University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN1664-705X(online)

Working Paper No. 16

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April 2011

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**Cournot Oligopoly and Concavo-Concave Demand**

**Abstract.** Using an expanded notion of concavity, the $N$-firm Cournot model is reviewed to obtain generalized and unified conditions for existence of a pure strategy Nash equilibrium. The main theorem imposes independent conditions on inverse demand (generalized concavity) and cost functions (monotonicity). No separate assumption for large outputs is needed. We also find new conditions for strict quasiconcavity and equilibrium uniqueness.

**Keywords.** Cournot competition; existence and uniqueness of Nash equilibrium; generalized concavity; supermodular games.

**JEL-Codes.** L13 - Oligopoly and other imperfect markets; C72 - Noncooperative games; C62 - Existence and stability conditions of equilibrium.

## 1. Introduction

Given that the prediction of the Cournot model is widely acknowledged as a useful benchmark in the analysis of antitrust regulation, it is a striking fact that the conditions under which a pure strategy Nash equilibrium exists are still not fully understood. Existence is known to depend on characteristics of market demand, on restrictions imposed by technology, and even on the question of equal access to technology (cf., e.g., Vives, 1999). But it is certainly desirable to fully clarify how existence depends on these primitives of the environment. Allowing for general cost specifications, the present paper derives unified and generalized conditions for equilibrium existence in an oligopolistic market for a single homogeneous good. The analysis investigates also a number of issues other than equilibrium existence, including strict quasiconcavity and equilibrium uniqueness.

In the literature, there are essentially two main types of theorems that ascertain existence from conditions on the primitives of the model. A first approach, associated with contributions by McManus (1962, 1964) and Roberts and Sonnenschein (1976), allows for general demand specifications and convex technologies. Existence can then be shown for any finite number of identical firms.[1] Amir and Lambson (2000) have rebuilt and generalized this theory for symmetric firms using the theory of supermodular games and Tarski's fixed point theorem. A second approach, developed by Novshek (1985) and Bamon and Fraysse (1985), applies to any finite collection of firms and arbitrary technologies, imposing instead restrictions on market de-

[1] Kukushkin (1993) relaxes the assumption of identical technologies to incorporate firm-specific capacity constraints.

mand. This approach has its origins in an observation by Selten (1970).[2] The state of the art is Amir's (1996) comprehensive derivation of results from the perspective of the theory of supermodular games.[3]

Despite this impressive progress, the inquiry into the existence question is by no means settled. In particular, in the second approach that allows for general technologies, two genuinely different conditions coexist in the literature. The so-called *cardinal test* evaluates the sign of a certain cross-partial derivative of the firm's profit function. If this sign is negative, then Cournot profits are submodular in own output and rivals' joint output, so that an equilibrium can be shown to exist. The so-called *ordinal test* requires that inverse demand is logconcave. In that case, profits are logsubmodular, which likewise allows to prove existence. In an important contribution, Amir (2005) has pointed out that neither of these tests is more stringent than the respective other. He also shows that, with linear costs, cardinal and ordinal tests can be seen as special cases of a cross-partial on transformed profits. However, no unified perspective has been available so far for general technologies, which is puzzling.

In this paper, we use a variant of generalized concavity (Caplin and Nalebuff, 1991a, 1991b)[4] to reformulate and generalize conditions for existence of a pure-strategy Nash equilibrium in Cournot games. The specific concavity notion employed, referred to as *biconcavity* in the sequel, is actually quite

[2] While Selten (1970) conceptualized the important "Einpassungsfunktion" and proved existence using Brouwer's fixed point theorem, the general result obtained by Novshek (1985) and Bamon and Frayssé (1985) requires an additional argument, which was distilled into a fixed-point theorem by Kukushkin (1994).

[3] A third, complementary approach assumes profits to be quasiconcave. See Section 5.

[4] Dierker (1991) independently stressed the role of generalized concavity assumptions in the context of the Bertrand model.

intuitive. It requires that the inverse demand function is concave following monotone transformations of the scales for both price and quantity. The intuitive nature of the biconcavity assumption turns out to be very useful. For instance, in the main existence theorem (cf. Theorem 3.2), we can do away with any additional assumption at large quantities, i.e., we need not assume that sufficiently large output levels are dominated choices. Instead, this is shown to be a consequence of biconcavity. In the smooth case with downward-sloping inverse demand, the relevant concavity condition for existence can be written as

$$1 + \varepsilon_{p'} + \beta(|\varepsilon_p| - 1) \geqq 0 \tag{1}$$

for some $\beta \in [0, 1]$, where $\varepsilon_p = Qp'(Q)/p(Q)$ and $\varepsilon_{p'} = Qp''(Q)/p'(Q)$ denote the elasticities of inverse market demand and of its first derivative.

To derive the general existence theorem, which does not impose smoothness nor convexity of choice sets, methods developed by Novshek (1985) and Amir (1996) are combined and enhanced. First, we formulate generalized concavity assumptions on inverse demand implying that suitably transformed revenues have weakly decreasing differences in the relevant domain. Then, a new lattice-theoretic argument is used to prove that this property of transformed revenues is sufficient for profits to satisfy the dual single-crossing property. Finally, it is shown that effective choice sets are compact, so that, invoking a generalized theorem of the maximum, firms' best-response correspondences necessarily allow a nonincreasing selection. By the virtue of Kukushkin's (1994) fixed point theorem, this ensures existence of a Cournot-

Nash equilibrium in pure strategies.[5] As will be explain further below, the resulting existence theorem contains virtually all earlier results for general technologies as special cases, and even makes an additional step ahead.

In addition to the existence theorem, we also find generalized and unified conditions for strict quasiconcavity of profits and for equilibrium uniqueness in Cournot games. For example, it follows from our results that the generalized concavity condition (1), together with

$$p' < 0, \text{ and } p' - C_i'' < 0 \text{ for } i = 1, ..., N, \tag{2}$$

implies strict quasiconcavity of profits and equilibrium uniqueness. This finding extends a well-known criterion (due to Vives, 1999), that assumes (2) as well as logconcave inverse demand.

The rest of the paper is structured as follows. Section 2 introduces the notion of biconcavity. The existence theorem is stated and discussed in Section 3. Section 4 explains the main steps of the proof. Conditions for strict quasiconcavity are derived in Section 5. Section 6 deals with equilibrium uniqueness. Technical arguments have been relegated to an Appendix.

## 2. Biconcavity

This section offers a brief account of the generalized concavity notion that is key to unifying and expanding prior results. The reader interested only in the main results should take note of the definition of $\alpha$-biconcavity, and continue with Section 3.

Write $\varphi_\alpha : \mathbb{R}_{++} \to \mathbb{R}$ for the mapping $x \mapsto x^\alpha/\alpha$ if $\alpha \neq 0$, and $\varphi_0$ for the logarithm. For constants $\alpha, \beta \in \mathbb{R}$, a function $p : \mathbb{R}_+ \to \mathbb{R}_+$ will be called

[5] For the class of Cournot games considered in this paper, Kukushkin's theorem is equally strong as its recent extension by Jensen (2010).

$(\alpha, \beta)$-*biconcave* [$(\alpha, \beta)$-*biconvex*] if the domain $I_p = \{Q > 0 : p(Q) > 0\}$ is an interval and $\varphi_\alpha(p(Q))$ is a concave [convex] function of $\varphi_\beta(Q)$ in $I_p$.[6] E.g., $p$ is $(0, 1)$-biconcave if and only if $p$ is logconcave. More generally, $p$ is $(\rho, 1)$-biconcave if and only if $p$ is $\rho$-concave.[7] For another example, $p$ is $(1, 0)$-biconcave [$(1, 1)$-biconcave] if and only if $I_p$ is an interval and $p(Q)$ is concave in $I_p$ as a function of $\log Q$ [as a function of $Q$]. In the important case $\beta = 1-\alpha$, we will simply drop the second entry and say that $p$ is $\alpha$-*biconcave* [$\alpha$-*biconvex*].

