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On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games*

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Abstract

We generalize Athey's (2001) and McAdams' (2003) results on the existence of monotone pure strategy equilibria in Bayesian games. We allow action spaces to be compact locally-complete metrizable semilattices and type spaces to be partially ordered probability spaces. Our proof is based upon contractibility rather than convexity of best reply sets. Several examples illustrate the scope of the result, including new applications to multi-unit auctions with risk-averse bidders.

1. Introduction

In an important paper, Athey (2001) demonstrates that a monotone pure strategy equilibrium exists whenever a Bayesian game satisfies a Spence-Mirlees single-crossing property. Athey's result is now a central tool for establishing the existence of monotone pure strategy equilibria in auction theory (see e.g., Athey (2001), Reny and Zamir (2004)). Recently, McAdams (2003) has shown that Athey's results, which exploit the assumed total ordering of the players' one-dimensional type and action spaces, can be extended to settings in which type and action spaces are multi-dimensional and only partially ordered. This permits new existence results in auctions with multi-dimensional types and multi-unit demands (see McAdams (2004)). The techniques employed by Athey and McAdams, while ingenious, have their limitations and do not appear to easily extend beyond the environments they consider. We therefore introduce a new approach.

The approach taken here exploits an important unrecognized property of a large class of Bayesian games. In these games, the players' pure-strategy best-reply sets, while possibly

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nonconvex, are always contractible.[1] This observation permits us to generalize the results of Athey and McAdams in several directions. First, we permit infinite-dimensional type spaces and infinite-dimensional action spaces. Both can occur, for example, in share-auctions where a bidder's type is a function expressing his marginal valuation at any quantity of the good, and where a bidder's action is a downward-sloping demand schedule. Second, even when type and action spaces are subsets of Euclidean space, we permit more general joint distributions over types, allowing one player to have private information about the support of another's private information, as well as permitting positive probability on lower dimensional subsets, which can be useful when modeling random demand in auctions. Third, our approach allows general partial orders on both type spaces and action spaces. This can be especially helpful because, while single-crossing may fail for one partial order, it might nonetheless hold for another, in which case our existence result can still be applied (see section 5 for two such applications). Finally, while single-crossing is helpful in establishing the hypotheses of our main theorem, it is not necessary; our hypotheses are satisfied even in instances where single-crossing fails.

The key to our approach is to employ a more powerful fixed point theorem than those employed in Athey (2001) and McAdams (2003). Both Athey and McAdams apply a fixed-point theorem to the product of the players' best-reply correspondences — Athey applies Kakutani's theorem, McAdams applies Glicksberg's theorem. In both cases, essentially all of the effort is geared toward proving that sets of monotone pure-strategy best replies are convex. Our central observation is that this impressive effort is unnecessary and, more importantly, that the additional structure imposed to achieve the desired convexity (i.e., *Euclidean* type spaces with the *coordinatewise* partial order, *Euclidean sublattice* action spaces, *absolutely continuous* type distributions), is unnecessary as well.

The fixed point theorem upon which our approach is based is due to Eilenberg and Montgomery (1946) and does not require the correspondence in question to be convex-valued. Rather, the correspondence need only be contractible-valued. Consequently, we need only demonstrate that monotone pure-strategy best-reply sets are contractible. While this task need not be straightforward in general, it turns out to be essentially trivial in the class of Bayesian games of interest here. To gain a sense of this, note first that a pure strategy — a function from types to actions — is a best reply for a player if and only if it is a pointwise interim best reply for almost every type of that player. Consequently, any piecewise combination of two best replies — i.e., a strategy equal to one of the best replies on some subset of types and equal to the other best reply on the remainder of types — is also a best reply. Thus, by reducing the set of types on which the first best reply is employed

[1] A set is contractible if it can be continuously deformed, within itself, to a single point. Convex sets are contractible, but contractible sets need not be convex (e.g., the symbol "+" viewed as a subset of $\mathbb{R}^2$).

and increasing the set of types on which the second is employed, it is possible to move from the first best reply to the second, all the while remaining within the set of best replies. With this simple observation, the set of best replies can be shown to be contractible.[2]

Because contractibility of best-reply sets follows almost immediately from the pointwise almost everywhere optimality of interim best replies, we are able to expand the domain of analysis well beyond Euclidean type and action spaces, and most of our additional effort is directed here. In particular, we require and prove two new results about the space of monotone functions from partially ordered probability spaces into compact metric semilattices. The first of these results (Lemma A.10) is a generalization of Helly's selection theorem stating that, under suitable conditions, any sequence of monotone functions possesses a pointwise almost everywhere convergent subsequence. The second result (Lemma A.16) provides conditions under which the space of monotone functions is an absolute retract, a property that, like convexity, renders a space amenable to fixed point analysis.

Our main result, Theorem 4.1, is as follows. Suppose that action spaces are compact convex semilattices or compact locally-complete metric semilattices, that type spaces are partially ordered probability spaces, that payoffs are continuous in actions for each type vector, and that the joint distribution over types induces atomless marginals for each player assigning positive probability only to sets that can be order-separated by a fixed countable set of his types.[3] If, whenever the others employ monotone pure strategies, each player's set of monotone pure-strategy best replies is nonempty and join-closed,[4] then a monotone pure strategy equilibrium exists.

We provide several applications yielding new existence results. First, we consider both uniform-price and discriminatory multi-unit auctions with independent private values. We depart from standard assumptions by permitting bidders to be risk averse. Under risk aversion, monotonicity of best replies is known to fail under the standard coordinatewise partial order over types. Nevertheless, by employing an alternative, yet natural, partial order over types, we are able to demonstrate the existence of a monotone pure strategy equilibrium with respect to this partial order. In the uniform-price auction no additional assumptions are required, while in the discriminatory auction each bidder is assumed to have CARA preferences. Another application considers a price-competition game between firms selling differentiated products. Firms have private information about their constant marginal

[2] Because we are concerned with *monotone* pure strategy best replies, some care must be taken to ensure that one maintains monotonicity throughout the contraction. Further, continuity of the contraction requires appropriate assumptions on the distribution over players' types. In particular there can be no atoms.

[3] One set is order-separated by another if the one set contains two points between which lies a point in the other.

[4] A subset of strategies is join-closed if the pointwise supremum of any pair of strategies in the set is also in the set.

cost as well as private information about market demand. While it is natural to assume that costs may be affiliated, in the context we consider it is less natural to assume that information about market demand is affiliated. Nonetheless, and again through a judicious choice of a partial order over types, we are able to establish the existence of a pure strategy equilibrium that is monotone in players' costs, but not necessarily monotone in their private information about demand. Our final application establishes the existence of monotone *mixed* strategy equilibria when type spaces have atoms.[5]

If the actions of distinct players are strategic complements – an assumption we do not impose – Van Zandt and Vives (2006) have shown that even stronger results can be obtained. They prove that monotone pure strategy equilibria exist under somewhat more general distributional and action-space assumptions than we employ here, and demonstrate that such an equilibrium can be obtained through iterative application of the best reply map.[6] Van Zandt and Vives (2006) obtain perhaps the strongest possible results for the existence of monotone pure strategy equilibria in Bayesian games when strategic complementarities are present. Of course, while many interesting economic games exhibit strategic complements, many do not. Indeed, many auction games satisfy the hypotheses required to apply our result here, but fail to satisfy the strategic complements condition.[7] The two approaches are therefore complementary.

The remainder of the paper is organized as follows. Section 2 presents the essential ideas as well as the corollary of Eilenberg and Montgomery's (1946) fixed point theorem that is central to our approach. Section 3 describes the formal environment, including semilattices and related issues. Section 4 contains our main result, section 6 contains its proof, and section 5 provides several applications.

2. The Main Idea

As already mentioned, the proof of our main result is based upon a fixed point theorem that permits the correspondence for which a fixed point is sought — here, the product of the players' monotone pure best reply correspondences — to have contractible rather than convex values.

In this section, we introduce this fixed point theorem and also illustrate the ease with

[5] A player's mixed strategy is monotone if every action in the totally ordered support of one of his types is greater or equal to every action in the totally ordered support of any lower type.

[6] Related results can be found in Milgrom and Roberts (1990) and Vives (1990).

[7] In a first-price IPV auction, for example, a bidder might increase his bid if his opponent increases her bid slightly when her private value is high. However, for sufficiently high increases in her bid at high private values, the bidder might be better off reducing his bid (and chance of winning) to obtain a higher surplus when he does win. Such strictly optimal nonmonotonic responses to increases in the opponent's strategy are not possible under strategic complements.

which contractibility can be established, focussing on the most basic case in which type spaces are $[0,1]$, action spaces are subsets of $[0,1]$, and the marginal distribution over each player's type space is atomless.

A subset X of a metric space is *contractible* if for some $x_0 \in X$ there is a continuous function $h : [0,1] \times X \to X$ such that for all $x \in X$, $h(0,x) = x$ and $h(1,x) = x_0$. We then say that h is a *contraction* for X.

Note that every convex set is contractible since, choosing any point x_0 in the set, the function $h(\tau, x) = (1-\tau)x + \tau x_0$ is a contraction. On the other hand, there are contractible sets that are not convex (e.g., the symbol "+"). Hence, contractibility is a strictly more permissive condition than convexity.

A subset X of a metric space Y is said to be a *retract* of Y if there is a continuous function mapping Y onto X leaving every point of X fixed. A metric space (X,d) is an *absolute retract* if for every metric space (Y,δ) containing X as a closed subset and preserving its topology, X is a retract of Y. Examples of absolute retracts include closed convex subsets of Euclidean space or of any metric space, and many nonconvex sets as well (e.g., any contractible polyhedron).[8] The fixed point theorem we make use of is the following corollary of an even more general result due to Eilenberg and Montgomery (1946).[9]

Theorem 2.1. *Suppose that a compact metric space (X,d) is an absolute retract and that $F : X \twoheadrightarrow X$ is an upper-hemicontinuous, nonempty-valued, contractible-valued correspondence.*[10] *Then F has a fixed point.*

For our purposes, the correspondence F is the product of the players' monotone pure strategy best reply correspondences and X is the product of their sets of monotone pure strategies. While we must eventually establish all of the properties necessary to apply Theorem 2.1, our modest objective for the remainder of this section is to show, with remarkably little effort, that in the simple environment considered here, F is contractible-valued, i.e., that monotone pure best reply sets are contractible.

