden states. The authors propose a log exponential specification, which, in the simple case of two possible hidden states, takes the form:

$$\begin{aligned}
\mathcal{T}^{hs}(\mathcal{R}) &= -\psi \log \left[\nu \exp \left(-\frac{\mathcal{R}|\xi=1}{\psi} \right) + (1-\nu) \exp \left(-\frac{\mathcal{R}|\xi=2}{\psi} \right) \right] \quad (3) \\
&= \min_{0 \leq \hat{\nu} \leq 1} \{ [\mathcal{R}|\xi=1 + \psi(\log \hat{\nu} - \log \nu)] \hat{\nu} + \\
&\quad [\mathcal{R}|\xi=2 + \psi(\log(1-\hat{\nu}) - \log(1-\nu))] (1-\hat{\nu}) \},
\end{aligned}$$

where ξ is an indicator variable for the hidden state, ν represent the full Bayesian beliefs (or ambiguity neutral probabilities), the parameter ψ is inversely related to the degree of probabilities misspecification aversion, so that when $\psi \rightarrow \infty$, $\mathcal{T}^{hs}(V) = V$. This operator produces twisted model probabilities, $\hat{\nu}$: it shifts probabilities towards the model that has the worse value at time t . Following Klibanoff (2005), we use an alternative specification. We start the description of smooth ambiguity model by presenting the static setting of a utility function in which a decision maker is ambiguous over consumption:

$$\left(\int_{\Pi} \eta \left(u^{-1} \left(\int_S u(C) d\theta \right) \right) d\pi \right) \equiv E_{\pi} \phi E_{\theta} u(C), \forall C : S \rightarrow \mathbb{R}_+,$$

where u is a von Neumann-Morgestern (vN-M) utility function, η is a function which captures ambiguity attitudes, π is a subjective prior over the set Π of probability measures on S that the decision maker think possible. There may be subjective uncertainty about what the right probability on S is: π is the decision maker's subjective prior over Π , the set of possible probabilities θ over S , and therefore measures the subjective relevance of a particular θ as the right probability. Letting $\phi = \eta \circ u^{-1}$, we see that while u characterizes risk aversion, ambiguity aversion is captured by ϕ : a concave ϕ characterizes ambiguity aversion. Notice that, unless ϕ is a linear function (ambiguity neutrality), there is in general no reduction between θ and π . The perception of ambiguity is characterized by the subjective set of measures Π while ambiguity aversion is completely described by the properties of the function ϕ .²

In our model, we consider the dynamic version of the smooth ambiguity preferences. and we consider a special case of constant relative ambiguity aversion utility:

$$\eta(x) = \frac{x^{1-\alpha}}{1-\alpha}, \quad \alpha > 0, \quad \neq 1, \quad (4)$$

where α is the ambiguity aversion parameter. Therefore, we consider the following recursion:

$$V_t = H(C, \mathcal{T}(\mathcal{R}(V_{t+1}))),$$

where the ambiguity operator, in the simple case of two states, takes the form:

²See Klibanoff et al. (2005) for a detailed discussion of the separation between ambiguity perception (beliefs) and ambiguity aversion (tastes).

$$\mathcal{T}(\mathcal{R}) = \left[(1 - \alpha) \left(\pi \frac{(\mathcal{R} | \xi = 1)^{1-\alpha}}{1 - \alpha} + (1 - \pi) \frac{(\mathcal{R} | \xi = 2)^{1-\alpha}}{1 - \alpha} \right) \right]^{\frac{1}{1-\alpha}}, \quad (5)$$

which can be compared to (3). Considering N generic hidden states and applying the ambiguity operator \mathcal{T} to (1), the recursive preferences become:

$$U(c_t, E_t V_{t+1}) = \left\{ (1 - \beta) C_t^{1-\rho} + \beta \left[\left\{ \sum_{j=1}^N \pi_{j,t} \left[E_{j,t} (V_{t+1})^{1-\gamma} \right]^{\frac{(1-\alpha)}{1-\gamma}} \right\}^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}} \right\}. \quad (6)$$

It is easy to see that if $\alpha = \gamma$, (6) reduces to (1) and the investor is ambiguity neutral. When $\alpha > \gamma$ the investor displays ambiguity aversion and no reduction is possible between the state beliefs and the conditional expectations.

3 The Model

3.1 Endowment

We consider a simple Lucas (1978) endowment economy. In order to capture the effect of the switching of volatility regimes, as in Lettau et al. (2008), we model the consumption growth rate process as a hidden Markov switching model of the form:

$$y_t = \log \frac{C_{t+1}}{C_t} = c_{t+1} - c_t = \Delta c_{t+1} = \mu(\xi_{t+1}) + \sigma(v_{t+1}) \varepsilon_{t+1}, \quad (7)$$

where the variables ξ_{t+1} and v_{t+1} are two independent two-state Markov chains with possible values 1 and 2. We suppose that $\mu(1) = \mu_h > \mu(2) = \mu_l$ and $\sigma(1) = \sigma_h > \sigma(2) = \sigma_l$. The driving process ε_{t+1} is a simple i.i.d. standard normal. The transition probability matrices of the Markov Chains, P^μ and P^σ are defined as follows:

$$P^\mu = \begin{bmatrix} p_{hh}^\mu & (1 - p_{ll}^\mu) \\ (1 - p_{hh}^\mu) & p_{ll}^\mu \end{bmatrix}, \quad (8)$$

$$P^\sigma = \begin{bmatrix} p_{hh}^\sigma & (1 - p_{ll}^\sigma) \\ (1 - p_{hh}^\sigma) & p_{ll}^\sigma \end{bmatrix}, \quad (9)$$

where p_{ii}^μ represents the probability of remaining in the mean-state i and p_{ii}^σ the probability of remaining on the volatility-state i . Therefore, we consider a 4 states economy, whose transition matrix can be computed by taking the Kronecker product between the transition matrices, (8) and (9):

$$P = P^\sigma \otimes P^\mu,$$

where \otimes denotes the Kronecker product operator. Consequently, we consider a Markov chain, Z_t , which switches among four states. State 1 is represented by high growth and high volatility $[\mu = \mu_h; \sigma = \sigma_h]$; state 2 is represented by low growth and high volatility $[\mu = \mu_l; \sigma = \sigma_h]$; state 3 denotes the state with high growth and low volatility $[\mu = \mu_h; \sigma = \sigma_l]$ and state 4 is the low-growth and low-volatility case $[\mu = \mu_l; \sigma = \sigma_l]$.