If $p$ is both $(\alpha, \beta)$-biconcave and weakly decreasing, then $p$ is also $(\alpha', \beta')$-biconcave for any $\alpha' \leqq \alpha$ and $\beta' \leqq \beta$.[8] For example, $(1, 1)$-biconcavity is more stringent than $(1, 0)$-biconcavity, which in turn is more stringent than $(0, 0)$-biconcavity. It is essential here, however, that $p$ is nonincreasing. Without this assumption, it is not generally true that higher values for $\beta$ correspond to more stringent variants of biconcavity.

In many cases, it will be convenient to work with the smooth criterion for $(\alpha, \beta)$-biconcavity, derived in the Appendix, which requires that $I_p$ is an

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[6] This definition is due to Avriel (1972).

[7] A function $p$ is $\rho$-*concave* for $\rho \in \mathbb{R}$ if $I_p$ is convex and $\varphi_\rho(p(Q))$ is concave in $I_p$ (cf. Caplin and Nalebuff, 1991a).

[8] The proof extends an argument in Caplin and Nalebuff (1991a). For $Q, \widehat{Q} > 0$, $\lambda \in [0, 1]$, define the $\rho$*th generalized mean* by

$$M_\rho(Q, \widehat{Q}, \lambda) = [(1-\lambda)Q^\rho + \lambda \widehat{Q}^\rho]^{1/\rho} \tag{3}$$

if $\rho \neq 0$, and by $M_0(Q, \widehat{Q}, \lambda) = Q^{1-\lambda}\widehat{Q}^\lambda$ if $\rho = 0$. Then, clearly, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is $(\alpha, \beta)$-biconcave if and only if $I_p$ is an interval and

$$M_\alpha(p(Q), p(\widehat{Q}), \lambda) \leqq p(M_\beta(Q, \widehat{Q}, \lambda)) \tag{4}$$

for all $Q, \widehat{Q} \in I_p$ and $\lambda \in [0, 1]$. By Hölder's inequality (cf. Hardy et al., 1934), $M_\rho(Q, \widehat{Q}, \lambda)$ is nondecreasing in $\rho$. Therefore, if $p$ is weakly decreasing, inequality (4) becomes more restrictive as either $\alpha$ or $\beta$ increases.

interval, and that

$$\Delta^p_{\alpha,\beta}(Q) \equiv Q\{(\alpha-1)p'(Q)^2 + p(Q)p''(Q)\} + (1-\beta)p(Q)p'(Q) \leqq 0 \quad (5)$$

holds for any $Q \in I_p$. As a matter of convention, $\Delta^p_\alpha(Q) = \Delta^p_{\alpha,1-\alpha}(Q)$.

When $p$ is strictly declining when positive and continuous, then $p$ is $(\alpha, \beta)$-biconcave if and only if the inverse of $p$ is $(\beta, \alpha)$-biconcave. Through this equivalence, generalized concavity conditions on inverse demand translate naturally into conditions on direct demand and vice versa. For example, inverse demand is $(1, -1)$-biconcave [$(1, 0)$-biconcave] if and only if direct demand is $(-1)$-concave [log-concave].[9]

## 3. The existence theorem

Consider an industry composed of $N \geqq 1$ firms producing a homogeneous good. Firm $i$ produces the good in quantity $q_i \in D_i$, where $D_i \subseteq \mathbb{R}_+$ denotes the firm's set of technologically feasible output levels. Aggregate output $Q = \sum_{i=1}^N q_i$ determines *inverse demand* $p(Q) \in \mathbb{R}_+$. Firm $i$'s profit reads

$$\Pi_i(q_i, Q_{-i}) = p(q_i + Q_{-i})q_i - C_i(q_i), \quad (6)$$

where $Q_{-i} = \sum_{j \neq i} q_j$ is the joint output of firm $i$'s competitors, and $C_i : D_i \to \mathbb{R}$ is firm $i$'s cost function. Denote by

$$r_i(Q_{-i}) = \{q_i \in D_i : \Pi_i(q_i, Q_{-i}) \geqq \Pi_i(\widetilde{q}_i, Q_{-i}) \text{ for all } \widetilde{q}_i \in D_i\} \quad (7)$$

the set of firm $i$'s *best responses* to an aggregate rival output $Q_{-i} \geqq 0$. A *Cournot-Nash equilibrium* is a vector of quantities $(q_1, ..., q_N)$ such that $q_i \in r_i(Q_{-i})$ for $i = 1, ..., N$.

[9] Such conditions on direct demand have been used, in particular, by Deneckere and Kovenock (1999). See also Anderson and Renault (2003).

Using the definition of biconcavity introduced in the previous section, we can state the main result of this paper.

**Theorem 3.2.** *Given a market for a single homogeneous good with inverse demand* $p$, *and* $N$ *firms with sets of technologically feasible output levels* $D_1, ..., D_N \subseteq \mathbb{R}_+$, *and cost functions* $C_1, ..., C_N$, *if*

*1.* $p : \mathbb{R}_+ \to \mathbb{R}_+$ *is non-constant, nonincreasing, left-continuous, as well as* $\alpha$*-biconcave for some* $\alpha \in [0, 1]$, *and*

*2. for* $i = 1, ..., N$, *the set* $D_i$ *is closed, and* $C_i : D_i \to \mathbb{R}$ *is left-continuous and nondecreasing,*

*then, for any firm* $i$, *the minimum best response function* $\min r_i$ *is well-defined and nonincreasing in rivals' output. Moreover, a Cournot-Nash equilibrium exists.*[10]

The proof is in the Appendix. See Section 4 for an exposition of the main steps. There, it is also explained why one can dispense with any compactness assumption.

It is easy to see that Theorem 3.2 contains the two main approaches to existence for general technologies. Indeed, inequality (5) reduces to Novshek's (1985) marginal revenue condition

$$p''(Q)Q + p'(Q) \leqq 0 \tag{8}$$

for $\alpha = 1$, and to Amir's (1996, Theorem 3.1) logconcavity assumption

$$p(Q)p''(Q) - p'(Q)^2 \leqq 0 \tag{9}$$

[10] There are two specific reasons to focus on the minimum best response. First, since strategy spaces may be unbounded, the maximum best response need not exist in general. Second, arbitrary selections need not be monotonic under the conditions of the theorem.

for $\alpha = 0$.[11]

The discussion above sheds additional light on a well-known link, explored by Amir (2005), between cardinal and ordinal approaches to the Cournot model. More specifically, one can show that profits transformed via $\varphi_\alpha$ exhibit nonincreasing differences in $(q_i, Q_{-i})$ if and only if

$$0 \geqq \widehat{p}(q_i + Q_{-i})[p'(q_i + Q_{-i}) + q_i p''(q_i + Q_{-i})] \qquad (10)$$
$$+ (\alpha - 1)p'(q_i + Q_{-i})[\widehat{p}(q_i + Q_{-i}) + q_i p'(q_i + Q_{-i})],$$

where $\widehat{p}(.) \equiv p(.) - c$. Amir (2005) noted that for $\alpha = 0$, condition (10) corresponds to the logconcavity of $\widehat{p}$, so that (8) and (9) are indeed special cases of the cross-partial. With regard to this link, Theorem 3.2 offers additional transparency by dropping the restriction to linear costs, and by surpressing the second argument in the condition.

Could monotone transformations $h$ other than $\varphi_\alpha$ lead to equally simple, but perhaps more general criteria for existence? E.g., Amir (2005) considers specific transformations including $h_1(x) = \log(ax + b)$ with $a, b > 0$, $h_2(x) = \exp(\alpha x)$ with $\alpha > 0$, and $h_3(x) = ax + b \log x$ with $a, b > 0$. A priori, it is not clear why $\varphi_\alpha$ should be privileged among those transformations. To clarify this issue, we make two observations. The first, which is actually at the heart of the present analysis, is that cross-partial conditions for zero costs are sufficient for ordinal submodularity with nonincreasing costs, provided that $h$ is *concave*. Formally, this is a mild generalization of Lemma 4.2 below, with $\varphi_\alpha$ replaced by $h$.[12]

[11] Clearly, Theorem 3.2 also takes account of the fact that standard existence theorems do not require strategy spaces to be convex (see, e.g., Dubey et al., 2006).