Suppose that player 1's type is drawn uniformly from the unit interval $[0,1]$. Fix monotone pure strategies for other players, and suppose that $\bar{s} : [0,1] \to A$ is a monotone best reply for player 1, where $A \subseteq [0,1]$ is player 1's compact action set. Indeed, suppose that $\bar{s}$ is player 1's *largest* monotone best reply in the sense that if s is any other monotone

[8] Indeed, a compact subset, X, of Euclidean space is an absolute retract if and only if it is contractible and locally contractible. The latter means that for every $x_0 \in X$ and every neighborhood U of x_0, there is a neighborhood V of x_0 and a continuous $h : [0,1] \times V \to U$ such that $h(0,x) = x$ and $h(1,x) = x_0$ for all $x \in V$.

[9] Theorem 2.1 follows directly from Eilenberg and Montgomery (1946) Theorem 1, because every absolute retract is a contractible absolute neighborhood retract (Borsuk (1966), V (2.3)) and every nonempty contractible set is acyclic (Borsuk (1966), II (4.11)).

[10] By upper-hemicontinuous, we shall always mean that the correspondence in question has a closed graph.

best reply, then $\bar{s}(t) \geq s(t)$ for every type t of player 1. We shall provide a contraction that shrinks player 1's entire set of monotone best replies, within itself, to the largest monotone best reply $\bar{s}$. The simple, but key, observation is that a pure strategy is a best reply for player 1 if and only if it is a pointwise best reply for almost every type $t \in [0,1]$ of player 1.

Consider the following candidate contraction. For $\tau \in [0,1]$ and any monotone best reply, s, for player 1, define $h(\tau, s) : [0,1] \to A$ as follows:

$$h(\tau, s)(t) = \begin{cases} s(t), & \text{if } t \leq 1 - \tau \text{ and } \tau < 1 \\ \bar{s}(t), & \text{otherwise.} \end{cases}$$

Note that $h(0,s) = s$, $h(1,s) = \bar{s}$, and $h(\tau, s)(t)$ is always either $\bar{s}(t)$ or $s(t)$ and so is a best reply for almost every t. Hence, by the key observation in the previous paragraph, $h(\tau, s)(\cdot)$ is a best reply. The pure strategy $h(\tau, s)(\cdot)$ is monotone because it is the smaller of two monotone functions for low values of t and the larger of them for high values of t. Moreover, because the marginal distribution over player 1's type is atomless, the monotone pure strategy $h(\tau, s)(\cdot)$ varies continuously in the arguments τ and s, when the distance between two strategies of player 1 is defined to be the integral with respect to his type distribution of their absolute pointwise difference (see section 6).[11] Consequently, h is a contraction under this metric, and so player 1's set of monotone best replies is contractible. It's that simple.

Figure 2.1 shows how the contraction works when player 1's set of actions A happens to be finite, so that his set of monotone best replies cannot be convex in the usual sense unless it is a singleton. Three monotone functions are shown in each panel, where 1's actions are on the vertical axis and 1's types are on the horizontal axis. The dotted line step function is s, the solid line step function is $\bar{s}$, and the thick solid line step function (red) is the step function determined by the contraction h.

In panel (a), $\tau = 0$ and h coincides with s. The position of the vertical line (blue) appearing in each panel represents the value of τ. The vertical line (blue) appearing in each panel intersects the horizontal axis at the point $1 - \tau$. When $\tau = 0$ the vertical line is at the far right-hand side, as shown in panel (a). As indicated by the arrow, the vertical line moves continuously toward the origin as τ moves from 0 to 1. The thick (red) step function determined by the contraction h is $s(t)$ for values of t to the left of the vertical line and is $\bar{s}(t)$ for values of t to the right; see panels (b) and (c). The step function h therefore changes continuously with τ because the areas between strategies change continuously. In panel (d), $\tau = 1$ and h coincides with $\bar{s}$. So altogether, as τ moves continuously from 0 to 1, the image of the contraction moves continuously from s to $\bar{s}$.

[11] This particular metric is important because it renders a player's payoff continuous in his strategy choice.

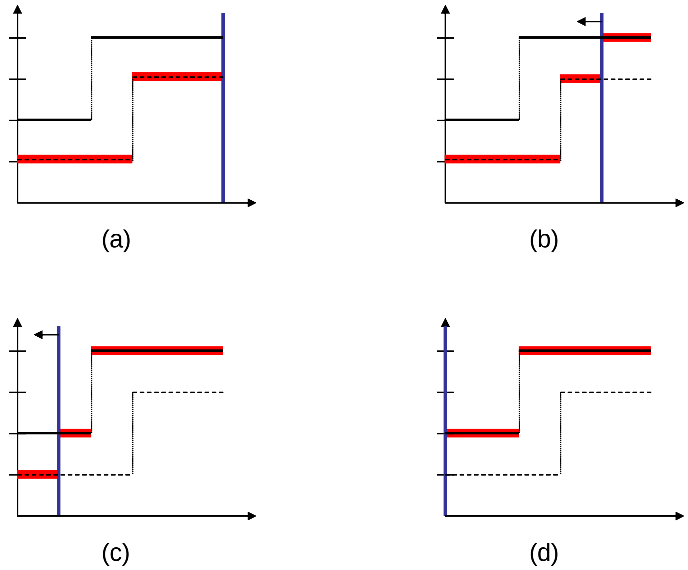

Figure 2.1: The Contraction

Two points are worth mentioning before moving on. First, single-crossing plays no role in establishing the contractibility of sets of monotone best replies. As we shall see, ensuring the existence of monotone pure strategy best replies is where single-crossing can be helpful. Thus, the present approach clarifies the role of single-crossing insofar as the existence of monotone pure strategy equilibrium is concerned.[12] Second, the action spaces employed in the above illustration are totally ordered, as in Athey (2001). Consequently, if two actions are optimal for some type of player 1, then the maximum of the two actions, being one or the other of them, is also optimal. The optimality of the maximum of two optimal actions is important for ensuring that a largest monotone best reply exists. When action spaces are only partially ordered (e.g., when actions are multi-dimensional with, say, the coordinatewise partial order), the maximum of two optimal actions need not even be well-defined, let alone optimal. Therefore, to also cover partially ordered action spaces, we assume in the sequel (see section 3) that action spaces are semilattices — i.e., that for every pair of actions there is a least upper bound (l.u.b.) — and that the l.u.b. of two optimal actions is optimal. Stronger versions of both assumptions are employed in McAdams (2003).

[12]Both Athey (2001) and McAdams (2003) employ single-crossing to help establish the existence of monotone best replies *and* to establish the convexity of the set of monotone best replies. Their single-crossing conditions are therefore more restrictive than necessary. See Subsection 4.1.

3. The Environment

3.1. Partial Orders, Lattices and Semilattices

Let A be a nonempty set partially ordered by $\geq$.[13] If A is endowed with a sigma-algebra of subsets $\mathcal{A}$, then the partial order $\geq$ on A is called *measurable* if $\{(a, b) \in A \times A : b \geq a\}$ is a member of $\mathcal{A} \times \mathcal{A}$.[14] If A is endowed with a topology, then the partial order $\geq$ on A is called *closed* if $\{(a, b) \in A \times A : b \geq a\}$ is closed in the product topology. The partial order $\geq$ on A is called *convex* if $\{(a, b) \in A \times A : b \geq a\}$ is convex. Note that if the partial order on A is convex then A is convex because $a \geq a$ for every $a \in A$. Say that A is *upper-bound-convex* if it contains the convex combination of any two members whenever one of them, $\bar{a}$ say, is an upper bound for A – i.e., $\bar{a} \geq a$ for every $a \in A$.[15] Every convex set is upper-bound-convex.

For $a, b \in A$, if the set $\{a, b\}$ has a least upper bound (l.u.b.) in A, then it is unique and will be denoted by $a \vee b$, the *join* of a and b. In general, such a bound need not exist. However, if every pair of points in A has an l.u.b. in A, then we shall say that A is a *semilattice.* It is straightforward to show that, in a semilattice, every finite set, $\{a, b, ..., c\}$, has a least upper bound, which we denote by $\vee\{a, b, ..., c\}$ or $a \vee b \vee ... \vee c$.

If the set $\{a, b\}$ has a greatest lower bound (g.l.b.) in A, then it too is unique and it will be denoted by $a \wedge b$, the *meet* of a and b. Once again, in general, such a bound need not exist. If every pair of points in A has both an l.u.b.. in A and a g.l.b. in A, then we say that A is a *lattice.*[16]

Clearly, every lattice is a semilattice. However, the converse is not true. For example, employing the coordinatewise partial order on vectors in $\mathbb{R}^m$, the set of vectors whose sum is at least one is a semilattice, but not a lattice.

A *metric semilattice* is a semilattice, A, endowed with a metric under which the join operator, $\vee$, is continuous as a function from $A \times A$ into A. In the special case in which A is a metric semilattice in $\mathbb{R}^m$ under the Euclidean metric, we say that A is a *Euclidean metric semilattice.* Note also that because in a semilattice $b \geq a$ if and only if $a \vee b = b$, a partial order in a metric semilattice is necessarily closed.[17]

A semilattice A is *complete* if every nonempty subset S of A has a least upper bound, $\vee S$, in A. A metric semilattice A is *locally complete* if for every $a \in A$ and every neighborhood

[13] Hence, $\geq$ is transitive ($a \geq b$ and $b \geq c$ imply $a \geq c$), reflexive ($a \geq a$), and antisymmetric ($a \geq b$ and $b \geq a$ imply $a = b$).

[14] Recall that $\mathcal{A} \times \mathcal{A}$ is the smallest sigma algebra containing all sets of the form $B \times C$ with B, C in $\mathcal{A}$.

[15] Sets without upper bounds are trivially upper-bound-convex.

[16] Defining a semilattice in terms of the join operator, $\vee$, rather than the meet operator, $\wedge$, is entirely a matter of convention.

[17] The converse can fail. For example, the set $A = \{(x, y) \in \mathbb{R}^2_+ : x + y = 1\} \cup \{(1, 1)\}$ is a semilattice with the coordinatewise partial order, and this order is closed under the Euclidean metric. But A is not a metric semilattice because whenever $a_n \neq b_n$ and $a_n, b_n \to a$, we have $(1, 1) = \lim(a_n \vee b_n) \neq (\lim a_n) \vee (\lim b_n) = a$.