3.2 Learning

Let define the conditional probabilities of a state j with $\pi_{j,t} = \Pr(Z_{t+1} = j|I_t)$ and the vector of the conditional probabilities $\pi_t = [\pi_t(1), \dots, \pi_t(4)]'$. Agents update their posterior beliefs after each observation via Bayes' rule:

$$\pi_{t+1} = P \frac{(\pi_t \odot f_t)}{\mathbf{1}'(\pi_t \odot f_t)} \equiv B(\Delta c_{t+1}, \pi_t), \quad (10)$$

where $f_t = [f_t(1), \dots, f_t(4)]$ and $f_t(j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left[-\frac{(\Delta c_{t+1} - \mu_j)^2}{2\sigma_j^2}\right]$ is the density function of the normal distribution with mean μ_j and σ_j , and the operator \odot represents element by element multiplication.

3.3 Asset pricing Implications

As it is standard in consumption based asset pricing models, the representative investor chooses to invest a fraction x_k of his disposable wealth $W_t - C_t$ on a risky asset k . The agent's budget constraint is given by

$$W_{t+1} = (W_t - C_t)(R_{w,t+1}), \quad (11)$$

where $R_{w,t+1}$ is the gross return on the wealth portfolio between period t and $t + 1$, which is also the return on the consumption claim. If we suppose that there are K traded assets, the return on the wealth portfolio, $R_{w,t+1}$, is equal to $\sum_{k=1}^K x_{kt} R_{k,t+1}$ where $\sum_{k=1}^K x_{kt} = 1$. We solve the investor's problem by dynamic programming methods. The state variables of our problem are the level of wealth and beliefs (W_t, π_t) . We guess the following functional form of the value function:

$$V_t(W_t, \pi_t) = W_t G(\pi_t), \quad (12)$$

and we write the Bellman equation that we substitute in (6):

$$V_t(W_t, \pi_t) = \max_{C_t, x} \left\{ (1 - \beta) C_t^{1-\rho} + \beta \left[\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} (W_{t+1} G(\pi_{t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}.$$

The optimization is subject to the following constraints:

$$\begin{aligned} W_{t+1} &= (W_t - C_t) R_{w,t+1}, \\ \pi_{t+1} &= B(\Delta c_{t+1}, \pi_t), \end{aligned} \quad (13a)$$

$$R_{w,t+1} = \sum_{k=2}^K x_{kt} R_{k,t+1} + \left(1 - \sum_{k=2}^K x_{kt}\right) R_{f,t+1}, \quad (13b)$$

where $R_{f,t+1}$ is the gross return on riskless asset between period t and $t + 1$. Substituting the first line of the budget constraint we get

$$Vt = \max_{C_t, x} \left\{ (1 - \beta) C_t^{1-\rho} + \beta (W_t - C_t)^{1-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} (R_{w,t+1} G(\pi_{t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}. \quad (14)$$

Proposition 1 *The first order condition with the respect to consumption plan leads to the following equilibrium condition:*

$$1 = \beta \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}. \quad (15)$$

Proof. See Appendix A.1 ■

Equation (15) can be manipulated in order to emphasize the effects of ambiguous beliefs in the equilibrium determination. Suppose for the moment that our investor is ambiguity neutral. Then the equilibrium condition corresponding to (15) would become

$$\begin{aligned} 1 &= \beta \left[\sum_{j=1}^4 \pi_{j,t} E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right], \quad (16) \\ &= \beta \left[E_t R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right]. \end{aligned}$$

Let us go back to eq. (15) and let us multiply and divide the expression inside the square bracket by the term $E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)$ so to get

$$1 = \beta \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma} - 1} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\alpha}}.$$

We can further multiply and divide the exponent of the square bracket by $\frac{1-\gamma}{1-\gamma}$ and get:

$$\begin{aligned}
1 &= \beta^{\frac{1-\alpha}{1-\gamma}} \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}-1} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\gamma}} \quad (17) \\
&= \beta \left[\sum_{j=1}^4 \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}-1} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\gamma}} \\
&= \beta \left[\sum_{j=1}^4 \hat{\pi}_{j,t} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\rho}{1-\gamma}},
\end{aligned}$$

where $\hat{\pi}_{j,t} = \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}-1}$ is the distorted belief used by the investor who fears ambiguity over the transition probabilities. Eq. (17), compared to eq. (16) shows how the ambiguous investor behaves as an ambiguity neutral investor with distorted beliefs³.

The first order condition with the respect to the trading startegy, x_k , permits us to characterize the pricing kernel for our model.

Proposition 2 *The pricing kernel conditioned on state j is given by*

$$M_{j,t+1} = \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \left[E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right]^{\frac{1-\alpha}{1-\gamma}-1}. \quad (18)$$

Proof. See Appendix A.2 ■

The pricing kernel is also a hidden stochastic variable. Eq. (18) shows that when $\alpha = \gamma$, we get the usual pricing kernel for time-non separable preferences without ambiguity concerns, and when we further impose time separability, $\gamma = \rho$, we get the standard pricing kernel $M = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}$ for CES utility functions. This equation reveals that optimal portfolio rules are affected by ambiguity. The gross risk free rate (or bond return) is given by

$$\frac{1}{R_{f,t+1}} = \sum_{j=1}^4 \pi_{j,t} (E_{j,t} M_{j,t+1}).$$

³Note, however, that the expected returns, being defined via the price-consumption and price-dividend ratios, depend on the degree of ambiguity aversion, in the general case in which utility functions are not logarithmic.

Let P_t^D denote the ex-dividend price of a claim to the dividend stream measured at the end of time t , and P_t^C denote the ex-dividend price of a share of a claim to the consumption stream. We conjecture that the price-dividend ratio is given by

$$P_t^D = \varphi(\pi_t) D_t,$$

and the price-consumption ratio

$$P_t^C = \varphi^C(\pi_t) C_t$$

where $\varphi(\cdot)$ and $\varphi^C(\pi_t)$ have to be determined. In equilibrium, we have that on average, the return on equity, R_e , is defined as follows

$$R_{e,t+1} \equiv \frac{P_{t+1}^D + D_{t+1}}{P_t^D} = \frac{D_{t+1}}{D_t} \frac{1 + \varphi(\pi_{1,t+1})}{\varphi(\pi_{1,t})}. \quad (19)$$

As in Campbell(1986) and Abel(1999) the dividend on equity, D_t , is modelled as consumption raised to a power λ :⁴

$$D_t = C_t^\lambda,$$

this specification implies that the volatility of dividends is proportional to the volatility of consumption and it is a convenient representation because, while keeping the number state variables limited, allows for higher volatility of the dividend dynamics compared to the consumption process. Substituting in (15) gives

$$1 = \beta \left[\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(\frac{1 + \varphi^C(\pi_{t+1})}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}. \quad (20)$$

The equilibrium condition (20) is a functional equation from which we can derive the price-consumption ratio $\varphi^C(\cdot)$. Further, substituting (33) into (32), we derive the equilibrium condition for the price-dividend ratio function.