[12] On the other hand, this argument does not hold for the strictly convex transformation $h_3$. See also Footnote 19.

The other observation identifies conditions under which one argument in the cross-partial can be dropped. To derive those conditions, note that for general $h$, transformed revenues exhibit nonincreasing differences provided that

$$0 \geqq (1-\widetilde{\beta})p'(q_i+Q_{-i})p(q_i+Q_{-i}) \qquad (11)$$
$$+ q_i p''(q_i+Q_{-i})p(q_i+Q_{-i}) - \widetilde{\beta} q_i p'(q_i+Q_{-i})^2,$$

where the coefficient $\widetilde{\beta}(R) = -Rh''(R)/h'(R)$ is a function of revenues $R \equiv q_i p(q_i+Q_{-i})$. Hence, if $\widetilde{\beta}(.)$ is nonincreasing and $\leqq 1$, one may surpress the argument $Q_{-i}$, so that (11) reduces to[13]

$$(1-\widetilde{\beta})p'(Q)p(Q) + Qp''(Q)p(Q) - \widetilde{\beta} Qp'(Q)^2 \leqq 0. \qquad (12)$$

E.g., the simplication works for $\varphi_\alpha$ if $\alpha \geqq 0$, and also for $h_2$ and $h_3$.[14]

These observations suggest that a one-parameter test allowing for general cost specifications is bound to have the form (12) for some nonincreasing function $\widetilde{\beta}$ with values in the unit interval. Biconcavity is a special case of (12) that avoids, in particular, the necessity of having to deal with large outputs. Thus, while the biconcavity condition corresponds to only a very specific family of transformations, it clearly stands out through desirable properties that are not necessarily shared by more flexible tests.

We now discuss Theorem 3.2 in more detail. The following example shows that none of the parameter values $\alpha \in [0,1]$ can be dropped without loss. It also illustrates that prior results are strictly generalized.

[13] Assume (12) holds. If, in addition, $p''(Q)p(Q) - \widetilde{\beta}p'(Q)^2 > 0$, then clearly, condition (11) holds at positive prices. But also if $p''(Q)p(Q) - \widetilde{\beta}p'(Q)^2 \leqq 0$, we may conclude that (11) holds provided that $\widetilde{\beta}(.)$ is nonincreasing and $\leqq 1$.

[14] On the other hand, when $\widetilde{\beta}(.)$ is locally increasing or $> 1$, it seems to be necessary to work with two arguments.

**Example 3.3.** For $0 < \gamma < 1$, let inverse demand be given by $p(Q) = (1 - Q^{1-\gamma})^{1/\gamma}$ if $Q \leqq 1$, and $= 0$ if $Q > 1$. Using the smooth criterion for biconcavity (5), one can check that $p$ is $\alpha$-biconcave for some $\alpha \in [0, 1]$ if and only if $\alpha = \gamma$. Indeed, a long but straightforward calculation delivers

$$\Delta_\alpha^p(Q) = (\alpha - \gamma)(Q^{1-\gamma} - \gamma)\frac{(1-\gamma)(1-Q^{1-\gamma})^{2/\gamma - 2}}{\gamma^2 Q^\gamma}. \tag{13}$$

Hence, $\Delta_\alpha^p(Q) \leqq 0$ for all $Q \in (0, 1)$ if and only if $\gamma = \alpha$. Thus, none of the parameter values $\alpha \in [0, 1]$ can be disposed of. Moreover, for $\gamma \neq 0, 1$, the marginal revenue condition fails, as does logconcavity of inverse demand.[15]

The next example demonstrates that it is not possible to relax the parameter restriction on the biconcavity condition.

**Example 3.4.** Consider the inverse demand function $p(Q) = (1 - Q^\beta)^{1/\alpha}$ for $Q < 1$ and $p(Q) = 0$ for $Q \geqq 0$, where $\alpha, \beta \in (0, 1)$ are constants. The graph of this function becomes concave (actually linear) in the relevant domain once the quantity scale is transformed using $\varphi_\beta$, and the price scale is transformed using $\varphi_\alpha$. If $\alpha + \beta < 1$, then $p$ fulfils concavity requirements somewhat weaker than those imposed in Theorem 3.2. Now, with zero costs, it follows from Theorem 5.1 below that profits are strictly quasiconcave. Implicit differentiation of the first-order condition, and subsequently exploiting the smooth condition for biconcavity as well as the first-order condition shows that the slope of the best response at $Q_{-i} = 0$ is given by

$$\left.\frac{dq_i}{dQ_{-i}}\right|_{Q_{-i}=0} = -\frac{q_i p''(q_i) + p'(q_i)}{q_i p''(q_i) + 2p'(q_i)} = \frac{1 - \alpha - \beta}{\alpha + \beta} > 0. \tag{14}$$

[15] Similar arguments can be used to show that the limit cases $\alpha = 0$ and $\alpha = 1$ are not redundant either. For this, one checks that $p(Q) = \exp(-Q)$ is $\alpha$-biconcave for some $\alpha \in [0, 1]$ if and only if $\alpha = 0$. Similarly, consider $p(Q) = -\ln(Q)$ for $Q \leqq 1$, and $= 0$ for $Q > 1$. This price function is $\alpha$-biconcave for some $\alpha \in [0, 1]$ if and only if $\alpha = 1$.

I.e., the conclusion of Theorem 3.2 ceases to hold for less stringent variants of biconcavity.

One can somewhat relax assumptions on demand if *all* firms face capacity constraints, i.e., if $D_i \subseteq [0, \overline{Q}_i]$ for each $i = 1, ..., N$. Then, the conditions on $p$ need hold only in the relevant domain $[0, \overline{Q}_1 + ... + \overline{Q}_N]$.[16]

*Discussion.* We have discussed above the biconcavity condition. Is there a way to relax the other conditions of Theorem 3.2 further? Restrictions on technology are extremely weak. The closedness of the set of feasible output levels and left-continuity of the cost function are minimum technical requirements to allow optimization. Weak monotonicity of costs is a standard assumption (satisfied in a world with free disposal, for instance). Assumptions on demand, given biconcavity, are also very weak. A constant price is not a very interesting case, and is excluded for convenience. The left-continuity assumption on inverse demand is, again, needed to allow for optimization. In fact, at positive prices, generalized concavity implies continuity, so left-continuity matters only when $p$ jumps to zero. The existence theorem can also be easily generalized to price functions $p : \mathbb{R}_{++} \to \mathbb{R}_+$ unbounded at $Q = 0$ provided that revenues go to zero as output approaches zero, i.e., provided that $\lim_{Q \to +0} p(Q)Q = 0$. Monotonicity of inverse demand is again a standard assumption.[17] Thus, the biconcavity assumption on inverse demand

[16] Two extensions of Theorem 3.2 are feasible for *linear* cost functions $C_i(q_i) = c_i q_i$, with $i = 1, ..., N$. Assume first that for each $i = 1, ..., N$, there is an $\alpha_i > 1$ such that firm $i$'s net-of-variable-cost demand $\widehat{p}_i \equiv p - c_i$ is $\alpha_i$-biconcave. Then, a Cournot-Nash equilibrium exists. See Footnote 19 for illustration. For another extension, consider an oligopoly with capacity constraints $\overline{Q}_1, ..., \overline{Q}_N$, and assume that for each $i = 1, .., N$, there is an $\alpha_i \leqq 0$ such that $\widehat{p}_i$ is $\alpha_i$-biconvex on $[0, \overline{Q}_1 + ... + \overline{Q}_N]$. Then, a Cournot-Nash equilibrium exists by Tarski's fixed point theorem. The latter finding extends Amir's (1996) Theorem 3.2, which corresponds to the case $\alpha_1 = ... = \alpha_N = 0$.

[17] As a matter of fact, for logconcave prices and linear costs, the proof of Theorem

is indeed the only significant restriction.

## 4. Proof of Theorem 3.2

The derivation of the existence theorem is based on three basic insights.