U of a, there is a neighborhood W of a contained in U such that every nonempty subset S of W has a least upper bound, $\vee S$, contained in U. Lemma A.18 establishes that a compact metric semilattice A is locally complete if and only if for every $a \in A$ and every sequence $a_n \to a$, $\lim_m(\vee_{n \geq m} a_n) = a$.[18] A distinct sufficient condition for local completeness is given in Lemma A.20.

Some examples of compact locally-complete metric semilattices are,

- finite semilattices
- compact sublattices of $\mathbb{R}^m$– because the join of any two points is their coordinatewise maximum
- compact Euclidean metric semilattices (Lemma A.19)
- compact upper-bound-convex semilattices in $\mathbb{R}^m$ endowed with the coordinatewise partial order (Lemmas A.17 and A.19)
- The space of continuous functions $f : [0,1] \to [0,1]$ satisfying for some $\lambda > 0$ the Lipschitz condition $|f(x) - f(y)| \leq \lambda |x - y|$, endowed with the maximum norm $\|f\| = \max_x |f(x)|$, and partially ordered by $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [0,1]$.

The last example is an infinite dimensional compact locally-complete metric semilattice. In general, and unlike compact Euclidean metric semilattices, infinite dimensional metric semilattices need not be locally complete even if compact and convex.[19]

Finally, if a, b, and c are members of a partially ordered set, we say that b *lies between* a *and* c if $a \geq b \geq c$.

3.2. A Class of Bayesian Games

There are N players, $i = 1, 2, ..., N$. Player i's type space is T_i and his action space is A_i, and both are nonempty and partially ordered. Unless a notational distinction is helpful, all partial orders, although possibly distinct, will be denoted by $\geq$. Player i's payoff function is $u_i : A \times T \to \mathbb{R}$, where $A = \times_{i=1}^N A_i$ and $T = \times_{i=1}^N T_i$. For each player i, $\mathcal{T}_i$ is a sigma-algebra of subsets of T_i, and members of $\mathcal{T}_i$ will often be referred to simply as *measurable sets*. The common prior over the players' types is a countably additive probability measure μ defined on $\mathcal{T}_1 \times ... \times \mathcal{T}_N$. Let G denote this Bayesian game.

[18]Hence, compactness and metrizability of a lattice under the order topology (see Birkohff (1967, p.244) are sufficient, but not necessary, for local completeness of the corresponding semilattice.

[19]No $\mathcal{L}_p$ space is locally complete when $p < +\infty$ and endowed with the pointwise partial order. See Hart and Weiss (2005) for a compact metric semilattice that is not locally complete. Their example can be modified so that the space is in addition convex and locally convex.

We shall make use of the following additional assumptions, where μ_i denotes the marginal of μ on T_i. Hence, the domain of μ_i is $\mathcal{T}_i$. For every player i,

G.1 The partial order on T_i is measurable.

G.2 The probability measure μ_i on $\mathcal{T}_i$ is atomless.

G.3 There is a countable subset T_i^0 of T_i such that every set in $\mathcal{T}_i$ assigned positive probability by μ_i contains two points between which lies a point in T_i^0.

G.4 A_i is a compact metric space and a semilattice with a closed partial order.[20]

G.5 Either (i) A_i is a convex and locally convex topological space and the partial order on A_i is convex, or (ii) A_i is a locally-complete metric semilattice.[21]

G.6 $u_i(a,t)$ is bounded, jointly measurable, and continuous in $a \in A$ for every $t \in T$.

Assumptions G.1-G.6 strictly generalize the assumptions in Athey (2001) and McAdams (2003) who assume that each A_i is a compact sublattice of Euclidean space and hence a compact locally-complete metric semilattice, that each T_i is a Euclidean cube $[0,1]^{m_i}$ endowed with the coordinatewise partial order, and that μ is absolutely continuous with respect to Lebesgue measure.[22,23] This additional structure is necessary for their Kakutani-Glicksberg-based approach.[24]

In addition to permitting infinite-dimensional type spaces, assumption G.1 permits the partial order on player i's type space to be distinct from the usual coordinatewise partial order when T_i is Euclidean. As we shall see, this flexibility is very helpful in providing new equilibrium existence results for multi-unit auctions with risk averse bidders.

[20] Note that G.4 does not require A_i to be a metric semilattice – its join operator need not be continuous.

[21] It is permissible for (i) to hold for some players and (ii) to hold for others. A topological space is convex if the operation of taking convex combinations of pairs of points yields a point in the space and is jointly continuous in the pair of points and in the weights on them. A topological space is locally convex if for every open set U, every point in U has a convex open neighborhood contained in U.

[22] McAdams (2003) assumes, further, that the joint density over types is everywhere strictly positive.

[23] If μ is absolutely continuous and each $T_i = [0,1]^{m_i}$, then let T_i^0 be the set of points in T_i with rational coordinates. Consequently, if $\mu_i(B) > 0$, then by Fubini's theorem there exists $t_i \in (0,1)^{m_i}$ such that $B \cap [0,1]t_i$ contains a continuum of members, any two of which define an interval containing a member of T_i^0. Hence, G.3 holds.

[24] Indeed, suppose a player's action set is the semilattice $A = \{(1,0),(1/2,1/2),(0,1),(1,1)\}$ in $\mathbb{R}^2$, with the coordinatewise partial order and note that A is not a sublattice of $\mathbb{R}^2$. It is not difficult to see that this player's set of monotone pure strategies from $[0,1]$ into A, endowed with the metric $d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$, is homeomorphic to three line segments joined at a common endpoint. Consequently, this strategy set is not homeomorphic to a convex set and so neither Kakutani's nor Glicksberg's theorems can be directly applied. On the other hand, this strategy set is an absolute retract (see Lemma A.16), which is sufficient for our approach.

Assumption G.2 is used to establish the contractibility of the players' sets of monotone best replies and in particular to construct an associated contraction that is continuous in a topology in which payoffs are continuous as well.

Assumption G.3 connects the partial order on a player's type space with his marginal distribution, and it implies in particular that no atomless subset of a player's type space having positive probability can be totally unordered. For example, if $T_i = [0,1]^2$ is endowed with the Borel sigma-algebra and the coordinatewise partial order, G.3 requires μ_i to assign probability zero to any atomless negatively sloped line in T_i. In fact, whenever T_i happens to be a separable metric space and $\mathcal{T}_i$ contains the open sets, G.3 holds if every atomless set having positive μ_i-measure contains two "strictly ordered" points (Lemma A.21).[25]

Together with G.1 and G.4, G.3 ensures the compactness of the players' sets of monotone pure strategies (Lemma A.10) in a topology in which payoffs are continuous.[26] Thus, although G.3 is logically unrelated to Milgrom and Weber's (1985) absolute-continuity assumption on the joint distribution over types, it plays the same compactness role for monotone pure strategies as the Milgrom-Weber assumption plays for distributional strategies.[27,28]

Assumption G.5 is used in ensuring that the set of monotone pure strategies is an absolute retract and therefore amenable to fixed point analysis.

Assumption G.6 is used to ensure that best replies are well defined and that best-reply correspondences are upper-hemicontinuous. Assumption G.6 is trivially satisfied when action spaces are finite. Thus, for example, it is possible to consider auctions here by supposing that players' bid spaces are discrete. We do so in section 5.

As functions from types to actions, best replies for any player i are determined only up to μ_i measure zero sets. This leads us to the following definitions. A *pure strategy* for player i is a function, $s_i : T_i \to A_i$, that is μ_i-a.e. (almost-everywhere) equal to a measurable function, and is *monotone* if $t_i' \geq t_i$ implies $s_i(t_i') \geq s_i(t_i)$ for all $t_i, t_i' \in T_i$.[29,30] Let S_i denote player

[25] Two points in a partially ordered metric space are strictly ordered if they are contained in disjoint open sets such that every point in one set is greater or equal to every point in the other.

[26] Indeed, without G.3, a player's type space could be the negative diagonal in $[0,1]^2$ endowed with the coordinatewise partial order. But then every measurable function from types to actions would be monotone because no two distinct types are ordered. Compactness in a useful topology is then effectively precluded.

[27] To see that even G.2 and G.3 together do not imply the Milgrom and Weber (1985) restriction that μ is absolutely continuous with respect to the product of its marginals $\mu_1 \times ... \times \mu_n$, note that G.2 and G.3 hold when there are two players, each with unit interval type space, and where types are drawn according to Lebesgue measure conditional on any one of finitely many positively or negatively sloped lines in the unit square.

[28] One might wonder whether G.3 can be weakened by requiring instead merely that every atomless set in $\mathcal{T}_i$ assigned positive probability by μ_i contains two distinct ordered points. The answer is "no," in the sense that this weakening permits examples in which every measurable function from $[0,1]$ into $[0,1]$ is monotone, precluding compactness of the set of monotone pure strategies in a useful topology.

[29] Recall that a property $P(t_i)$ holds μ_i-a.e. if the set of t_i for which $P(t_i)$ holds contains a measurable subset having μ_i-measure one.

[30] Because under G.4 A_i is a metric space, we willl always endow A_i with the Borel sigma-algebra. Thus

i's set of pure strategies and let $S = \times_{i=1}^{N} S_i$.

A vector of pure strategies, $(\hat{s}_1, ..., \hat{s}_N) \in S$ is an *equilibrium* if for every player i and every pure strategy s_i' for player i,

$$\int_T u_i(\hat{s}(t), t) d\mu(t) \geq \int_T u_i(s_i'(t_i), \hat{s}_{-i}(t_{-i}), t) d\mu(t),$$

where the left-hand side, henceforth denoted by $U_i(\hat{s})$, is player i's payoff given the joint strategy $\hat{s}$, and the right-hand side is his payoff when he employs s_i' and the others employ $\hat{s}_{-i}$.

It will sometimes be helpful to speak of the payoff to player i's type t_i from the action a_i given the strategies of the others, s_{-i}. This payoff, which we will refer to as i's *interim* payoff, is

$$V_i(a_i, t_i, s_{-i}) \equiv \int_T u_i(a_i, s_{-i}(t_{-i}), t) d\mu_i(t_{-i}|t_i),$$

where $\mu_i(\cdot|t_i)$ is a version of the conditional probability on T_{-i} given t_i. A single such version is fixed for each player i once and for all.