Proposition 3 *The price-dividend ratio of a claim to the dividend stream satisfies*

$$\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} M_{j,t+1} \left((1 + \varphi(\pi_{t+1})) \frac{D_{t+1}}{D_t} \right) \right) = \varphi(\pi_t). \quad (21)$$

We solve these functional equations numerically on a grid of values for the state variables π_t .⁵

⁴Abel (1999) shows that this specification is a good approximation to represent leverage equity.

⁵In the Appendix we describe the functional equations in more details. The Matlab codes and numerical algorithms are available under request.

4 Comparative Analysis

In this section we present a comparative analysis on the behaviour of prices and the equity premium. These types of studies are not new. Veronesi (1999), Cechetti et al. (2000) or Ju and Miao (2007) study equilibrium prices when the endowment is characterized by a hidden growth rate of the consumption process. Prices are analyzed as a function of the high mean state. These studies typically find that the equilibrium prices of the asset are an increasing and convex function of the high mean state belief. The intuition is quite straightforward and has been first proposed in Veronesi (1999). Suppose we believe we are in the low growth state. When a positive new arrives, the probability of being in a high growth state increases. This causes the investor to expect higher return in the future, to buy more assets increasing, therefore, the price. At the same time, however, the closer is the probability of being in a high growth state to 0.5, the higher the uncertainty of the investor about which state is realizing. This uncertainty may cause the investor to wait in order to get more information. The price-dividend ratio timidly increases, suggesting that the former effect dominates the latter. Let us consider the other way around. We believe we are in the high growth state and a bad new occurs. Again the effect is twofold. On one side, investors are incentivated to sell the asset, on the other, again, the uncertainty increases, because our belief is now closer to 0.5. Therefore both effects lower the equilibrium prices and the decrease is more accentuated. This fact causes the price-dividend ratio to be a convex function. Similar findings have been documented by Ju and Miao (2007) who include also ambiguity with the respect the transition probabilities between high and low growth states. They show that ambiguity does not significantly modify the characteristics of the price dividend ratio, except that it accentuates the curvature, helping to explain the high volatility of asset prices. Also Lettau et al. (2008) document an increasing and convex price dividend ratio.

Figure (3) represents an example of such comparative study. It proposes the log price dividend ratio as function of the beliefs of being in a low volatility state (upper panels) and as a function of the high-mean state belief (lower panels). Horizontally, we differentiate the conditional cases considered (respectively, high and low mean and low and high volatility). The solid line corresponds to the ambiguity neutrality case ($\alpha = \gamma = 25$), the dashed and dotted line represents positive levels of ambiguity aversion, respectively $\alpha - \gamma = 5$ and $\alpha - \gamma = 10$.

We start our discussion focusing on the lower panels, so to be able to make a comparison with existing studies. In all cases considered the price dividend function is increasing and convex. It follows that the model predicts a pro-cyclical price-dividend function, consistently with the empirical observations. Second, we observe that higher levels of ambiguity aversion always imply lower prices since our investor requires higher premium to hold risky and ambiguous asset. Further, comparing the two panels horizontally, the effect of ambiguity aversion appears much stronger in low-volatility regimes: high volatility attenuates the distortion due to ambiguity concerns. In high-volatility states, it becomes more difficult to distinguish high-growth and low-growth regimes from dividend obser-

vations. That is, in high-volatility regimes the investor perceives the increased risk, but his perception of ambiguity is reduced because the worst-case model is not too "far", given some defined norms to measure closeness between data generating processes. The panels above also reveals interesting features. As could be expected, a convex and increasing price dividend ratio is also obtained when plotting against the low-volatility state beliefs. However, the curvature is flatter for higher levels of ambiguity aversion, which, while implying always lower prices, has a stronger effect when the investor believes that the low-valuation state is very likely. This further implies that the volatility of prices is also reduced, as if the ambiguity-averse investor, being an insuring agent, reacts less to news. In this respect, ambiguity aversion is providing a rationale for price underreaction to news about fundamentals. Indeed, as we move from the belief of the high volatility state to the low-volatility state, the ambiguity averse investor will continue requiring a compensation for holding the ambiguous asset, while risk aversion will play an attenuated role. This is because our investor is pessimistic and the effects of such distortion of beliefs appear much more accentuated in low-volatility states where the low-mean state is perceived worse than the low-valuation state with high volatility. The stronger effect of ambiguity aversion in low-volatility cases causes the price-dividend ratio to be much flatter suggesting that high levels of ambiguity aversion will not help explaining the high volatility of equity premium observed in the data (volatility puzzle). Our result is in contrast with Ju and Miao (2007) who claim that, in a world where the switching is only allowed among mean states, ambiguity aversion always increases the volatility of equity returns. The authors focus on the early post-war sample (until 1993) which does not allow to appreciate a sensible switching among volatility regimes.

Figure(4) also compares the dynamics of the expected equity premium for different level of ambiguity. Again, not surprisingly, the equity premium is always higher for higher levels of ambiguity aversion. When the uncertainty about the state is higher, also the ambiguity premium is higher. This is because ambiguity-averse preferences imply a distortion of beliefs towards the lower continuation state injecting pessimism in the economy.

Figure (5) shows that, for given level of ambiguity aversion, uncertainty about volatility state also requires higher expected equity premium. However, increasing ambiguity aversion implies lower levels of equity premium, as if the ambiguity averse investor, not being able to differentiate among worst and best model, perceives less ambiguity. Even if the level of volatility is ambiguous, the ambiguity averse investor is not requiring a compensation for not knowing in which volatile state is the economy. This aspect of the model implies that switching among volatility states does not necessarily imply important effects on prices when the investor is ambiguity averse.

5 Quantitative Analysis

In this section we present the quantitative analysis. Since the state beliefs are the only state variables of the model, we proceed estimating the filtered state probabilities so to obtain the implied time series of the economic variables of interest, prices and equity premia. Our goal is not only engaging in matching the moments, as it is usually done in asset pricing studies, but we want also to compare the implied time series with real data and, in particular, we want to assess the ability of the model to replicate long and medium frequency fluctuations in prices.

Studies who typically focus on the moments are aimed at explaining financial markets' puzzles like the high equity returns together with very low risk free rates. The observed values are typically very hard to replicate via simple consumption based asset pricing models. Further, from Shiller (1981) the literature has also documented very high volatility of the equity returns (see also Campbell, 1999 for a survey). Table (1) reproduces sample moments from annual US data. Other important stylized facts emphasized by the literature are the countercyclicality of conditional volatility of stock return, the pro-cyclicality of the price-dividend ratio and the counter-cyclicality of the conditional expected equity premia.