*Generalized concavity of demand maps into submodularity of transformed revenues.* The first step in the existence proof is to acknowledge that any $\alpha$-biconcave price function $p$, for nonnegative $\alpha$, has the property that suitably transformed revenues exhibit weakly decreasing differences. To state a corresponding result, we write $R(q_i, Q_{-i}) = q_i p(q_i + Q_{-i})$ for revenues. The following result extends an argument used by Novshek (1985).

**Lemma 4.1.** *Let $p$ be nonincreasing and $\alpha$-biconcave for some $\alpha \geqq 0$ [$\alpha$-biconvex for some $\alpha \leqq 0$]. Then for any $\widehat{q_i} > q_i > 0$ and for any $\widehat{Q}_{-i} > Q_{-i} \geqq 0$ such that $p(\widehat{q_i} + \widehat{Q}_{-i}) > 0$,*

$$\varphi_\alpha(R(q_i, Q_{-i})) - \varphi_\alpha(R(q_i, \widehat{Q}_{-i})) \underset{[\geqq]}{\leqq} \varphi_\alpha(R(\widehat{q_i}, Q_{-i})) - \varphi_\alpha(R(\widehat{q_i}, \widehat{Q}_{-i})). \quad (15)$$

**Proof.** We will prove the claim for biconcave demand only. The bracketed case is completely analogous. By assumption, $\varphi_\alpha(p(q_i))$ is concave in $\varphi_{1-\alpha}(q_i)$ in the interval where $p > 0$. Since $\alpha \geqq 0$, by Lemma A.3 in the Appendix, $\varphi_\alpha(p(q_i + \widetilde{Q}_{-i}))$ is concave in $\varphi_{1-\alpha}(q_i)$ in the interval where $p > 0$, for any $\widetilde{Q}_{-i} \geqq 0$. Therefore, the derivative $\partial\varphi_\alpha(p(q_i + \widetilde{Q}_{-i}))/\partial\varphi_{1-\alpha}(q_i)$ is a.e. well-defined, and nonincreasing in $\varphi_{1-\alpha}(q_i)$. But, using the properties of the

3.2 goes through without monotonicity of demand. See Section 4 in Amir (1996) for a discussion of nonmontonic demand in the presence of general equilibrium effects.

functions $\varphi_\alpha$ and $\varphi_{1-\alpha}$, one has a.e.

$$\frac{\partial \varphi_\alpha(p(q_i + \widetilde{Q}_{-i}))}{\partial \varphi_{1-\alpha}(q_i)} = \frac{\varphi'_\alpha(p(q_i + \widetilde{Q}_{-i}))p'(q_i + \widetilde{Q}_{-i})}{\varphi'_{1-\alpha}(q_i)} \tag{16}$$

$$= \varphi'_\alpha(R(q_i, \widetilde{Q}_{-i}))q_i p'(q_i + \widetilde{Q}_{-i}) \tag{17}$$

$$= \frac{\partial \varphi'_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}}. \tag{18}$$

So the derivative in (18) is a.e. well-defined and nonincreasing in $q_i$. By the fundamental theorem of calculus,[18]

$$\varphi_\alpha(R(q_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) = \int_{Q_{-i}}^{\widehat{Q}_{-i}} \frac{\partial \varphi'_\alpha(R(q_i, \widetilde{Q}_{-i}))}{\partial \widetilde{Q}_{-i}} d\widetilde{Q}_{-i}. \tag{19}$$

It follows that the left-hand side of equation (19) is nonincreasing in $q_i$. This proves the lemma. □

To get an intuition for this result, recall that if inverse demand is actually concave and declining over the relevant domain, then the drop in price resulting from a given increase in rivals' output will be more pronounced as own output increases. Lemma 4.1 follows a similar logic. Under the assumptions made, the loss in (transformed) revenues resulting from a given increase in rivals' output will be larger as own output increases. However, to take account of the quantity dimension of revenues, concavity of demand needs to be replaced by the biconcavity assumption.

*Submodularity of transformed revenues maps into dual single crossing of profits.* The second observation that needs to be made in the proof of Theorem 3.2 is of a lattice-theoretic nature. The result says that provided suitably transformed revenues exhibit weakly decreasing differences, and provided

[18] See Royden (1988, Chapter 5). The usual regularity assumption follows here from the generalized concavity of inverse demand.

that costs are nondecreasing, profits satisfy the dual single-crossing property. The proof given below extends an argument due to Amir (1996).

**Lemma 4.2.** *Let* $\alpha \in [0,1]$. *Take some* $\widehat{q}_i > q_i \geqq 0$, $\widehat{Q}_{-i} > Q_{-i} \geqq 0$, *such that* $p(\widehat{q}_i + \widehat{Q}_{-i}) > 0$. *Assume that* $p$ *is nonincreasing, and that* $C_i$ *is nondecreasing. Assume also that*

$$\varphi_\alpha(R(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) \geqq \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, \widehat{Q}_{-i})). \quad (20)$$

*Then,*

$$\Pi_i(q_i, Q_{-i}) - \Pi_i(\widehat{q}_i, Q_{-i}) \geqq 0 \Rightarrow \Pi_i(q_i, \widehat{Q}_{-i}) - \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) \geqq 0. \quad (21)$$

*If in addition,* $\alpha < 1$, *and* $p$ *is strictly decreasing when positive, then* $\Pi_i$ *satisfies (21) with a strict inequality in the concluding inequality.*

**Proof.** To verify dual single crossing, assume that $\Pi_i(q_i, Q_{-i}) \geqq \Pi_i(\widehat{q}_i, Q_{-i})$. There are two cases. Assume first that, in fact, $R(q_i, Q_{-i}) > R(\widehat{q}_i, Q_{-i})$. Clearly, since $\varphi \circ R$ is submodular (we drop the index $\alpha$), and $\varphi$ is strictly increasing, $R$ satisfies the dual single-crossing property (cf. Milgrom and Shannon, 1994). Hence, $R(q_i, \widehat{Q}_{-i}) > R(\widehat{q}_i, \widehat{Q}_{-i})$. Therefore, since costs are nondecreasing, $\Pi_i(q_i, \widehat{Q}_{-i}) > \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$, as desired. Assume now that $R(q_i, Q_{-i}) \leqq R(\widehat{q}_i, Q_{-i})$. We will show that then

$$R(q_i, Q_{-i}) - R(q_i, \widehat{Q}_{-i}) \leqq R(\widehat{q}_i, Q_{-i}) - R(\widehat{q}_i, \widehat{Q}_{-i}), \quad (22)$$

which is sufficient for the dual single-crossing property. For this, write $u = \varphi(R(\widehat{q}_i, \widehat{Q}_{-i}))$ and $w = \varphi(R(\widehat{q}_i, Q_{-i})) - \varphi(R(\widehat{q}_i, \widehat{Q}_{-i}))$. Define also $v = \varphi(R(q_i, Q_{-i})) - w$. See Figure 1 for illustration. As $\varphi$ is strictly in- Fig. 1 here creasing, it follows from the case assumption that $u > v$. Moreover, since $p$ is nonincreasing,

$$R(\widehat{q}_i, Q_{-i}) = \widehat{q}_i p(\widehat{q}_i + Q_{-i}) \geqq \widehat{q}_i p(\widehat{q}_i + \widehat{Q}_{-i}) = R(\widehat{q}_i, \widehat{Q}_{-i}), \tag{23}$$

which implies that $w \geqq 0$. Denote by $\varphi^{-1}$ the inverse function of $\varphi$. Since $\varphi^{-1}$ is convex,

$$\varphi^{-1}(v + w) - \varphi^{-1}(v) \leqq \varphi^{-1}(u + w) - \varphi^{-1}(u). \tag{24}$$