4. The Main Result

Call a subset of player i's pure strategies *join-closed* if for any pair of strategies, s_i, s_i', in the subset, the strategy taking the action $s_i(t_i) \vee s_i'(t_i)$ for each $t_i \in T_i$ is also in the subset.[31] We can now state our main result, whose proof is provided in section 6.

Theorem 4.1. *If G.1-G.6 hold, and each player's set of monotone pure best replies is non-empty and join-closed whenever the others employ monotone pure strategies, then G possesses a monotone pure strategy equilibrium.*

A strengthening of Theorem 4.1 can be helpful when one wishes to demonstrate not merely the existence of a monotone pure strategy equilibrium but the existence of a monotone pure strategy equilibrium within a particular subset of strategies. For example, in a uniform-price auction for m units, a strategy mapping a player's m-vector of marginal values into a vector of m bids is undominated only if his bid for a kth unit is no greater than his marginal value for a kth unit. As formulated, Theorem 4.1 does not directly permit one to

measurable subsets of A_i are the Borel subsets.

[31] Note that when the join operator is continuous, as it is in a metric semilattice, the resulting function is a.e.-measurable, being the composition of a.e.-measurable and continuous functions. But even when the join operator is not continuous, because the join of two monotone pure strategies is monotone, it is a.e.-measurable under the hypotheses of Lemma A.11.

demonstrate the existence of an undominated equilibrium.[32] The next result takes care of this. Its proof is a straightforward extension of the proof of Theorem 4.1, and is provided in Remark 8.

A subset of player i's pure strategies is called *pointwise-limit-closed* if whenever $s_i^1, s_i^2, \ldots$ are each in the set and $s_i^n(t_i) \to_n s_i(t_i)$ for μ_i almost-every $t_i \in T_i$, then s_i is also in the set. A subset of player i's pure strategies is called *piecewise-closed* if whenever s_i and s_i' are in the set, then so is any strategy s_i'' such that for every $t_i \in T_i$ either $s_i''(t_i) = s_i(t_i)$ or $s_i''(t_i) = s_i'(t_i)$.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, if for each player i, C_i is a join-closed, piecewise-closed and pointwise-limit-closed subset of pure strategies containing at least one monotone pure strategy, and the intersection of C_i with i's set of monotone pure best replies is nonempty whenever every other player j employs a monotone pure strategy in C_j, then G possesses a monotone pure strategy equilibrium in which each player i's pure strategy is in C_i.*

Remark 1. *When player i's action space is a semilattice with a closed partial order (as implied by G.4) and C_i is defined by any collection of weak inequalities, i.e., if $\mathcal{F}_i$ and $\mathcal{G}_i$ are arbitrary collections of measurable functions from T_i into A_i and $C_i = \cap_{f \in \mathcal{F}_i, g \in \mathcal{G}_i} \{s_i \in S_i : g(t_i) \leq s_i(t_i) \leq f(t_i)$ for μ_i a.e. $t_i \in T_i\}$, then C_i is join-closed, piecewise-closed and pointwise-limit-closed.*

The next section provides conditions that are sufficient for the hypotheses of Theorem 4.1.

4.1. Sufficient Conditions

Both Athey (2001) and McAdams (2003), within the confines of a lattice, make use of quasisupermodularity and single-crossing conditions on interim payoffs. We now provide weaker versions of both of these conditions, as well as single condition that is weaker than their combination.

Suppose that player i's action space, A_i, is a lattice. We say that player i's interim payoff function V_i is *weakly quasisupermodular* if for all monotone pure strategies s_{-i} of the others, all $a_i, a_i' \in A_i$, and every $t_i \in T_i$,

$$V_i(a_i, t_i, s_{-i}) \geq V_i(a_i \wedge a_i', t_i, s_{-i}) \text{ implies } V_i(a_i \vee a_i', t_i, s_{-i}) \geq V_i(a_i', t_i, s_{-i}).$$

[32] Note that it is not possible to restrict the action space alone to ensure that the player chooses an undominated strategy since the bids that he must be permitted to choose will depend upon his private type, i.e., his vector of marginal values.

McAdams (2003) imposes the stronger assumption of quasisupermodularity – due to Milgrom and Shannon (1994) – which requires, in addition, that the second inequality must be strict if the first happens to be strict.[33] It is well-known that V_i is supermodular in actions – hence weakly quasisupermodular – when the coordinates of a player's *own* action vector are complementary, i.e., when $A_i = [0,1]^K$ is endowed with the coordinatewise partial order and the second cross-partial derivatives of $V_i(a_{i1}, ..., a_{iK}, t_i, s_{-i})$ with respect distinct action coordinates are nonnegative.[34]

We say that i's interim payoff function V_i satisfies *weak single-crossing* if for all monotone pure strategies s_{-i} of the others, for all player i action pairs $a_i' \geq a_i$, and for all player i type pairs $t_i' \geq t_i$,

$$V_i(a_i', t_i, s_{-i}) \geq V_i(a_i, t_i, s_{-i})$$

implies

$$V_i(a_i', t_i', s_{-i}) \geq V_i(a_i, t_i', s_{-i}).$$

Athey (2001) and McAdams (2003) assume that V_i satisfies the slightly more stringent single-crossing condition in which, in addition to the above, the second inequality is strict whenever the first one is.[35] We next present a condition that will be shown to be weaker than the combination of weak quasisupermodularity and weak single-crossing.

Return now to the case in which A_i is merely a semilattice. For any joint pure strategy of the others, player i's interim best reply correspondence is a mapping from his type into the set of optimal actions – or interim best replies – for that type. Say that player i's interim best reply correspondence is *monotone* if for every monotone joint pure strategy of the others, whenever action a_i is optimal for player i when his type is t_i, and a_i' is optimal when his type is $t_i' \geq t_i$, then $a_i \vee a_i'$ is optimal when his type is t_i'.[36]

The following result relates the above conditions to the hypotheses of Theorem 4.1.

Proposition 4.3. *The hypotheses of Theorem 4.1 are satisfied if G.1-G.6 hold, and if for each player i and for each monotone joint pure strategy of the other players, at least one of the following three conditions is satisfied.*[37]

[33] When actions are totally ordered, as in Athey (2001), interim payoffs are automatically supermodular, and hence both quasisupermodular and weakly quasisupermodular.

[34] Complementarities between the actions of distinct *players* is not implied. This is useful because, for example, many auction games satisfy only own-action complementarity.

[35] For conditions on the joint distribution of types, μ, and the players' payoff functions, $u_i(a,t)$, that imply the more stringent condition, see Athey (2001, pp.879-81), McAdams (2003, p.1197) and Van Zandt and Vives (2005).

[36] This is strictly weaker than requiring the interim best reply correspondence to be increasing in the strong set order, which in any case requires the additional structure of a lattice (see Milgrom and Shannon (1994)).

[37] Which of the three conditions is satisfied is permitted to depend both on the player, i, and on the joint pure strategy employed by the others.

1. *Player i's action space is a lattice and i's interim payoff function is weakly quasisupermodular and satisfies weak single-crossing.*

2. *Player i's interim best reply correspondence is nonempty-valued and monotone.*

3. *Player i's set of monotone pure strategy best replies is nonempty and join-closed.*

Furthermore, the three conditions are in increasing order of generality, i.e., $1 \Longrightarrow 2 \Longrightarrow 3$.

Proof. Because, under G.1-G.6, the hypotheses of Theorem 4.1 hold if condition 3 holds for each player i, it suffices to show that $1 \Longrightarrow 2 \Longrightarrow 3$. So, fix some player i and some monotone pure strategy for every player but i for the remainder of the proof.

($1 \Longrightarrow 2$). Suppose i's action space is a lattice. By G.4 and G.6, for each of i's types, his interim payoff function is continuous on his compact action space. Player i therefore possesses an optimal action for each of his types and so his interim best reply correspondence is nonempty-valued. Suppose that action a_i is optimal for i when his type is t_i and a_i' is optimal when his type is $t_i' \geq t_i$. Then because $a_i \wedge a_i'$ is no better than a_i when i's type is t_i, weak quasisupermodularity implies that $a_i \vee a_i'$ is at least as good as a_i' when i's type is t_i. Weak single-crossing then implies that $a_i \vee a_i'$ is at least as good as a_i' when i's type is t_i'. Since a_i' is optimal when i's type is t_i' so too must be $a_i \vee a_i'$. Hence, i's interim best reply correspondence is monotone.

($2 \Longrightarrow 3$). Let $B_i : T_i \twoheadrightarrow A_i$ denote i's interim best reply correspondence. If a_i and a_i' are in $B_i(t_i)$, then $a_i \vee a_i'$ is also in $B_i(t_i)$ by the monotonicity of $B_i(\cdot)$ (set $t_i = t_i'$ in the definition of a monotone correspondence). Consequently, $B_i(t_i)$ is a subsemilattice of i's action space for each t_i and therefore i's set of monotone pure strategy best replies is join-closed (measurability of the pointwise join of two strategies follows as in footnote 31). It remains to show that i's set of monotone pure best replies is nonempty.

Let $\bar{a}_i(t_i) = \vee B_i(t_i)$, which is well-defined because G.4 and Lemma A.6 imply that A_i is a complete semilattice. Because i's interim payoff function is continuous in his action, $B_i(t_i)$ is compact. Hence $B_i(t_i)$ is a compact subsemilattice of A_i and so $B_i(t_i)$ is itself complete by Lemma A.6. Therefore, $\bar{a}_i(t_i)$ is a member of $B_i(t_i)$ implying that $\bar{a}_i(t_i)$ is optimal for every t_i. It remains only to show that $\bar{a}_i(t_i)$ is monotone (measurability in t_i can be ensured by Lemma A.11).