5.1 The Estimated Model

We estimate the model (7)-(8)-(9) by *Expectation-Maximization* (EM) algorithm. Estimated coefficients are reproduced in Table (2). The switching between the mean states is more frequent than the switching between volatility states. The persistence of the high growth state is about 7 years, the persistence of a low growth state is almost 5 years, while the persistence of, respectively, the high volatility state and the low volatility state, is 41 years and 27 years. The switching between volatility regimes is a low frequency event and it has been associated to the decline of *macroeconomic risk*, also termed *the great moderation*.

Figure (6) plots the filtered state probabilities of a low volatility state along with the probabilities of a high mean state. Consumption exhibits a considerable reduction in volatility starting from the 90s and the filtered state probabilities approach one. The figure confirms that the perceived switching among mean states occurs at business cycle frequencies while the switching among volatilities is a low frequency event.

5.2 Matching the Moments

Table (3) refers to the ambiguity neutrality case when we set $\gamma = 30$, such that the model corresponds to the one calibrated in Lettau et al. (2008) which do not consider ambiguity aversion motives. In our model this case corresponds in setting $\gamma = 30$ and $\alpha = 30$. Figure (8) compares the observed price dividend ratios with the one implied by the model and suggests that the calibration considered implies quite low equilibrium prices, therefore, a sustained equity

premium. The risk aversion parameter is, however, much higher than values typically considered acceptable by macroeconomists (γ less than 10 and close to 2). Next, we consider the role of ambiguity motives ($\alpha > \gamma$). Tables (4)-(7) collect the results from simulations for different values of the discount factor β . The tables show different interesting aspects of ambiguity motives. First, ambiguity aversion always increases the equity return. This is due to pessimistic distortion that ambiguity averse motives place on the investor's beliefs, so that he requires more compensation to hold a risky asset in an ambiguous economy. Further increasing values of ambiguity aversion lower the risk free rate, helping explaining the equity premium puzzle. On the same time our model is able to generate very low volatility of the riskfree rate together with sustained volatility of the equity returns.

We can also observe that when the risk aversion parameter is quite sustained (e.g. $\gamma = 30$ or $\gamma = 25$) the effect of an increment of the degree of ambiguity aversion are very moderate. Different is the case when risk aversion appears contained (e.g. $\gamma = 15$). In general we can observe that ambiguity aversion allows matching the mean risk free rate and risky rates with a relatively low value of the risk aversion parameter, like $\gamma = 10$, as it shown by the last rows of table (7): it is suggested that a value of ambiguity aversion between 20 and 40 could match the moments.

5.3 Time Series Implications

We consider a calibration which is quite satisfactory in matching the moments described above: $\beta = 0.9925$, $\gamma = 15$ and $\alpha = 35$. As in all calibrations, we set ρ equal to $1/1.5$ and $\lambda = 4.5$. We are interested in understanding the time series performance of the equilibrium prices and we plot the time series of the price dividend ratio from data with the time series implied by the model. Figure (9) shows that the comparison is not satisfactory. As it has been already observed in the comparative study, the switching from the high-volatility state to the low-volatility state implies a moderation in prices which has not been observed in the data. Extensive simulations show that low values of risk aversion associated with high values of ambiguity aversion imply similar qualitative dynamics. As we already noticed indeed, the effects of ambiguity aversion is much more effective in low volatility states rather than very volatile ones. Therefore, when a very high level of ambiguity aversion is imposed on the investor, the different intensity of the effect of distortion causes prices to decrease during the last part of the sample. In figure (10) we compare the observed time series of prices with the ones implied by our model with γ equal to 25 and α , respectively, equal to 25 and 35. The figure makes clear that, while ambiguity aversion is very effective in lowering the prices explaining, therefore the high equity premium, high levels of ambiguity aversion attenuate the effects of uncertainty on prices in unstable times. Therefore, increased riskiness of the economy attenuates the premium due to ambiguity aversion. This effect worsens the performance of the model for the last part of the sample, the so called *great moderation* during which prices have increased considerably. In other words, ambiguity aversion with

the respect to mean states reinforces risk aversion, but ambiguity aversion with the respect to volatility states does not. This tension was never emphasized previously in the literature and we believe it constitutes an important warning for the growing literature on ambiguity aversion and asset prices.

6 Conclusions

In this paper we consider a smooth ambiguity averse representative investor in a standard consumption based asset pricing model. We model consumption and dividend growth rate as a regime switching process. The switching is imposed both on the mean and on the volatility values. We estimate the Markov switching models and show that switching between volatility states is a low frequency event: the first part of the postwar sample is characterized by high volatility, while, starting from the 1990s we observe the so called *great moderation* period. The aim of our study is to understand how ambiguity concerns interact with beliefs about volatility and mean regimes. Our study shows that ambiguity aversion is certainly helpful in order to explain the observed high level of equity premia even with low values of risk aversion. Indeed, higher ambiguity aversion lowers risk free rates while increasing the equity returns. Further, this effect is more accentuated, the lower the level of risk aversion. Not surprisingly, indeed, ambiguity aversion motives have had a great success on the asset pricing literature in recent years.

However, we also show some additional implications of ambiguity concerns that, up to our knowledge, have not been emphasized previously. We uncover the results summarized in the exercises aimed at matching the moments and we analyze the time series implications of the smooth ambiguity averse preferences. Even if successful calibrations can match the observed moments of the equity premia and the risk free rate, the implied time series do not appear satisfactory for the levels of ambiguity aversion required to explain the equity premium puzzle. This is due to the second-order distortion that ambiguity aversion places among volatility states. Indeed, as it has already been widely documented, the ambiguity averse investor behaves as a pessimistic agent with distorted state probabilities among good and bad states: she removes mass probabilities from good states in favour of bad states. However, ambiguity aversion with respect to volatility states plays a second order effect: the pessimistic agent considers more likely that a good new is of low quality or that a bad new is of good quality, implying that during unstable times the distortion between mean states is less accentuated. Therefore, high levels of ambiguity aversion often reduce the volatility of the equity returns, due to the attenuated response of the investor to the new observation. This effect produces counterfactual price-dividend ratio time series. Further, the second order effect of ambiguity aversion causes prices to decline, rather than increase, during the last part of the sample, characterized by a decline of volatility.

This observations constitute an important warning for the growing literature on ambiguity aversion and asset prices and suggest the necessity to deal with

such preferences with caution. For instance, new frameworks can be studied which are able to emphasize the role of ambiguity aversion only in certain contexts where it appears to explain successfully stylized facts (for instance during recession times), allowing time dependence in the ambiguity aversion literature. Another interesting analysis may consider time variation in the perception of the ambiguity (rather than the variation in the ambiguity aversion) exploiting the peculiar feature of the smooth ambiguity averse preferences: the separation between ambiguity aversion (beliefs) and ambiguity (perception of ambiguity). We leave such extensions for future research.