Plugging the expressions for $u$, $v$, and $w$ into (24) and simplifying yields

$$R(q_i, Q_{-i}) - \varphi^{-1}(v) \leqq R(\widehat{q}_i, Q_{-i}) - R(\widehat{q}_i, \widehat{Q}_{-i}), \tag{25}$$

as illustrated in Figure 1. On the other hand, from the submodularity property of $\varphi \circ R$,

$$\varphi(R(q_i, Q_{-i})) - \varphi(R(q_i, \widehat{Q}_{-i})) \leqq \varphi(R(\widehat{q}_i, Q_{-i})) - \varphi(R(\widehat{q}_i, \widehat{Q}_{-i})), \tag{26}$$

or equivalently,

$$R(q_i, \widehat{Q}_{-i}) \geqq \varphi^{-1}(v). \tag{27}$$

Combining (25) with (27), one arrives at (22), as desired. To prove the second assertion of the lemma, assume that $p$ is strictly decreasing when positive, and that $\alpha < 1$. Then, for the first case considered above, it has been shown that $\Pi_i(q_i, \widehat{Q}_{-i}) > \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$. In the second case, $w > 0$, because $p$ is strictly decreasing when positive, and $\varphi^{-1}$ is strictly convex, so again, $\Pi_i(q_i, \widehat{Q}_{-i}) - \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) > 0$. This proves the second assertion, whence the lemma. $\square$

To understand why Lemma 4.2 holds, it is important to acknowledge that there are two possible regimes. In the first, marginal revenues are zero or negative when the output of other firms is low. Under this regime, the fact that revenues are "submodularizable" directly implies that marginal revenues remain zero or negative when the output of other firms increases. Clearly then, a fortiori, marginal profits must be zero or negative when rivals' joint output is high, as desired. In the second regime, marginal revenues are positive when rivals' joint output is low. But then, since revenues are declining in the other firms' output, and since in addition, concavely transformed revenues exhibit weakly decreasing differences, marginal revenues must be smaller when the output of other firms increases. Thus, under either regime, an expansion of output cannot become attractive just because competing firms increase their output.[19]

*Generalized concavity of demand maps into compactness of the game.* The final step of the proof derives the boundedness of the effective choice set from the biconcavity assumption on inverse demand. Novshek (1985, Section 5.1) noted that the marginal revenue condition implies that the market price reaches zero provided that monopoly output is not maximized at infinite output. Our argument is similar. Specifically, we assume that inverse demand

[19]With concave transformations of revenue (i.e., with $\alpha \leqq 1$), the presence of nondecreasing costs unambiguously reinforces that best responses are weakly declining. With strictly convex transformation of revenues ($\alpha > 1$), this is no longer true. To see why, consider $p(Q) = \sqrt{(1-Q)/Q}$ for $0 < Q \leqq 1$ and $p(Q) = 0$ for $Q > 1$. One can check that $p$ is 2-biconcave. Hence, by Lemma 4.1, squared revenues $R(q_i, Q_{-i})^2$ are submodular at positive prices. The best response with zero variable costs is a nonincreasing function. With positive variable costs, this need not be true anymore. For instance, for $q_i = 0.1$, $\widehat{q}_i = 0.5$, $Q_{-i} = 0$, $\widehat{Q}_{-i} = 0.1$, and costs $C_i(q_i) = 0.5 \cdot q_i$, the single-crossing condition is violated. Moreover, the best response with costs satisfies $r_i(Q_{-i}) \approx 0.276 < 0.286 \approx r_i(\widehat{Q}_{-i})$, i.e., it is not weakly decreasing. Thus, the restriction to concave transformations of revenues is really needed for general cost specifications.

is non-constant, nonincreasing, and $\alpha$-biconcave for some $\alpha \in [0, 1]$. Now consider the graph of the inverse demand function with the scales for own output $Q$ and price $P$ transformed by $\varphi_\alpha$ and $\varphi_{1-\alpha}$, respectively. Since $p$ is nonincreasing and non-constant, the same is true for the transformed function, so the graph with transformed axes has a negative slope somewhere at positive prices. Moreover, from biconcavity, this graph has a concave shape at positive prices. Hence, there is a finite $\overline{Q}$ such that transformed prices are zero (or negative) for any output level $Q \geqq \overline{Q}$. Provided that $\alpha > 0$, this means that prices itself are zero for some finite output level $\overline{Q}$.

The argument is somewhat more involved for logconcave inverse demand, i.e., for $\alpha = 0$. In this case, assume that inverse demand is strictly positive for all output levels (otherwise there is nothing to show). Consider now the slope of firm $i$'s log-revenue curve, which is given by

$$\frac{d \ln(q_i p(q_i + Q_{-i}))}{dq_i} = \frac{1}{q_i} + \frac{1}{p(q_i + Q_{-i})} \frac{dp(q_i + Q_{-i})}{dq_i}. \tag{28}$$

For large quantities $q_i$, the first term on the right-hand side of (28) becomes arbitrarily small. Moreover, since inverse demand $p$ is non-constant and non-increasing, the second term must be negative somewhere. Furthermore, since $p$ is logconcave, the second term is nonincreasing. Therefore, for sufficiently large quantities $q_i$, the slope of the firm's log-revenue function is negative and bounded away from zero, so that revenues go to zero at large quantities at an exponential rate, *regardless of rivals' output* $Q_{-i}$.[20] Thus, with nondecreasing costs, there exists a finite $\overline{Q}$ such that any output $q_i > \overline{Q}$ is dominated.

[20] For instance, when $p(Q) = \exp(-Q)$, then the slope of the firm's log-revenue function is $(1/q_i) - 1$, so that output levels $q_i > 1$ are clearly suboptimal.

## 5. Strict quasiconcavity

The literature offers a diversity of criteria for strict quasiconcavity of Cournot profits. One test checks if $Qp(Q)$ is strictly concave, $p$ is strictly decreasing, and costs are convex (cf. Murphy et al., 1982). Another test checks $pp'' - 2p'^2 < 0$ (cf. Amir, 1996). Still another test, adapted from Caplin and Nalebuff (1991b, Prop. 11), requires strict logconcavity of direct demand in log-price as well as linear costs. For nonconvex technologies, one checks logconcavity of $p$ and $p' - C_i'' < 0$ (cf. Vives, 1999).

The notion of biconcavity can be used to unify and generalize such criteria. Consider firm $i$'s problem, keeping $Q_{-i} \geqq 0$ fixed.

**Theorem 5.1.** *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be twice continuously differentiable when positive, and nonincreasing. Furthermore, let $C_i$ be twice continuously differentiable on $\mathbb{R}_+$, nondecreasing, and strictly increasing in the domain where the monopoly price is zero. For some $\alpha \leqq 1$, $\beta \leqq 1$, assume that (i) $\Delta^p_{\alpha,\beta} \leqq 0$ and (ii) $(\alpha+\beta)p' - C_i'' \leqq 0$ at positive prices. If in addition, at least one of the inequalities (i) and (ii) holds strictly, then $\Pi_i(q_i, Q_{-i})$ is strictly quasiconcave in $q_i$.*

**Proof.** The proof is an extension of an argument in Vives (1999). It is shown first that $\Pi_i(., Q_{-i})$ is strictly quasiconcave in the largest open interval $\widetilde{I}$ where $q_i > 0$ and $p > 0$. It suffices to show that $\partial^2\Pi_i/\partial q_i^2 < 0$ at any $q_i \in \widetilde{I}$ with $\partial\Pi_i/\partial q_i = 0$ (see, e.g., Diewert et al., 1981). So take $q_i \in \widetilde{I}$ with

$$0 = q_i p'(Q) + p(Q) - C_i'(q_i), \tag{29}$$

where $Q = q_i + Q_{-i}$. With $p(Q) > 0$ satisfied within $\widetilde{I}$, it follows from the assumption that

$$0 \geqq (\alpha - 1)p'(Q)^2 Q + p(Q)p''(Q)Q + (1-\beta)p(Q)p'(Q). \qquad (30)$$

Multiplying (29) through with $(1-\alpha)p'(Q)$, and (30) with $q_i/Q$, a simple addition yields

$$0 \geqq -(1-\alpha)p'(Q)C_i'(q_i) + p(Q)(q_i p''(Q) + p'(Q)\{(1-\alpha) + \frac{q_i}{Q}(1-\beta)\}). \quad (31)$$

For $\alpha \leqq 1$, the first term is nonnegative. Hence, noting that $q_i/Q \leqq 1$, and exploiting the remaining assumptions, one finds the second-order condition $0 > p''(Q)q_i + 2p'(Q) - C_i''(q_i)$. Thus, $\Pi_i$ is indeed strictly quasiconcave on $\widetilde{I}$. Since $\Pi_i$ is continuous in $q_i = 0$, the function $\Pi_i$ must also be strictly quasiconcave on $\widetilde{I} \cup \{0\}$. This proves the assertion in the case where $p > 0$. If, however, $p$ reaches zero at a finite output level, costs must be strictly increasing by assumption, so again $\Pi_i$ is strictly quasiconcave in $q_i$. The continuity of the best response follows now from the continuity of the profit function at positive prices. □

Using the smooth criterion, it is easy to check that concavity of $Qp(Q)$ is equivalent to $\Delta^p_{1,-1}(Q) \leqq 0$, while log-concavity of direct demand in log-price amounts to $\Delta^p_{0,0}(Q) \leqq 0$ (in fact, for arbitrary convex costs). Moreover, $pp'' - 2p'^2 \leqq 0$ captures $\Delta^p_{-1,1}(Q) \leqq 0$. Finally, log-concavity of inverse demand is tantamount to $\Delta^p_{0,1}(Q) \leqq 0$. Thus, ignoring regularity assumptions and boundary cases, Theorem 5.1 indeed covers and extends the various criteria mentioned at the beginning of this section.