So, suppose that $t_i' \geq_i t_i$. Because $\bar{a}_i(t_i) \in B_i(t_i)$ and $\bar{a}_i(t_i') \in B_i(t_i')$, the monotonicity of $B_i(\cdot)$ implies that $\bar{a}_i(t_i) \vee \bar{a}_i(t_i') \in B_i(t_i')$. Therefore, because $\bar{a}_i(t_i')$ is the largest member of $B_i(t_i')$ we have $\bar{a}_i(t_i') = \bar{a}_i(t_i) \vee \bar{a}_i(t_i') \geq \bar{a}(t_i)$, as desired. ■

Remark 2. *The environments considered in Athey (2001) and McAdams (2003) are strictly more restrictive than G.1-G.6 permit. Moreover, their conditions on interim payoffs are* *strictly more restrictive than condition 1 of Proposition 4.3. Theorem 4.1 is therefore a strict generalization of their main results.*

When G.1-G.6 hold, it is often possible to apply Theorem 4.1 by verifying condition 1 of Proposition 4.3. But there are important exceptions. For example, Reny and Zamir (2004) have shown in the context of asymmetric first-price auctions that, when bidders have distinct and finite bid sets, monotone best replies exist even though weak single-crossing fails. Further, since action sets (i.e., real-valued bids) there are totally ordered, best reply sets are necessarily join-closed and so the hypotheses of Theorem 4.1 are satisfied even though condition 1 of Proposition 4.3 is not. A similar situation arises in the context of multi-unit discriminatory auctions with risk averse bidders (see subsection 5 below). There, under CARA utility weak quasisupermodularity fails but sets of monotone best replies are nonetheless non-empty and join-closed because condition 2 of Proposition 4.3 is satisfied.

We now turn to several applications of our results.

5. Applications

5.1. Uniform-Price Multi-Unit Auctions with Risk Averse Bidders

Consider a uniform-price auction with n bidders and m homogeneous units of a single good for sale. Each bidder i simultaneously submits a bid, $b = (b_1, ..., b_m)$, where $b_{i1} \geq ... \geq b_{im}$ and each b_{ik} is taken from the finite set $B \subseteq [0, 1]$. Call b_{ik} bidder i's kth unit-bid. The uniform price, p, is the $m + 1$st highest of all nm unit-bids. Each unit-bid above p wins a unit at price p, and any remaining units are awarded to unit-bids equal to p according to a random-bidder-order tie-breaking rule.[38]

Bidder i's private type is his vector of nonincreasing marginal values, so that his type space is $T_i = \{t_i \in [0, 1]^m : t_{i1} \geq ... \geq t_{im}\}$. Bidder i is risk averse with utility function for money $u_i : [-m, m] \to \mathbb{R}$, where $u_i' > 0$, $u_i'' \leq 0$. If bidder i's type is t_i and he wins k units at price p, his payoff is $u_i(t_{i1} + ... + t_{ik} - kp)$. Types are chosen independently across bidders and bidder i's type-vector is chosen according to the density f_i, which need not be positive on all of T_i.[39]

Multi-unit uniform-price auctions always have trivial equilibria in weakly dominated strategies in which some player always bids very high on all units and all others always

[38] The tie-breaking rule is as follows. Bidders are ordered randomly and uniformly. Then, one bidder at a time according to this order, each bidder's *total* remaining demand (i.e., his number of bids equal to p), or as much as possible, is filled at price p per unit until supply is exhausted.

[39] It is possible to permit a bidder's total demand to be stochastic in the sense that, for each $k > 1$, his marginal value for a kth and higher unit may be zero with positive probability, as might occur if a bidder's endowment of the good were private information. We will not pursue this here.

bid zero. We wish to establish the existence of monotone pure strategy equilibria that are not trivial in this sense. But observe that, because the set of feasible bids is finite, bidding above one's marginal value on some unit need not be weakly dominated. Indeed, it might be a strict best reply for bidder i of type t_i to bid $b_k > t_{ik}$ for a kth unit so long as no feasible bid is in $[t_{ik}, b_k)$. Such a kth unit-bid might permit bidder i to win a kth unit and earn a surplus with high probability rather than risk losing the unit by bidding below t_{ik}. On the other hand, in this instance there is never any gain, and there might be a loss, from bidding above b_k on a kth unit.

Call a monotone pure strategy equilibrium *nontrivial* if for each bidder i, for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible bid greater than or equal to t_{ik}. As shown by McAdams (2006), under the coordinatewise partial order on type and action spaces, nontrivial monotone pure strategy equilibria need not exist when bidders are risk averse, as we permit here. Nonetheless, we will demonstrate that a nontrivial monotone pure strategy equilibrium does exist under an economically motivated partial order on type spaces that differs from the coordinatewise partial order; we maintain the coordinatewise partial order the action space B^m of m-vectors of unit-bids.

Before introducing the new partial order, it is instructive to see what goes wrong with the coordinatewise partial order on types. The heart of the matter is that single-crossing fails. To see why, it is enough to consider the case of two units. Fix monotone pure strategies for the other bidders and consider two bids for bidder i, $\bar{b} = (\bar{b}_1, \bar{b}_2)$ and $\underline{b} = (\underline{b}_1, \underline{b}_2)$, where $\bar{b}_k > \underline{b}_k$ for $k = 1, 2$. Suppose that when bidder i employs the high bid, $\bar{b}$, he is certain to win both units and pay $\bar{p}$ for each, while he is certain to win only one unit when he employs the low bid, $\underline{b}$. Further, suppose that the low bid yields a price for the one unit he wins that is either $\underline{p}$ or $\underline{p}' > \underline{p}$, each being equally likely. Thus, the expected difference in his payoff from employing the high bid versus the low one can be written as,

$$\frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p}')\right] + \frac{1}{2}\left[u_i(t_{i1} + t_{i2} - 2\bar{p}) - u_i(t_{i1} - \underline{p})\right].$$

Single-crossing requires this difference, when nonnegative, to remain nonnegative when bidder i's type increases according to the coordinatewise partial order, i.e., when t_{i1} and t_{i2} increase. But this can fail when risk aversion is strict because, whenever $t_{i1} + t_{i2} - 2\bar{p} > t_{i1} - \underline{p}'$, the first utility difference above strictly falls when t_{i1} increases. Consequently, the expected difference can become negative if the second utility difference is negative to start with.

The economic intuition for the failure of single-crossing is straightforward. Under risk aversion, the marginal utility of winning a second unit falls when the dollar value of a first unit rises, giving the bidder an incentive to reduce his second unit bid so as to reduce the price paid on the first unit. We now turn to the new partial order, which ensures that a

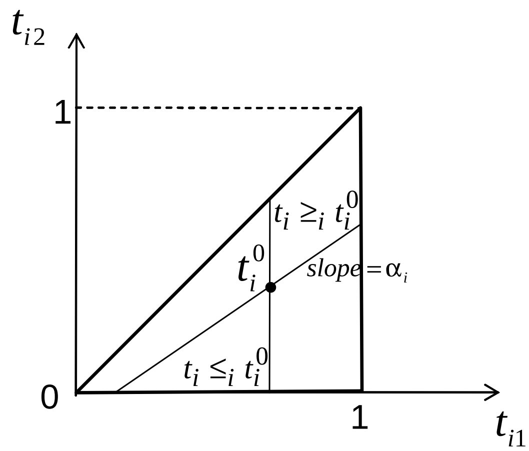

Figure 5.1: Types that are ordered with t_i^0 are bounded between two lines through t_i^0, one being vertical, the other having slope α_i.

higher type is associated with a higher marginal utility of winning each additional unit.

For each bidder i, let $\alpha_i = \frac{u_i'(-m)}{u_i'(m)} - 1 \geq 0$, and consider the partial order, $\geq_i$, on T_i defined as follows: $t_i' \geq_i t_i$ if,

$$\begin{aligned} &1.\ t_{i1}' \geq t_{i1},\ \text{and} \\ &2.\ t_{ik}' - \alpha_i(t_{i1}' + ... + t_{ik-1}') \geq t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1}),\ \text{for all } k \in \{2, ..., m\}. \end{aligned} \tag{5.1}$$

Figure 5.1 shows the types that are greater than and less than a typical type, t_i^0, when types are two-dimensional, i.e., when $m = 2$.

Under the Euclidean metric and Borel sigma-algebra on the type space, the partial order $\geq_i$ defined by (5.1) is clearly closed so that G.1 is satisfied. Because the marginal distribution of each player's type has a density, G.2 is satisfied as well. To see that G.3 is satisfied, let T_i^0 be the set of points in T_i with rational coordinates and suppose that $\int_B f_i(t_i)dt_i > 0$ for some Borel subset B of T_i. Then B must have positive Lebesgue measure in $\mathbb{R}^m$. Consequently, by Fubini's theorem, there exists $z \in \mathbb{R}^m$ (indeed there is a positive Lebesgue measure of such z's) such that the line defined by $z + \mathbb{R}((1+\alpha_i), (1+\alpha_i)^2, ..., (1+\alpha_i)^m)$ intersects B in a set of positive one-dimensional Lebesgue measure on the line. Therefore we may choose two distinct points, t_i and t_i' in B that are on this line. Hence, $t_i' - t_i = \beta((1+\alpha_i), (1+\alpha_i)^2, ..., (1+\alpha_i)^m)$, where we may assume without loss that $\beta > 0$. But then, $t_{i1}' - t_{i1} = \beta(1 + \alpha_i) > 0$ and for

$k \in \{2, ..., m\}$,

$$\begin{aligned} t'_{ik} - t_{ik} &= \beta(1+\alpha_i)^k \\ &= \beta\{1 + \alpha_i[1 + (1+\alpha_i) + (1+\alpha_i)^2 + ... + (1+\alpha_i)^{k-1}]\} \\ &= \beta(1+\alpha_i) + \alpha_i[\beta(1+\alpha_i) + \beta(1+\alpha_i)^2 + ... + \beta(1+\alpha_i)^{k-1}] \\ &= \beta(1+\alpha_i) + \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + ... + (t'_{ik-1} - t_{ik-1})] \\ &> \alpha_i[(t'_{i1} - t_{i1}) + (t'_{i2} - t_{i2}) + ... + (t'_{ik-1} - t_{ik-1})]. \end{aligned}$$

Consequently, for any $t_i^0 \in T_i^0$ close enough to $(t'_i + t_i)/2$,

$$t'_i \geq_i t_i^0 \geq_i t_i,$$

according to the partial order $\geq_i$ defined by (5.1). Hence, G.3 is satisfied.