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A Appendix

A.1 Euler 1

Taking the first order condition with the respect to consumption leads to the following:

$$(1 - \beta) C_t^{-\rho} = \beta (W_t - C_t)^{-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left[E_t (R_{w,t+1} G(\pi_{1,t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}. \quad (22)$$

We conjecture a consumption rule $C_t = a_t W_t$ and we substitute it in (14) and (22) to get, respectively,

$$G(\pi_t) = \left\{ (1 - \beta) a_t^{1-\rho} + \beta (1 - a_t)^{1-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} (R_{w,t+1} G(\pi_{t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} \quad (23)$$

and

$$(1 - \beta) a_t^{-\rho} = \beta (1 - a_t)^{-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left[E_t (R_{w,t+1} G(\pi_{1,t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}, \quad (24)$$

from which it follows that

$$\frac{G(\pi_t)^{1-\rho} - (1 - \beta) a_t^{1-\rho}}{1 - a_t} = (1 - \beta) a_t^{-\rho},$$

therefore,

$$G(\pi_t) = (1 - \beta)^{\frac{1}{1-\rho}} \left(\frac{C_t}{W_t} \right)^{-\frac{\rho}{1-\rho}}, \quad (25)$$

or, alternatively,

$$a_t = \left[\frac{G(\pi_t)^{1-\rho}}{1 - \beta} \right]^{-\frac{1}{\rho}}. \quad (26)$$

Substituting (26) in (23), we get the following equilibrium condition for $G(\cdot)$:

$$\begin{aligned} G(\pi_t)^{1-\rho} &= (1 - \beta) \left(\frac{G(\pi_t)^{1-\rho}}{1 - \beta} \right)^{-\frac{1-\rho}{\rho}} + \\ &+ \beta \left[1 - \left(\frac{G(\pi_t)^{1-\rho}}{1 - \beta} \right)^{-\frac{1}{\rho}} \right]^{1-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} [R_{w,t+1} G(\pi_{t+1})]^{1-\gamma} \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \end{aligned} \quad (27)$$

Using (25) into (24), we get

$$a_t^{-\rho} = \beta (1 - a_t)^{-\rho} \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} \left[R_{w,t+1}^{1-\gamma} \left(\frac{C_{t+1}}{W_{t+1}} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right] \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \quad (28)$$

From the budget constraint, we derive

$$\begin{aligned} W_{t+1} &= (W_t - C_t) R_{m,t+1} = W_t (1 - a_t) R_{m,t+1} = \\ &= C_t \frac{1 - a_t}{a_t} R_{m,t+1} \end{aligned} \quad (29)$$

which, substituted in (28) and after few simplifications, brings to condition (15)

A.2 The pricing kernel

Substituting the expression for $R_w = \sum_{k=1}^N x_k R_k$ in (14) and taking the foc with respect to the trading strategy, we have

$$\begin{aligned} 0 &= \left(\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} (W_{t+1} G(\pi_{t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}-1} \times \\ &\times \left[E_{j,t} (W_{t+1} G(\pi_{t+1}))^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{-\gamma} G(\pi_{t+1})^{1-\gamma} (R_k - R_f) \right) \\ &\qquad\qquad\qquad k = 2, \dots, N \\ &\qquad\qquad\qquad j = 1, \dots, 4 \end{aligned}$$

Substituting (25) for $G(\cdot)$, we get:

$$\begin{aligned} 0 &= \left(\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}} \right)^{\frac{1-\rho}{1-\alpha}-1} \times \\ &\times \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{-\gamma} G(\pi_{t+1})^{1-\gamma} (R_k - R_f) \right) \\ &\qquad\qquad\qquad k = 2, \dots, N \\ &\qquad\qquad\qquad j = 1, \dots, 4 \end{aligned}$$

Given (15) we can simplify the first row of the above expression, which equals a constant term, $\beta^{\frac{1-\rho}{\alpha-\rho}}$, and, further multiplying both terms by $\beta^{\frac{(\gamma-\alpha)(\alpha-\rho)}{(1-\rho)^2}}$, we get :

$$0 = \beta^{\frac{\gamma-\alpha}{1-\rho}} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} (R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} (R_{k,t+1} - R_f)$$

$$k = 2, \dots, N$$

$$j = 1, \dots, 4$$

Multiplying each term by π_j and summing up over the j s,

$$0 = \sum_{j=1}^4 \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} (R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} (R_{k,t+1} - R_f)$$

$$(30)$$

$$k = 2, \dots, N$$

$$j = 1, \dots, 4$$

which we can rewrite as

$$0 = \sum_{j=1}^4 \hat{\pi}_{j,t} E_{j,t} \left((R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} (R_{k,t+1} - R_f) \right)$$

$$(31)$$

$$k = 2, \dots, N$$

$$j = 1, \dots, 4$$

where $\hat{\pi}_{j,t} = \pi_{j,t} \beta^{\frac{\gamma-\alpha}{1-\rho}} \left(E_{j,t} R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}-1}$ is the distortion of the belief of state j due to ambiguity aversion. Indeed, notice that when $\alpha = \gamma$ we are back to the ambiguity neutral belief. Eq. (31) shows how the ambiguity averse investor behaves as an ambiguity neutral with distorted beliefs.

From eq. (30),

$$\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) R_{f,t+1} =$$

$$= \sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} R_{k,t+1} \right)$$

$$k = 2, \dots, N$$

so that, multiplying by x_k and summing over the k , we have:

$$\begin{aligned}
& \sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) R_{f,t+1} = \\
& = \sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right)
\end{aligned}$$

so that, recalling (15), we have

$$\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(R_{w,t+1}^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}-1} E_{j,t} \left((R_{w,t+1})^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \right) R_{f,t+1} = \left(\frac{1}{\beta} \right)^{\frac{1-\alpha}{1-\rho}}. \quad (33)$$

Therefore, since

$$\frac{1}{R_{f,t+1}} = \sum_{j=1}^4 \pi_{j,t} (E_{j,t} M_{j,t+1}),$$

eq. (18) must hold.

and, further substituting for the return on the wealth portfolio, we get:

$$\begin{aligned}
M_{j,t+1} & = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \times \\
& \left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right) \right]^{\frac{1-\alpha}{1-\gamma}} \times \\
& \left(E_{j,t} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \times \\
& \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{\frac{\rho-\gamma}{1-\rho}} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{\rho-\gamma}{1-\rho}}
\end{aligned}$$

that is,

$$\begin{aligned}
M_{j,t+1} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \times \\
&\left[\sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right)^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{\alpha-\rho}{1-\alpha}} \times \\
&\left(E_{j,t} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right)^{\frac{\gamma-\alpha}{1-\gamma}} \times \\
&\left(\frac{C_{t+1}}{C_t} \right)^{\rho-\gamma} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{\rho-\gamma}{1-\rho}}
\end{aligned}$$

Using (15) to simplify the second line of the above expression, we finally have

$$\begin{aligned}
M_{j,t+1} &= \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{\rho-\gamma}{1-\rho}} \times \\
&\left(E_{j,t} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{\varphi^C(\pi_{t+1}) + 1}{\varphi^C(\pi_t)} \right)^{\frac{1-\gamma}{1-\rho}} \right)^{\frac{\gamma-\alpha}{1-\gamma}},
\end{aligned}$$

which corresponds to (18).