Under very weak additional assumptions, the result above turns into an existence theorem complementing Theorem 3.2 in cases where profits are not necessarily ordinally submodular. Indeed, it is well-known that existence of a pure strategy equilibrium follows from (strict) quasiconcavity provided profits are continuous in $(q_i, Q_{-i})$ and effective choice sets are compact. These conditions are satisfied, e.g., if $p$ is continuous and zero above some finite $\overline{Q}$. In particular, Theorem 5.1 implies the main existence results currently known for convex costs, which require either $(-1)$-concave inverse demand or $(-1)$-concave direct demand (cf. Deneckere and Kovenock, 1999).

## 6. Uniqueness

The notion of biconcavity can also be used to unify and generalize various sufficient conditions for equilibrium uniqueness.[21] Indeed, as will be shown below, with suitably restricted biconcavity parameters, uniqueness obtains in the smooth model with strictly quasiconcave payoffs.[22] As in the previous section, profits are not necessarily ordinally submodular.

**Theorem 6.1.** *Let $p : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, twice continuously differentiable when positive with $p' < 0$, and $p(0) > 0$. Furthermore, for $i = 1, ..., N$, let $C_i$ be twice continuously differentiable and nondecreasing on $\mathbb{R}_+$. Assume that either $p > 0$ or that all $C_i$ are strictly increasing. For some $\alpha \leqq 1$, $\beta \leqq 1$ such that either (i) $\alpha + \beta = 1$ or (ii) $\alpha = 1$ and $\beta < 0$, assume that $\Delta^p_{\alpha,\beta} \leqq 0$ and $(\alpha + \beta)p' - C_i'' < 0$ at positive prices, for $i = 1, ..., N$. Then there is at most one Cournot-Nash equilibrium.*

**Proof.** We claim first that $p(Q) > 0$ in any equilibrium. Indeed, if $p(Q) = 0$, then necessarily $Q > 0$, so that some firm $i$ must be active, i.e., $q_i > 0$ for

[21] Necessary and sufficient conditions for uniqueness are derived by Kolstad and Mathiesen (1987) and Gaudet and Salant (1991).

[22] Szidarovszky and Yakowitz (1982) show that minimal differentiability conditions on demand are needed to ensure uniqueness.

some $i$. However, as inverse demand may vanish, $C_i$ is assumed strictly increasing. Hence, firm $i$ could profitably reduce own output, which is a contradiction. Thus, in any equilibrium, $p(Q) > 0$. Consider now an equilibrium $(q_1, ..., q_N)$. Since profit functions are differentiable at positive prices, there exist multipliers $u_1, ..., u_N$, such that

$$p(Q) + q_i p'(Q) - C_i'(q_i) + u_i = 0 \qquad (i = 1, ..., N), \tag{32}$$

$$u_i q_i = 0 \qquad (i = 1, ..., N), \tag{33}$$

$$u_i, q_i \geqq 0 \qquad (i = 1, ..., N), \tag{34}$$

$$\textstyle\sum_{i=1}^N q_i = Q, \text{ and } p(Q) > 0. \tag{35}$$

To prove uniqueness, it suffices to show that there is at most one solution $(q_1, ..., q_N; u_1, ..., u_N; Q)$ to the system (32-35). We exclude initially the possibility that $Q = 0$. For a given $Q > 0$ such that $p(Q) > 0$, consider the problem

$$\mathcal{P}(Q): \qquad \max_{q_1,...,q_N} p(Q)\textstyle\sum_{i=1}^N q_i + \frac{1}{2}p'(Q)\sum_{i=1}^N q_i^2 - \sum_{i=1}^N C_i(q_i)$$

$$\text{s.t.} \quad \text{(A)} \quad \textstyle\sum_{i=1}^N q_i = Q$$

$$(\text{B}_i) \quad q_i \geqq 0, \quad i = 1, ..., N.$$

Since by assumption, $p' - C_i'' < 0$ at positive prices, the objective function in problem $\mathcal{P}(Q)$ has a negative definite Hessian relative to $(q_1, ..., q_N)$. In particular, problem $\mathcal{P}(Q)$ has a unique solution $\{q_i^*(Q)\}_{i=1}^N$. This solution is characterized by the property that there exist multipliers $\lambda = \lambda(Q)$ and

$\widehat{u}_i = \widehat{u}_i(Q)$, $i = 1, ..., N$, such that Kuhn-Tucker conditions

$$p(Q) + q_i p'(Q) - C_i'(q_i) - \lambda + \widehat{u}_i = 0 \qquad (i = 1, ..., N) \tag{36}$$

$$\textstyle\sum_{i=1}^{N} q_i = Q \tag{37}$$

$$\widehat{u}_i q_i = 0 \qquad (i = 1, ..., N) \tag{38}$$

$$\widehat{u}_i, q_i \geqq 0 \qquad (i = 1, ..., N) \tag{39}$$

hold. Given our current assumption $Q > 0$, the multipliers $\lambda, \widehat{u}_1, ..., \widehat{u}_N$ are unique. Indeed, since $q_i > 0$ for some $i$, the corresponding $\widehat{u}_i$ vanishes, so that problem $\mathcal{P}(Q)$ defines $\lambda$, and also $\widehat{u}_j$ for $j \neq i$. By Gauvin and Janin (1988, Theorem 5.1), the optimal solution and Lagrangian multipliers are continuous as a function of $Q$, in the domain where $Q > 0$ and $p(Q) > 0$, and directional derivatives exist. Denote by $D_Q^+(.)$ and $D_Q^-(.)$, respectively, the right-hand and left-hand derivatives of the argument. It will be shown now that $D_Q^+(\lambda) < 0$ and $D_Q^-(\lambda) < 0$ whenever $\lambda = 0$. We confine ourselves to the right-hand derivative, since the argument for the left-hand derivative is very similar. The right-hand derivative of

$$\lambda(Q) = p(Q) + q_i^*(Q)p'(Q) - C_i'(q_i^*(Q)) + \widehat{u}_i(Q) \tag{40}$$

reads

$$D_Q^+(\lambda) = p'(Q) + q_i p''(Q) + (p'(Q) - C_i''(q_i^*(Q)))D_Q^+(q_i^*) + D_Q^+(\widehat{u}_i). \tag{41}$$

Write $K = \{i : D_Q^+(q_i^*) > 0\}$, and let $k = |K|$. Note that $k \geqq 1$ as a consequence of condition (A) in problem $\mathcal{P}(Q)$. From condition (38) and the continuity of $\widehat{u}_i(Q)$, it follows that $D_Q^+(\widehat{u}_i) = 0$ for any $i \in K$. Therefore, if

$i \in K$,

$$D_Q^+(\lambda) = p'(Q)\left[1 + (1-\alpha-\beta)D_Q^+(q_i^*)\right] + p''(Q)q_i \tag{42}$$
$$+ \left[(\alpha+\beta)p'(Q) - C_i''(q_i(Q))\right] D_Q^+(q_i^*)$$
$$< p'(Q)\left[1 + (1-\alpha-\beta)D_Q^+(q_i^*)\right] + p''(Q)q_i \tag{43}$$