As noted in section 4.1, actions spaces, being finite sublattices, are locally complete compact metric semilattices. Hence, G.4 and G.5 (ii) hold. Also, G.6 holds because action spaces are finite. Thus, we have so far verified G.1-G.6.

McAdams (2004) shows that each bidder's interim payoff function is modular and hence quasisupermodular. By condition 1 of Proposition 4.3, the hypotheses of Theorem 4.1 will be satisfied if interim payoffs satisfy weak single crossing, which we now demonstrate. It is here where the new partial order $\geq_i$ in (5.1) is fruitfully employed.

To verify weak single crossing it suffices to show that ex-post payoffs satisfy increasing differences. So, fix the strategies of the other bidders, a realization of their types, and an ordering of the players for the purposes of tie-breaking. With these fixed, suppose that the bid, $\bar{b}$, chosen by bidder i of type t_i wins k units at the price $\bar{p}$ per unit, while the coordinatewise-lower bid, $\underline{b}$, wins $j \leq k$ units at the price $\underline{p} \leq \bar{p}$ per unit. The difference in i's ex-post utility from bidding $\bar{b}$ versus $\underline{b}$ is then,

$$u_i(t_{i1} + ... + t_{ik} - k\bar{p}) - u_i(t_{i1} + ... + t_{ij} - j\underline{p}). \tag{5.2}$$

Assuming that $t'_i \geq_i t_i$ in the sense of (5.1), it suffices to show that (5.2) is weakly greater at t'_i than at t_i. Noting that (5.1) implies that $t'_{il} \geq t_{il}$ for every l, it can be seen that, if $j = k$, then (5.2), being negative, is weakly greater at t'_i than at t_i by the concavity of u_i. It therefore remains only to consider the case in which $j < k$, where we have,

$$\begin{aligned} u_i(t'_{i1} + ... + t'_{ik} - k\bar{p}) - u_i(t_{i1} + ... + t_{ik} - k\bar{p}) &\geq u'_i(m)[(t'_{i1} - t_{i1}) + ... + (t'_{ik} - t_{ik})] \\ &\geq u'_i(m)[(t'_{i1} - t_{i1}) + ... + (t'_{ij+1} - t_{ij+1})] \\ &\geq u'_i(-m)[(t'_{i1} - t_{i1}) + ... + (t'_{ij} - t_{ij})] \\ &\geq u_i(t'_{i1} + ... + t'_{ij} - j\underline{p}) - u_i(t_{i1} + ... + t_{ij} - j\underline{p}), \end{aligned}$$

where the first and fourth inequalities follow from the concavity of u_i and because a bidder's surplus lies between m and $-m$, and the third inequality follows because $t'_i \geq_i t_i$ in the sense of (5.1). We conclude that weak single crossing holds and so the hypotheses of Theorem 4.1 are satisfied.

Finally, for each player i, let C_i denote the subset of his pure strategies such that for f_i almost-every t_i, and for every k, bidder i's kth unit-bid does not exceed $\phi(t_{ik})$, the smallest feasible unit-bid greater than or equal to t_{ik}. By Remark 1, each C_i is join-closed, piecewise-closed and pointwise-limit-closed. Further, because the hypotheses of Theorem 4.1 are satisfied, whenever the others employ monotone pure strategies player i has a monotone best reply, $b'_i(\cdot)$, say. Defining $b_i(t_i)$ to be the coordinatewise minimum of $b'_i(t_i)$ and $(\phi(t_{i1}), ..., \phi(t_{im}))$ for all $t_i \in T_i$ implies that $b_i(\cdot)$ is a monotone best reply contained in C_i. This is because, ex-post, any units won by employing $b'_i(\cdot)$ that are also won by employing $b_i(\cdot)$ are won at a weakly lower price with $b_i(\cdot)$, and any units won by employing $b'_i(\cdot)$ that are not won by employing $b_i(\cdot)$ cannot be won at a positive surplus. Hence, the hypotheses of Theorem 4.2 are satisfied and we conclude that a nontrivial monotone pure strategy equilibrium exists. We may therefore state the following proposition.

Proposition 5.1. *Consider an independent private value uniform-price multi-unit auction with the random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse.*

Then, there is a pure strategy equilibrium of the auction with the following properties. For each bidder i,

(i) the equilibrium is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids, and

(ii) the equilibrium is nontrivial in the sense that for f_i almost-all of his types, and for every k, bidder i's kth unit-bid does not exceed the smallest feasible unit-bid greater than or equal to his marginal value for a kth unit.

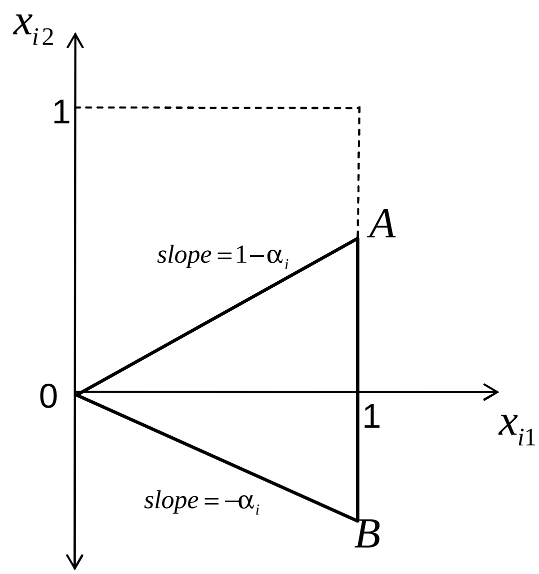

Figure 5.2: After performing the change of variable from t_i to x_i as described in Remark 4 bidder i's new type space is triangle OAB and it is endowed with the coordinatewise partial order. The figure is drawn for the case in which $\alpha_i \in (0, 1)$.

Remark 3. *The partial order defined by (5.1) reduces to the usual coordinatewise partial order under risk neutrality (i.e., when $\alpha_i = 0$), but is distinct from the coordinatewise partial order under strict risk aversion (i.e., when $\alpha_i > 0$), in which case McAdams (2003) does not apply since he employs the coordinatewise partial order.*

Remark 4. *The partial order defined by (5.1) can instead be thought of as a change of variable from t_i to say x_i, where $x_{i1} = t_{i1}$ and $x_{ik} = t_{ik} - \alpha_i(t_{i1} + ... + t_{ik-1})$ for $k > 1$, and where the coordinatewise partial order is applied to the new type space. Our results apply equally well using this change-of-variable technique. In contrast, McAdams (2003) still does not apply because the resulting type space is not the product of intervals, an assumption he maintains together with a strictly positive joint density.*[40] *See Figure 5.2 for the case in which $m = 2$.*

Remark 5. *One can use the above technique to obtain the existence of a nontrivial monotone pure strategy equilibrium when bidders' types remain independent but their payoffs are interdependent. For example, one can permit $u_i(\sum_{j=1}^{k} v_{ij}(t_{ij}, t_{-i}) - kp)$ to be bidder i's ex-post utility of winning k units at price p when the joint type vector is t and where bidder i's dollar*

[40] Indeed, starting with the partial order defined by (5.1) there is no change of variable that, when combined with the coordinatewise partial order, is order-preserving and maps to a product of intervals. This is because, in contrast to a product of intervals with the coordinatewise partial order, under the new partial order there is never a smallest element of the type space and there is no largest element when $\alpha_i > 1$.

value for a jth unit, $v_{ij}(t_{ij}, t_{-i})$, is strictly increasing in the jth coordinate of i's type vector and can depend in any way on all coordinates of the other bidders' type vectors.

Finally, by considering finer and finer finite grids of bids, one can permit unit-bids to be any nonnegative real number. The proof of the following corollary of Proposition 5.1 is in the appendix.

Corollary 5.2. *The conclusions of Proposition 5.1 remain valid even when the bidders' unit-bids are permitted to be any nonnegative real number.*

5.2. Discriminatory Multi-Unit Auctions with CARA Bidders

Consider the same setup as in Subsection 5.1 with two exceptions. First, change the payment rule so that each bidder pays his kth unit-bid for a kth unit won. Second, assume that each bidder's utility function, u_i, exhibits constant absolute risk aversion.

Despite these two changes, single-crossing still fails under the coordinatewise partial order on types for the same underlying reason as in a uniform-price auction with risk averse bidders. Nonetheless, just as in the previous section it can be shown here that assumptions G.1-G.6 hold and each bidder i's interim expected payoff function satisfies weak single-crossing under the partial order $\geq_i$, defined in (5.1).[41]

For the remainder of this section, we employ the type-space partial order $\geq_i$, defined in (5.1) and the coordinatewise partial order on the space of feasible bid vectors. Monotonicity of pure strategies is then defined in terms of these partial orders.

If it can be shown that interim expected payoffs are quasisupermodular, condition 1 of Proposition 4.3 would permit us to apply Theorem 4.1. However, quasisupermodularity does not hold in discriminatory auctions with strictly risk averse bidders – even CARA bidders.

The intuition for the failure of quasisupermodularity is as follows. Suppose there are two units, and let b_k denote a kth unit-bid. Fixing b_2, suppose that b_1 is chosen to maximize a bidder's interim payoff when his type is (t_1, t_2), namely,

$$P_1(b_1)[u(t_1 - b_1) - u(0)] + P_2(b_2)[u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)],$$

where $P_k(b_k)$ is the probability of winning at least k units.

There are two benefits from increasing b_1. First, the probability, $P_1(b_1)$, of winning at least one unit increases. Second, when risk aversion is strict, the marginal utility, $u((t_1 - b_1) + (t_2 - b_2)) - u(t_1 - b_1)$, of winning a second unit increases. The cost of increasing b_1 is that the marginal utility, $u(t_1 - b_1) - u(0)$, of winning a first unit decreases. Optimizing

[41] This statement remains true with any risk averse utility function. The CARA utility assumption is required for a different purpose which will be revealed shortly.

over the choice of b_1 balances this cost with the two benefits. For simplicity, suppose that the optimal choice of b_1 satisfies $b_1 > t_2$.

Now suppose that b_2 increases. Indeed, suppose that b_2 increases to t_2. Then the marginal utility of winning a second unit vanishes. Consequently, the second benefit from increasing b_1 is no longer present and the optimal choice of b_1 may fall — even with CARA utility.