A.3 Numerical Methods

Eq.(21) which can be rewritten

$$\begin{aligned}
\varphi(\pi_t) &= \sum_{j=1}^4 \pi_{j,t} \left(E_{j,t} M_{j,t+1} \left((1 + \varphi(\pi_{t+1})) \frac{D_{t+1}}{D_t} \right) \right) \\
&= \sum_{j=1}^4 \pi_{j,t} \left(\int_{-\infty}^{\infty} M_{j,t+1} ((1 + \varphi(B(\pi_t, y))) \exp((1-\gamma)\lambda y)) f(y, j) dy \right)
\end{aligned}$$

where $y = \log\left(\frac{C_{t+1}}{C_t}\right)$ and $f(y, j)$ is the density function of a normal distribution with mean μ_j and variance σ_j . The posterior probabilities π_t are the only state variables in this framework, so the price-dividend ratio is a function only of the vector π_t . We solve these functional equations numerically on a grid of values for the state variables π_t . In order to solve for the price-dividend ratio we first need to solve for the price-consumption ratio using eq. (20), which can be rewritten as

Return	Mean	Std
r_e	9.08	15.36
r_f	1.28	1.22

Table 1: Asset Market data: Annualized sample moments from quarterly US data 1948:II-2005:IV, r^e is the return on the value-weighted NYSE portfolio and r^f is the return on the three-months Treasury bill. Returns are measured in percent per quarter. Source: Hansen and Sargent (2008), page 311.

μ_h	μ_l	σ_h^2	σ_l^2	p_{hh}^μ	p_{ll}^μ	p_{hh}^σ	p_{ll}^σ
0.926	-0.047	0.890	0.208	0.970	0.813	0.992	0.998

Table 2: Estimates for the Markov Switching model of consumption growth. Numbers in the fist four columns are in percentage. Data are quarterly and span the period between 1947:2-2009:2. Estimation by EM algorithm

$$\begin{aligned}
\varphi^C(\pi_t) &= \beta \left[\sum_{j=1}^4 \pi_{j,t} \left[E_{j,t} \left(1 + \varphi^C(\pi_{t+1}) \right)^{\frac{1-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}} \\
&= \beta \left[\sum_{j=1}^4 \pi_{j,t} \left[\int_{-\infty}^{\infty} (1 + \varphi^C(B(\pi_t, y)))^{\frac{1-\gamma}{1-\rho}} \exp((1-\gamma)y) f(y, j) dy \right]^{\frac{1-\alpha}{1-\gamma}} \right]^{\frac{1-\rho}{1-\alpha}}
\end{aligned}$$

This which, again, is solved numerically on a grid of values for the state variables π_t .

B Tables and Figures

std(ret)	av. prem	av. rf(quarterly)	std(rf)
11.89	9.91	3.09	0.33

Table 3: Standard deviation of returns, average premium, average risk free rate and standard deviation in the absence of ambiguity case. Numbers are in percentage

Comparative Statistics: $\beta = 0.9925$				
α	r_f	$\sigma(r_f)$	$E(r_m)$	$\sigma(r_m)$
Panel A: $\gamma = 30$				
30	3.02	0.33	14.06	11.89
35	2.97	0.34	14.10	11.62
40	2.91	0.35	14.31	11.49
Panel B: $\gamma = 28$				
28	3.14	0.33	13.29	12.07
35	3.07	0.35	13.41	11.66
40	3.01	0.35	13.70	11.57
Panel C: $\gamma = 25$				
25	3.32	0.32	12.20	12.57
30	3.27	0.34	12.15	11.87
35	3.21	0.35	12.40	11.70
Panel D: $\gamma = 20$				
25	3.55	0.32	10.28	13.09
30	3.49	0.33	10.33	11.62
35	3.43	0.35	10.78	11.91
Panel E: $\gamma = 15$				
30	3.69	0.33	6.74	18.95
35	3.63	0.34	9.19	16.62
40	3.56	0.36	10.54	16.68
45	3.49	0.38	7.62	16.93

Table 4: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$, and $\beta = 0.9925$ in all cases.

Comparative Statistics: $\beta = 0.986$				
α	r_f	$\sigma(r_f)$	$E(r_m)$	$\sigma(r_m)$
Panel A: $\gamma = 30$				
30	5.72	0.34	15.65	11.49
35	5.65	0.35	16.01	11.40
40	5.58	0.36	16.39	11.35

Table 5: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$, and $\beta = 0.986$ in all cases.

Comparative Statistics: $\beta = 0.994$				
α	r_f	$\sigma(r_f)$	$E(r_m)$	$\sigma(r_m)$
Panel A: $\gamma = 30$				
30	2.40	0.32	14.08	12.05
35	2.37	0.34	13.85	11.70
Panel A: $\gamma = 25$				
25	2.72	0.32	12.34	13.17
30	2.68	0.33	11.84	12.01
35	2.64	0.35	11.97	11.74
Panel B: $\gamma = 20$				
20	3.00	0.31	9.35	59.01
25	2.96	0.32	9.47	22.37
30	2.92	0.33	9.83	11.66
Panel B: $\gamma = 15$				
15	3.12	0.33	4.56	22.70
35	3.07	0.35	6.64	25.28
Panel B: $\gamma = 10$				
25	3.34	0.30	4.54	22.74
30	3.29	0.32	3.42	27.55
40	3.23	0.34	3.09	33.15

Table 6: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$, and $\beta = 0.994$ in all cases.