Assume first $\alpha + \beta = 1$. Then, as in the proof of Theorem 5.1, the first-order condition for individual profit maximization combined with the biconcavity assumption implies $q_i p''(Q) + p'(Q) \leqq 0$. This settles the claim if $\alpha + \beta = 1$. Assume now $\alpha = 1$ and $\beta < 0$. Then, adding (41) over all $i \in K$, one obtains

$$kD_Q^+(\lambda) < p'(Q)\left[k - \beta \textstyle\sum_{i\in K} D_Q^+(q_i^*)\right] + p''(Q)\textstyle\sum_{i\in K} q_i \tag{44}$$
$$\leqq p'(Q)(1-\beta) + p''(Q)\textstyle\sum_{i\in K} q_i, \tag{45}$$

because $\sum_{i\in K} D_Q^+(q_i^*) \geqq 1$. If $\sum_{i\in K} q_i = 0$, then $D_Q^+(\lambda) < 0$. But if $\sum_{i\in K} q_i > 0$, then

$$\frac{Q}{\sum_{i\in K} q_i} kD_Q^+(\lambda) < (1-\beta)p'(Q) + Qp''(Q) \leqq 0, \tag{46}$$

where the second inequality follows from $\Delta_{1,\beta}^p \leqq 0$. Therefore, also in this case, $D_Q^+(\lambda) < 0$, proving the assertion. By the intermediate value theorem, $\lambda(Q)$ has at most one zero in the interval where $Q > 0$ and $p(Q) > 0$. Thus, there exists at most one equilibrium with $Q > 0$. Assume now that there is an equilibrium with aggregate output zero as well as an equilibrium with $Q > 0$. In the equilibrium with strictly positive aggregate output, there must be some firm $i$ with $q_i > 0$. By Theorem 5.1, profits are strictly quasi-concave in own output. In particular, $q_i p(q_i + Q_{-i}) - C_i(q_i) > -C_i(0)$, which implies $q_i p(q_i) - C_i(q_i) > -C_i(0)$. Hence, provided there is an equilibrium with strictly positive aggregate output, it cannot be optimal for a firm to remain inactive if all other firms are inactive. Thus, there is at most one Cournot-Nash equilibrium. □

Murphy et al. (1982) find conditions under which the Lagrangian multiplier $\lambda(Q)$ defined in the proof above is strictly decreasing in $Q$. The somewhat weaker conditions of Theorem 6.1 merely imply that the continuous function $\lambda(Q)$ is strictly declining at any zero.

Theorem 6.1 offers a unifying perspective on uniqueness results by Szidarovszky and Yakowitz (1982) for concave inverse demand, by Murphy et al. (1982) and Watts (1996) for concave industry revenues, by Amir (1996) and Deneckere and Kovenock (1999) for logconcave inverse demand, and by Vives (1999) for nonconvex costs.[23] An implication of Theorem 6.1 not covered by earlier work is the fact that the uniqueness condition $p' - C_i'' < 0$ works for equilibria with merely biconcave inverse demand, as captured by Theorem 3.2.

## Appendix

This appendix contains auxiliary results and the proof of Theorem 3.2.

**Lemma A.1.** *Consider a function $p$ that is twice differentiable when positive. Assume that the set $I_p = \{Q > 0 : p(Q) > 0\}$ is an interval. Then, for $\alpha, \beta \in \mathbb{R}$, the function $\varphi_\alpha(p(Q))$ is concave as a function of $\varphi_\beta(Q)$ if and only if the inequality*

$$(\alpha - 1)p'(Q)^2 Q + p(Q)p''(Q)Q + (1 - \beta)p(Q)p'(Q) \leqq 0 \qquad (47)$$

*holds for any $Q \in I_p$.*

[23] The reader is referred to the original papers for precise statements of these results.

**Proof.** By the standard differential condition, $\varphi_\alpha(p(Q))$ is concave as a function of $\varphi_\beta(Q)$ if and only if

$$\frac{d\varphi_\alpha(p(Q))}{d\varphi_\beta(Q)} = \frac{\varphi'_\alpha(p(Q))p'(Q)}{\varphi'_\beta(Q)} \tag{48}$$

is nonincreasing in $\varphi_\beta(Q)$, or equivalently, nonincreasing in $Q$. Differentiating the right-hand side of (48) with respect to $Q$ delivers that $\varphi_\alpha(p(Q))$ is concave as a function of $\varphi_\beta(Q)$ if and only if

$$\varphi'_\beta(Q)\varphi''_\alpha(p(Q))p'(Q)^2 + \varphi'_\beta(Q)\varphi'_\alpha(p(Q))p''(Q) - \varphi'_\alpha(p(Q))p'(Q)\varphi''_\beta(Q) \leqq 0 \tag{49}$$

for all $Q \in I_p$. Replacing the derivatives of $\varphi_\alpha$ and $\varphi_\beta$ by the respective explicit expressions, and subsequently multiplying through with $p(Q)^{2-\alpha}Q^{2-\beta}$ delivers (47). □

**Lemma A.2.** *Let $Y \geqq 0$. Then for any $\alpha \leqq 1$ [$\alpha \geqq 1$], the transformed translation mapping $\tau_Y : y \mapsto \varphi_\alpha(y + Y)$, defined on $\mathbb{R}_+$, is convex [concave] in $\varphi_\alpha(y)$.*

**Proof.** Using Lemma A.1, $\tau_Y$ is concave [convex] in $\varphi_\alpha(y)$ if and only if $(1-\alpha)Y \leqq 0$ [$\geqq 0$]. Thus, if $\alpha \leqq 1$, then $\tau_Y$ is convex in $\varphi_\alpha(y)$, and if $\alpha \geqq 1$, then $\tau_Y$ is concave in $\varphi_\alpha(y)$. □

**Lemma A.3.** *Assume that the function $p$ is $\alpha$-biconcave [$\alpha$-biconvex] and nonincreasing, for $\alpha \geqq 0$ [for $\alpha \leqq 0$]. Then $p(q_i + Q_i)$ is $\alpha$-biconcave [$\alpha$-biconvex] in $q_i$ for any $Q_i \geqq 0$.*

**Proof.** Assume first that $p$ is $\alpha$-biconcave with $\alpha \geqq 0$. Then $\varphi_\alpha(p(Q))$ is concave and nonincreasing in $\varphi_{1-\alpha}(Q)$ on $I_p$. Substituting $Q$ by $q_i + Q_i$, it follows trivially that $\varphi_\alpha(p(q_i+Q_i))$ is concave and nonincreasing in $\varphi_{1-\alpha}(q_i+$

$Q_i$) on $I_p - Q_i$. But by Lemma A.2, $\varphi_{1-\alpha}(q_i + Q_i)$ is convex in $\varphi_{1-\alpha}(q_i)$ provided $Q_i \geqq 0$. Since any composition of a concave and nonincreasing function with a convex function is concave, $\varphi_\alpha(p(q_i + Q_i))$ must be concave in $\varphi_{1-\alpha}(q_i)$ on $I_p$. Hence, $p(q_i+Q_i)$ is indeed $\alpha$-biconcave in $q_i$. The assertion for biconvex functions is proved in an analogous way. □