This illustrates that the change in utility from increasing one's first unit-bid may be positive when one's second unit-bid is low, but negative when one's second unit-bid is high. Thus, the different coordinates of a bidder's bid are not necessarily complementary, and weak quasisupermodularity can fail. We therefore cannot appeal to condition 1 of Proposition 4.3.

Fortunately, we can instead appeal to condition 2 of Proposition 4.3 owing to the following lemma, whose proof is in the appendix. It is here where we employ the assumption of CARA utility.

Lemma 5.3. *Fix any monotone pure strategies for other bidders and suppose that the vector of bids b_i is optimal for bidder i when his type vector is t_i, and that b_i' is optimal when his type is $t_i' \geq_i t_i$, where $\geq_i$ is the partial order defined in (5.1). Then the vector of bids $b_i \vee b_i'$ is optimal when his type is t_i'.*

Because Lemma 5.3 establishes condition 2 of Proposition 4.3, we may apply Theorem 4.1 to conclude that a monotone pure strategy equilibrium exists. Thus, despite the failure – even with CARA utilities – of both single-crossing with the coordinatewise partial order on types and of weak quasisupermodularity with the coordinatewise partial order on bids, we have established the following.

Proposition 5.4. *Consider an independent private value discriminatory multi-unit auction with the random-bidder-order tie-breaking rule and in which bids are restricted to a finite grid. Suppose that each bidder i's vector of marginal values is decreasing and chosen according to the density f_i, and that each bidder is weakly risk averse and exhibits constant absolute risk aversion.*

Then, there is a pure strategy equilibrium that is monotone under the type-space partial order $\geq_i$ defined by (5.1) and under the usual coordinatewise partial order on bids.

Remark 6. *As in Remark 5, similar techniques can be used to obtain the existence of a monotone pure strategy equilibrium when bidders' types remain independent but their payoffs are interdependent.*

The proof of the following corollary is in the appendix.

Corollary 5.5. *The conlcusions of Proposition 5.4 remain valid even when the bidders' unit bids are permited to be any nonnegative real number.*

The two applications provided so far demonstrate that it is useful to have flexibility in defining the partial order on the type space since the mathematically natural partial order (in this case the coordinatewise partial order on the original type space) may not be the partial order that corresponds best to the economics of the problem. The next application shows that even when single crossing cannot be established for all coordinates of the type space jointly, it is enough for the existence of a pure strategy equilibrium if single-crossing holds strictly even for a single coordinate of the type space.

5.3. Price Competition with Non-Substitutes

Consider an n-firm differentiated-product price-competition setting. Firm i chooses price $p_i \in [0,1]$, and receives two pieces of private information – his constant marginal cost, $c_i \in [0,1]$, and information $x_i \in [0,1]$ about the state of demand in each of the n markets. The demand for firm i's product is $D_i(p,x)$ when the vector of prices chosen by all firms is $p \in [0,1]^n$ and when their joint vector of private information about market demand is $x \in [0,1]^n$. Demand functions are assumed to be twice continuously differentiable, strictly positive when own-price is less than one, and strictly downward-sloping, i.e., $\partial D_i(p,x)/\partial p_i < 0$.

Some products may be substitutes, but others need not be. More precisely, the n firms are partitioned into two subsets N_1 and N_2.[42] Products produced by firms within each subset are substitutes, and so we assume that $D_i(p,x)$ and $\partial D_i(p,x)/\partial p_i$ are nondecreasing in p_j whenever i and j are in the same N_k. In addition, marginal costs are affiliated among firms within each N_k and are independent across the two subsets of firms. The joint density of costs is given by the continuously differentiable density $f(c)$ on $[0,1]^n$. Information about market demand may be correlated across firms, but is independent of all marginal costs and has continuously differentiable joint density $g(x)$ on $[0,1]^n$. We do not assume that market demands are nondecreasing in x because we wish to permit the possibility that information that increases demand for some products might decrease it for others.

Given pure strategies $p_j(c_j,x_j)$ for the others, firm i's interim expected profits are,

$$v_i(p_i,c_i,x_i) = \int (p_i - c_i) D_i(p_i, p_{-i}(c_{-i},x_{-i}), x) g_i(x_{-i}|x_i) f_i(c_{-i}|c_i) dx_{-i} dc_{-i}, \tag{5.3}$$

so that,

$$\frac{\partial^2 v_i(p_i,c_i,x_i)}{\partial c_i \partial p_i} = -E\left(\frac{\partial D_i}{\partial p_i}\middle| c_i, x_i\right) + \frac{\partial}{\partial c_i} E(D_i|c_i,x_i) + (p_i - c_i)\frac{\partial}{\partial c_i} E\left(\frac{\partial D_i}{\partial p_i}\middle| c_i, x_i\right). \tag{5.4}$$

[42] The extension to any finite number of subsets is straightforward.

Therefore, if $p_j(c_j, x_j)$ is nondecreasing in c_j for each firm $j \neq i$ and every x_j, then,

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} \geq -E\left(\left.\frac{\partial D_i}{\partial p_i}\right| c_i, x_i\right) > 0 \tag{5.5}$$

for all $p_i, c_i, x_i \in [0,1]$ such that $p_i \geq c_i$, where the weak inequality follows because both partial derivatives with respect to c_i on the right-hand side of (5.4) are nonnegative. For example, consider the expectation in the first partial derivative (the second is similar). If $i \in N_1$, then

$$E(D_i|c_i, x_i) = E\left[E(D_i(p_i, p_{-i}(c_{-i}, x_{-i}), x)|c_i, x_i, (c_j, x_j)_{j\in N_2})|c_i, x_i\right].$$

The inner expectation is nondecreasing in c_i because the vector of marginal costs for firms in N_1 are affiliated, their prices are nondecreasing in their costs, and their goods are substitutes. That the entire expectation is nondecreasing in c_i now follows from the independence of (c_i, x_i) and $(c_j, x_j)_{j\in N_2}$.

Thus, according to (5.5), when $p_i \geq c_i$ single-crossing holds strictly for the marginal cost coordinate of the type space. On the other hand, single-crossing need not hold for the market-demand coordinate, x_i, since we have made no assumptions about how x_i affects demand.[43] Nonetheless, we shall now define a partial order on firm i's type space $T_i = [0,1]^2$ under which a monotone pure strategy equilibrium exists.

Note that, because $-\partial D_i/\partial p_i$ is positive and continuous on its compact domain, it is bounded strictly above zero with a bound that is independent of the pure strategies, $p_j(c_j, x_j)$ employed by other firms. Hence, because our continuity assumptions imply that $\partial^2 v_i(p_i, c_i, x_i)/\partial x_i \partial p_i$ is bounded, there exists $\alpha_i > 0$ such that for all $\beta \in [0, \alpha_i]$ and all pure strategies $p_j(c_j, x_j)$ nondecreasing in c_j,

$$\frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial c_i \partial p_i} + \beta \frac{\partial^2 v_i(p_i, c_i, x_i)}{\partial x_i \partial p_i} > 0, \tag{5.6}$$

for all $p_i, c_i, x_i \in [0,1]$ such that $p_i \geq c_i$.

Inequality (5.6) implies that when $p_i \geq c_i$, the marginal gain from increasing one's price, namely,

$$\frac{\partial v_i(p_i, c_i, x_i)}{\partial p_i},$$

is strictly increasing along lines in (c_i, x_i)-space with slope $\beta \in [0, \alpha_i]$. This provides a basis

[43] We cannot simply restrict attention to strategies $p_i(c_i, x_i)$ that are monotone in c_i and jointly measurable in (c_i, x_i) because this set of pure strategies is not compact in a topology rendering ex-ante payoffs continuous.

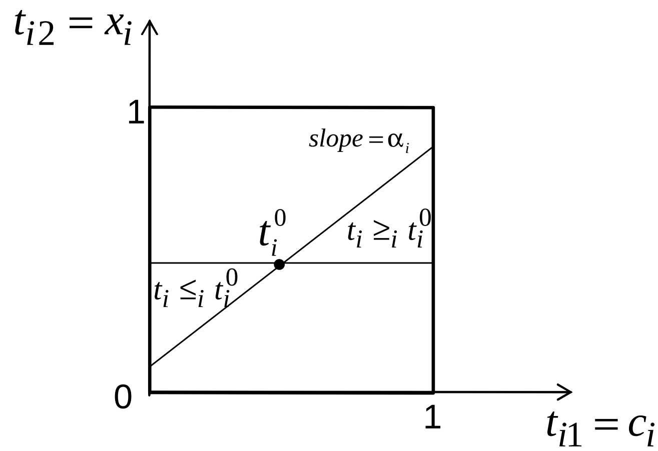

Figure 5.3: Types that are greater than and less than t_i^0 are bounded between two lines through t_i^0, one being horizontal, the other having slope α_i.

for defining a partial order under which players possess monotone best replies.

For each player i, define the partial order $\geq_i$ on $T_i = [0,1]^2$ as follows: $(c_i', x_i') \geq_i (c_i, x_i)$ if $\alpha_i c_i' - x_i' \geq \alpha_i c_i - x_i$ and $x_i' \geq x_i$. Figure 5.3 shows those types greater than and less than a typical type $t_i^0 = (c_i^0, x_i^0)$.

Under the partial order $\geq_i$, assumptions G.1-G.3 hold as in Example 5.1. The action-space assumption G.4 clearly holds while G.5 (ii) holds by Lemma A.19 given the usual partial order over the reals. Assumption G.6 holds by our continuity assumption on demand. Also, because the action space $[0,1]$ is totally ordered, the set of monotone best replies is join-closed because the join of two best replies is, at every t_i, equal to one of them or to the other. Finally, as is shown in the Appendix (see Lemma A.22), under the type-space partial order, $\geq_i$, firm i possesses a monotone best reply when the others employ monotone pure strategies.

Therefore, by Theorem 4.1, there exists a pure strategy equilibrium in which each firm's price is monotone in (c_i, x_i) according to $\geq_i$. In particular, there is a pure strategy equilibrium in which each firm's price is nondecreasing in his marginal cost, the coordinate in which strict single-crossing holds.

5.4. Type Spaces with Atoms

When type spaces contain atoms, assumption G.2 fails and there may not exist a pure strategy equilibrium, let alone a monotone pure strategy equilibrium. Thus, one must permit mixing and we show here how our results can be used to ensure the existence of a monotone mixed strategy equilibrium.