Comparative Statistics: $\beta = 0.996$				
α	r_f	$\sigma(r_f)$	$E(r_m)$	$\sigma(r_m)$
Panel A: $\gamma = 25$				
25	1.81	0.31	14.63	15.50
30	1.81	0.32	11.83	12.38
35	1.80	0.34	11.41	11.81
Panel A: $\gamma = 20$				
20	2.14	0.30	-0.4	59.17
25	2.12	0.32	4.98	42.04
30	2.10	0.33	9.14	11.70
Panel B: $\gamma = 15$				
15	2.42	0.28	-0.02	50.68
20	2.39	0.30	6.95	28.98
25	2.36	0.31	6.96	16.16
30	2.31	0.33	4.74	25.53
Panel B: $\gamma = 10$				
10	2.64	0.26	2.45	38.52
20	2.57	0.29	6.53	18.25
40	2.21	0.42	9.75	55.70

Table 7: Unconditional moments and comparative statistics. Except for the numbers in the first column, all other numbers are in percentage. Columns 2-5 present the mean and the standard deviation of the risk free rate and of the equity returns. We set $\rho = 1/1.5$, and $\beta = 0.996$ in all cases.

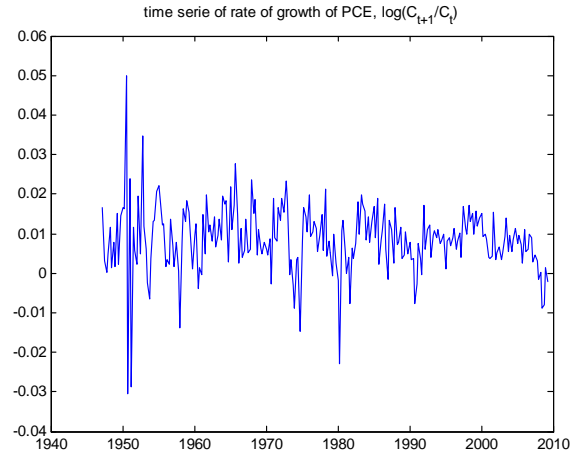


Figure 1: Time serie of the rate of growth of Total Personal Consumption Expenditure. Data are quarterly and span the period 1947:2-2009:2. Source: BEA.

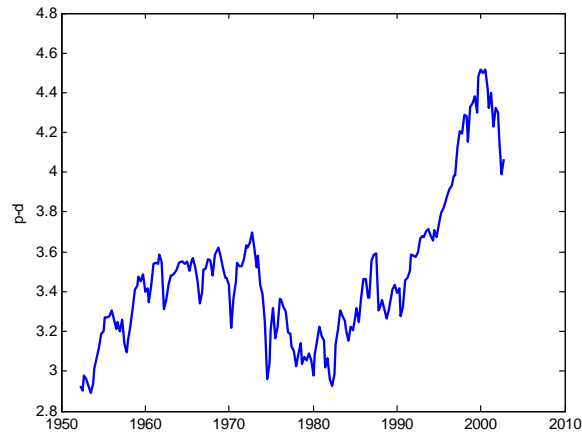


Figure 2: The CRSP-VW price-dividend ratio.

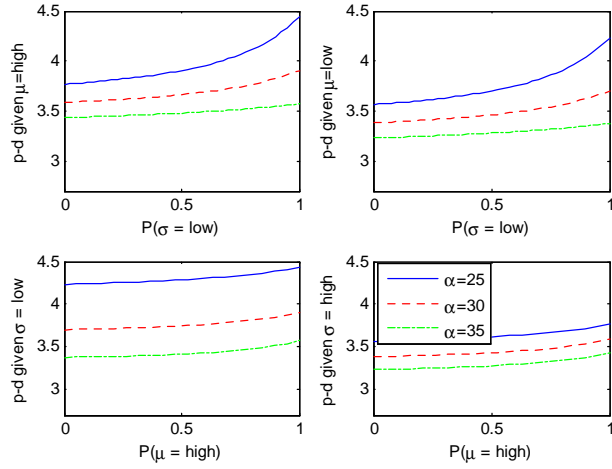


Figure 3: Price-dividend ratios ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

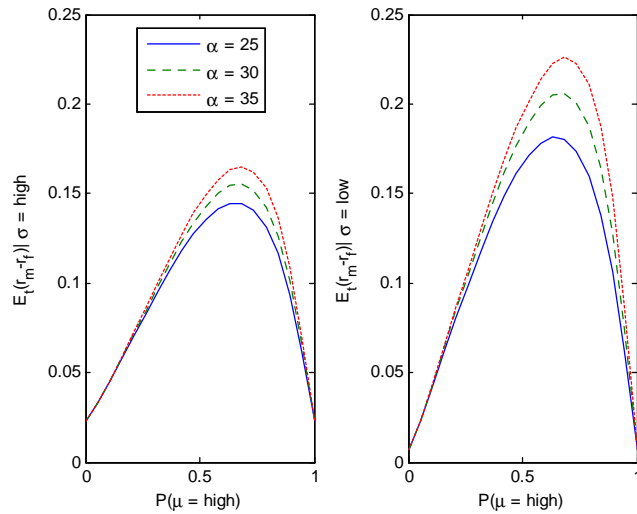


Figure 4: Conditional equity premium plotted as a function of the probability of high growth state. The left panel conditions on high volatility state and the right panel on low volatility state ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

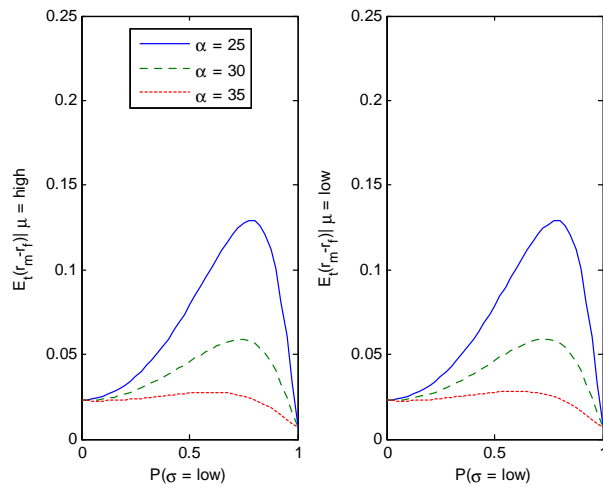


Figure 5: Conditional equity premium plotted as a function of the probability of low volatility state. The left panel conditions on high mean state and the right panel on low mean state ($\gamma = 25, \rho = 1/1.5, \lambda = 4.5, \beta = 0.9925$).