**Proof of Theorem 3.2.** To apply Kukushkin's (1994) theorem, we need to show that the best-response correspondence has a nonincreasing single-valued selection.[24] The proof has three steps. We prove first that $\min r_i(Q_{-i})$ is well-defined. We have argued in Section 4 that the effective choice set of the firms can w.l.o.g. be assumed bounded by some $\overline{Q} > 0$. I.e., given rivals' output $Q_{-i}$, a deviation to a choice outside the interval $[0, \overline{Q}]$ does never yield strictly higher profits. Clearly, the profit function is upper semicontinuous (u.s.c.) in own output, for any given level of rivals' output. Since any u.s.c. function on a compact set attains a maximum, and moreover, the set of maximizers is compact, $r_i(Q_{-i})$ is compact and non-empty for any $Q_{-i} \geqq 0$. Hence, $\min r_i(Q_{-i})$ is indeed well-defined. Next, we show that $\min r_i$ is nonincreasing. For this, take arbitrary quantities $\widehat{Q}_{-i} > Q_{-i} \geqq 0$ such that $p(\widehat{Q}_{-i}) > 0$ (obviously, for $p(\widehat{Q}_{-i}) = 0$, we have $\min r_i(\widehat{Q}_{-i}) = 0$, so there is nothing to show). Let $\widehat{q_i} = \min r_i(\widehat{Q}_{-i})$ and $q_i = \min r_i(Q_{-i})$. With nondecreasing costs, a positive production level leading to a zero market price is never a best response to $\widehat{Q}_{-i}$. Therefore, $p(\widehat{q_i} + \widehat{Q}_{-i}) > 0$. By assumption, there exists a parameter $\alpha \in [0, 1]$ such that $p$ is $\alpha$-biconcave. Lemma 4.1

[24] A direct application of Milgrom and Shannon's (1994) results does not seem feasible.

implies

$$\varphi_\alpha(R(\widehat{q}_i, Q_{-i})) - \varphi_\alpha(R(q_i, Q_{-i})) \geqq \varphi_\alpha(R(\widehat{q}_i, \widehat{Q}_{-i})) - \varphi_\alpha(R(q_i, \widehat{Q}_{-i})). \quad (50)$$

Using Lemma 4.2, one obtains

$$\Pi_i(q_i, Q_{-i}) - \Pi_i(\widehat{q}_i, Q_{-i}) \geqq 0 \Rightarrow \Pi_i(q_i, \widehat{Q}_{-i}) - \Pi_i(\widehat{q}_i, \widehat{Q}_{-i}) \geqq 0. \quad (51)$$

We wish to show that $q_i \geqq \widehat{q}_i$. To provoke a contradiction, assume $q_i < \widehat{q}_i$. Then, since $\widehat{q}_i$ is the smallest best response to $\widehat{Q}_{-i}$, necessarily $\Pi_i(q_i, \widehat{Q}_{-i}) < \Pi_i(\widehat{q}_i, \widehat{Q}_{-i})$. Hence, using the dual single-crossing property (51), it follows that $\Pi_i(q_i, Q_{-i}) < \Pi_i(\widehat{q}_i, Q_{-i})$, in contradiction to $q_i \in r_i(Q_{-i})$. Consequently, $q_i \geqq \widehat{q}_i$, as desired. Thus, $\min r_i(Q_{-i})$ is indeed nonincreasing in $Q_{-i}$. Finally, we prove existence. For this, one notes that, since $C_i$ is left-continuous and nondecreasing, $C_i$ is lower semi-continuous. Moreover, either $R$ is continuous in $(q_i, Q_{-i})$, viz. when $p$ is continuous at positive quantities, or $R$ is continuous in $(q_i, Q_{-i})$ in the closed domain where $p(q_i + Q_{-i}) > 0$, viz. when $p$ jumps to zero. In either case, profits are quasi-transfer upper continuous in $(q_i, Q_{-i})$ with respect to $[0, \overline{Q}]$. To see why, assume that

$$\Pi_i(\widehat{q}_i, Q_{-i}) > \Pi_i(q_i, Q_{-i}), \quad (52)$$

and consider a small perturbation of $(q_i, Q_{-i})$. If costs jump upwards through the perturbation, inequality (52) still holds. On the other hand, if revenues drop to zero through the perturbation, then necessarily $R(\widehat{q}_i, Q_{-i}) > 0$, hence $\widehat{q}_i > 0$. Since revenues are continuous in the domain of positive prices, we may replace $\widehat{q}_i$ by a smaller value so that (52) still holds. This shows that $\Pi_i$ is indeed quasi-transfer upper continuous. It follows now from Theorem 3 in

Tian and Zhou (1995) that $r_i$ has a closed graph when firms are restricted to choose a quantity from the compact interval $[0, \overline{Q}]$. Consequently, the assumptions of Kukushkin's (1994) fixed point theorem are satisfied in the restricted game. But a choice above $\overline{Q}$ can never increase profits. Thus, an equilibrium exists also in the unconstrained game. □

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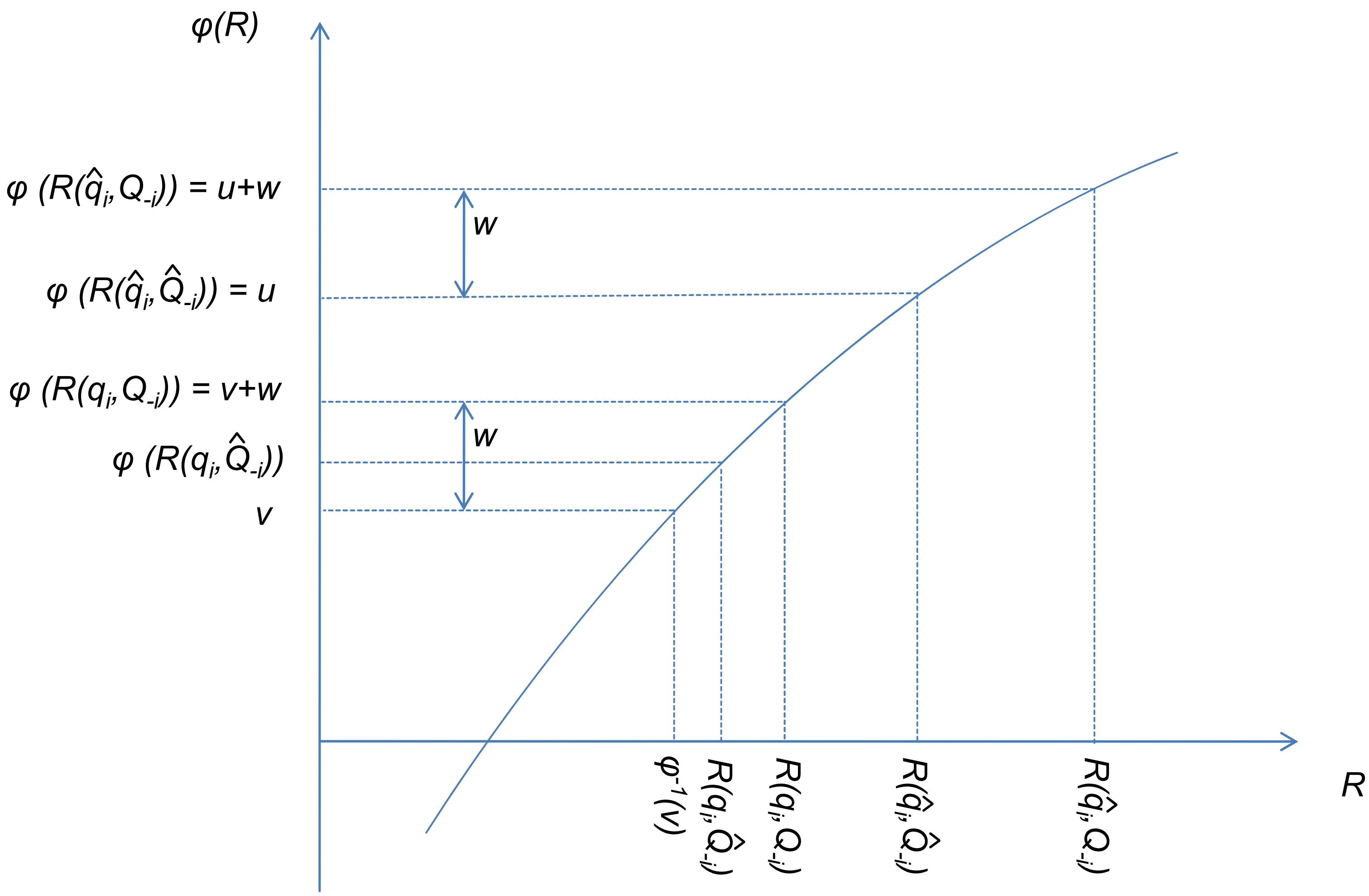

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Figure 1