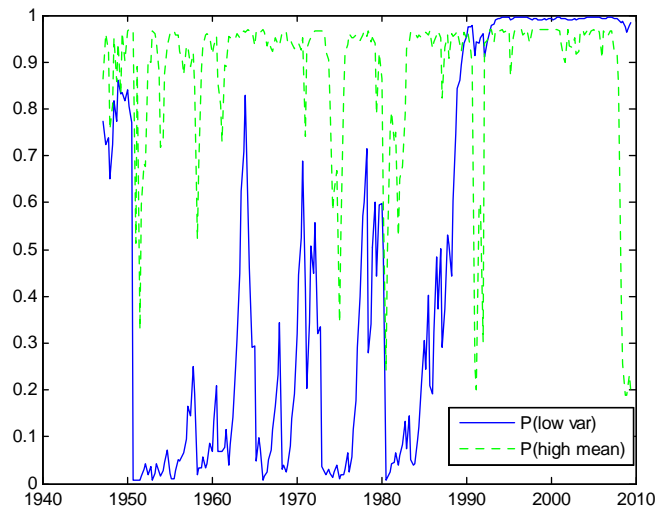


Figure 6: The figure plots the time series of estimated state probabilities. $P(\text{low variance})$ is the unconditional probability of being in a low consumption volatility state next period (solid line), calculated by summing the probability of being in a low volatility and high mean state and the probability of a low volatility and low mean state. $P(\text{high mean})$ is calculated analogously. The data are quarterly and span the period 1947:2-2009:2.

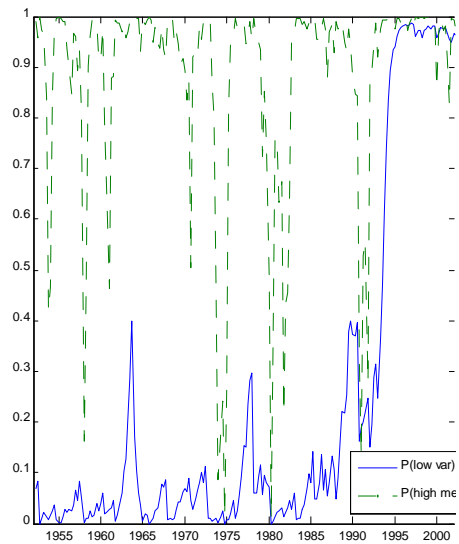


Figure 7: Source: Lettau et al., RFS (2008) The figure plots the time series of estimated state probabilities. $P(\text{low variance})$ is the unconditional probability of being in a low consumption volatility state next period (solid line), calculated by summing the probability of being in a low volatility and high mean state and the probability of a low volatility and low mean state. $P(\text{high mean})$ is calculated analogously. The data are quarterly and span the period 1952:1-2002:4.

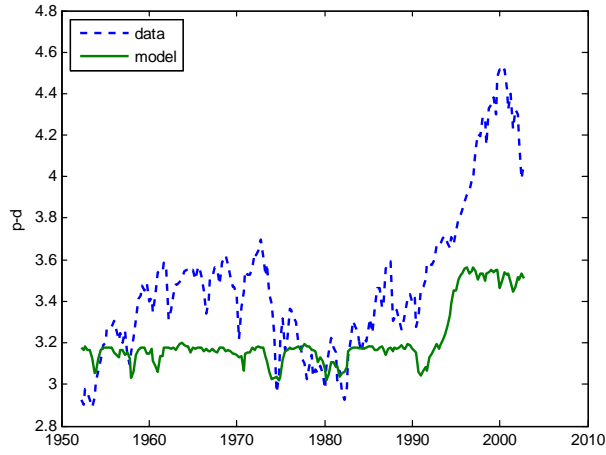


Figure 8: Price-dividend ratio (ambiguity neutrality).

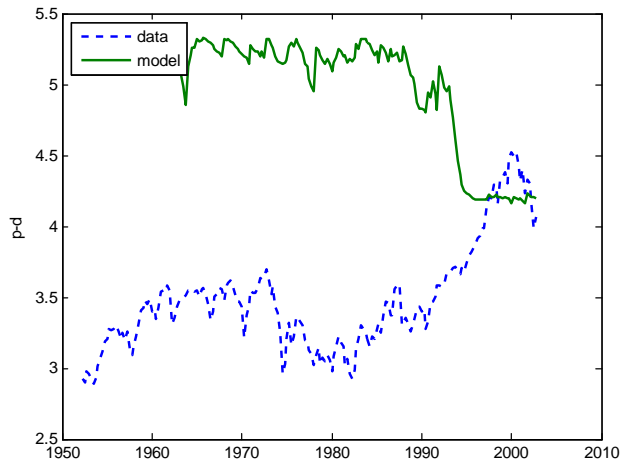


Figure 9: Time Series of the Price-Dividend ratio. Time series of the log price-dividend ratio from the data and implied by the model. The risk aversion parameter and the ambiguity aversion parameters are set, respectively to $\gamma = 15$ and $\alpha - \gamma = 20$. The rate of time preferences $\beta = 0.9925$ and the elasticity of intertemporal substitution $1/\rho = 1.5$ and leverage $\lambda = 4.5$. The data are quarterly and span the period 1952:1-2002:4.

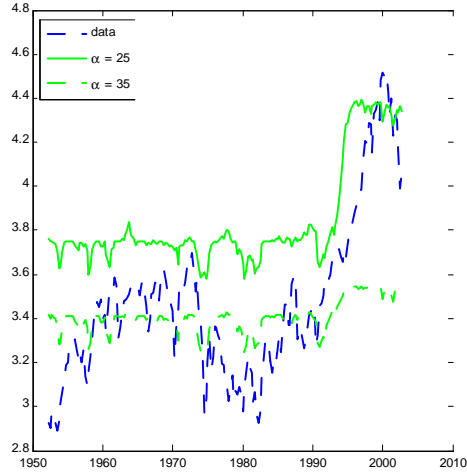


Figure 10: Time series of the log price-dividend ratio from the data and implied by the model for different ambiguity aversion values. In both simulations we set $\gamma = 25$, $\rho = 1/1.5$, $\beta = 0.9925$ and $\lambda = 4.5$.

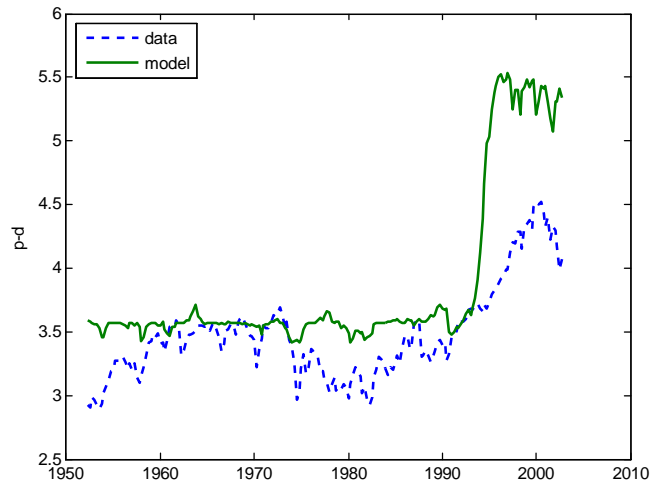


Figure 11: Price-dividend ratio. Ambiguity loving ($\gamma = 30$, $\alpha = 15$)