Because we do not make the Milgrom and Weber (1985) assumption that the joint distribution of types is absolutely continuous with respect to the product of its marginals, it is not useful to define mixed strategies as distributional strategies. For our purposes, the most direct route is to instead follow Aumann (1964) and define a mixed strategy for player i to be a measurable function, $m_i : T_i \times [0,1] \to A_i$, where $[0,1]$ is endowed with Borel sigma-algebra $\mathcal{B}$, and $T_i \times [0,1]$ is endowed with product sigma-algebra $\mathcal{T}_i \times \mathcal{B}$. As in Aumann (1964), mixed strategies $m_1, ..., m_N$ for the N players are implemented as follows. The players' types $t_1, ..., t_N$ are drawn jointly according to μ and then, independently, each player i privately draws ω_i from $[0,1]$ according to a uniform distribution. Player i knowing t_i and ω_i takes the action $m_i(t_i, \omega_i)$. Player i's payoff given the mixed strategies $m_1, ..., m_N$ is therefore, $\int_T \int_{[0,1]^N} u_i(m(t,\omega), t) d\omega d\mu$, where $m(t,\omega) = (m_1(t_1,\omega_1), ..., m_1(t_N,\omega_N))$.

Call a mixed strategy $m_i : T_i \times [0,1] \to A_i$ *monotone* if the image of $m_i(t_i, \cdot)$, i.e., the set $m_i(t_i, [0,1])$, is a totally ordered subset of A_i for every $t_i \in T_i$ and if every member of the image of $m_i(t_i, \cdot)$ is greater than or equal to every member of the image of $m_i(t_i', \cdot)$ whenever $t_i \geq t_i'$.[44] The following result permits a player's marginal type-distribution to contain atoms, even countably many.

Theorem 5.6. *If G.1 and G.3-G.6 hold, and each player's set of monotone pure best replies is nonempty and join-closed whenever the others employ monotone mixed strategies, then G possesses a monotone mixed strategy equilibrium.*

Proof. For each player i, let T_i^* denote the set of atoms of μ_i.[45] Consider the following surrogate Bayesian game. Player i's type space is $Q_i = [(T_i \backslash T_i^*) \times \{0\}] \cup (T_i^* \times [0,1])$ and the sigma-algebra on Q_i is generated by all sets of the form $(B \backslash T_i^*) \times \{0\}$ and $(B \cap T_i^*) \times C$, where $B \in \mathcal{T}_i$ and C is a Borel subset of $[0,1]$. The joint distribution on types, ν, is determined as follows. Nature first chooses $t \in T$ according to the original type distribution μ. Then, for each i, Nature independently and uniformly chooses $x_i \in [0,1]$ if $t_i \in T_i^*$, and chooses $x_i = 0$ if $t_i \in T_i \backslash T_i^*$.[46] Hence, ν_i, the marginal distribution on Q_i is atomless. Player i is informed of $q_i = (t_i, x_i)$. Action spaces are unchanged. The x_i are payoff irrelevant and so payoff functions are as before. This completes the description of the surrogate game.

The partial order on Q_i is the lexicographic partial order. That is, $q_i' = (t_i', x_i') \geq (t_i, x_i) = q_i$ if either $t_i' \geq t_i$ and $t_i' \neq t_i$, or $t_i' = t_i$ and $x_i' \geq x_i$. The metrics and partial orders on the players' action spaces are unchanged.

[44] A subset of a partially ordered space is totally ordered if any two members are ordered. Such a subset is sometimes also called a chain.

[45] For every $t_i \in T_i$, the singleton set $\{t_i\}$ is in $\mathcal{T}_i$ by G.1. See Section A.1 in the Appendix.

[46] In particular, if for each player i, $B_i \in \mathcal{T}_i$ and C_i is a Borel subset of $[0,1]$, and $D = \times_{i \in I}[(B_i \backslash T_i^*) \times \{0\}] \times \times_{i \in I^c}[(B_i \cap T_i^*) \times C_i]$, then $\nu(D) = \mu([\times_{i \in I} (B_i \backslash T_i^*)] \times [\times_{i \in I^c} (B_i \cap T_i^*)]) \Pi_{i \in I^c} \lambda(C_i)$, where λ is Lebesgue measure on $[0,1]$.

It is straightforward to show that under the hypotheses above, all the hypotheses of Theorem 4.1 but perhaps G.3 hold in the surrogate game.[47] We now show that G.3 too holds in the surrogate game.

For each player i, let T_i^0 denote the countable subset of T_i that can be used to verify G.3 in the original game and define the countable set $Q_i^0 = [T_i^0 \times \{0\}] \cup [T_i^* \times R]$, where R denotes the set of rationals in $[0,1]$. Suppose that for some player i, $\nu_i(B) > 0$ for some measurable subset B of Q_i. Then either $\nu_i(B \cap [(T_i \backslash T_i^*) \times \{0\}]) > 0$ or $\nu_i(B \cap (\{t_i^*\} \times [0,1])) > 0$ for some $t_i^* \in T_i^*$. In the former case, $\mu_i(\{t_i \in T_i \backslash T_i^* : (t_i, 0) \in B\}) > 0$ and G.3 in the original game implies the existence of t_i' and t_i'' in $\{t_i \in T_i \backslash T_i^0 : (t_i, 0) \in B\}$ and t_i^0 in T_i^0 such that $t_i'' \geq t_i^0 \geq t_i'$ according to the partial order on T_i. But then $(t_i'', 0) \geq (t_i^0, 0) \geq (t_i', 0)$ according to the lexicographic partial order on Q_i and where $(t_i'', 0)$ and $(t_i', 0)$ are in B and $(t_i^0, 0)$ is in Q_i^0. In the latter case there exist x_i', x_i in $[0,1]$ with $x_i' > x_i > 0$, such that (t_i^*, x_i) and (t_i^*, x_i') are in B. But for any rational r between x_i' and x_i we have $(t_i^*, x_i') \geq (t_i^*, r) \geq (t_i^*, x_i)$ according to the lexicographic order on Q_i and where (t_i^*, r) is in Q_i^0. Thus, the surrogate game satisfies G.3 and we may conclude, by Theorem 4.1, that it possesses a monotone pure strategy equilibrium. But any such equilibrium induces a monotone mixed strategy equilibrium of the original game. ■

Remark 7. *The proof of Theorem 5.6 in fact demonstrates that players need only randomize when their type is an atom.*

6. Proof of Theorem 4.1

Let M_i denote the nonempty set of monotone functions from T_i into A_i, and let $M = \times_{i=1}^N M_i$. By Lemma *A*.11, every element of M_i is equal μ_i almost-everywhere to a measurable monotone function, and so M_i coincides with player i's set of monotone pure strategies. Let $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ denote player i's best-reply correspondence when all players must employ monotone pure strategies. Because, by hypothesis, each player possesses a monotone best reply (among all strategies) when the others employ monotone pure strategies, any fixed point of $\times_{i=1}^n \mathbf{B}_i : M \twoheadrightarrow M$ is a monotone pure strategy equilibrium. The following steps demonstrate that such a fixed point exists.

STEP I. (M is a nonempty, compact, metric, absolute retract.) Without loss, we may assume for each player i that the metric d_i on A_i is bounded.[48] Given d_i, define a metric δ_i

[47] Observe that a monotone pure strategy in the surrogate game induces a monotone mixed strategy in the original game, and that a monotone pure strategy in the original game defines a monotone pure strategy in the surrogate game by viewing it to be constant in x_i.

[48] For any metric, $d(\cdot,\cdot)$, a topologically equivalent bounded metric is $\min(1, d(\cdot,\cdot))$.

on M_i as follows:[49]

$$\delta_i(s_i, s_i') = \int_{T_i} d_i(s_i(t_i), s_i'(t_i))d\mu_i(t_i).$$

By Lemmas A.13 and A.16, each (M_i, δ_i) is a compact absolute retract.[50] Consequently, under the product topology – metrized by the sum of the δ_i – M is a nonempty compact metric space and, by Borsuk (1966) IV (7.1), an absolute retract.

STEP II. ($\times_{i=1}^n \mathbf{B}_i$ is nonempty-valued and upper-hemicontinuous.) We first demonstrate that, given the metric spaces (M_j, δ_j), each player i's payoff function, $U_i : M \to \mathbb{R}$, is continuous under the product topology. To see this, suppose that s^n is a sequence of joint strategies in M, and that $s^n \to s \in M$. By Lemma A.12, for each player i, $s_i^n(t_i) \to s_i(t_i)$ for μ_i almost every $t_i \in T_i$. Consequently, $s^n(t) \to s(t)$ for μ almost every $t \in T$.[51] Hence, since u_i is bounded, Lebesgue's dominated convergence theorem yields

$$U_i(s^n) = \int_T u_i(s^n(t), t)d\mu(t) \to \int_T u_i(s(t), t)d\mu(t) = U_i(s),$$

establishing the continuity of U_i.

Because each M_i is compact, Berge's theorem of the maximum implies that $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is nonempty-valued and upper-hemicontinuous. Hence, $\times_{i=1}^n \mathbf{B}_i$ is nonempty-valued and upper-hemicontinuous as well.

STEP III. ($\times_{i=1}^n \mathbf{B}_i$ is contractible-valued.) According to Lemma A.3, for each player i, assumptions G.1-G.3 imply the existence of a monotone and measurable function $\Phi_i : T_i \to [0,1]$ such that $\mu_i\{t_i \in T_i : \Phi_i(t_i) = c\} = 0$ for every $c \in [0,1]$. Fixing such a function Φ_i permits the construction of a contraction map.[52]

Fix some monotone pure strategy, s_{-i}, for players other than i, and consider player i's set of monotone pure best replies, $\mathbf{B}_i(s_{-i})$. Because $\mathbf{B}_i(\cdot)$ is upper-hemicontinuous, it is closed-valued and therefore $\mathbf{B}_i(s_{-i})$ is compact, being a closed subset of the compact metric space M_i. By hypothesis, $\mathbf{B}_i(s_{-i})$ is join-closed, and so $\mathbf{B}_i(s_{-i})$ is a compact semilattice under the partial order defined by $s_i \geq s_i'$ if $s_i(t_i) \geq s_i'(t_i)$ for μ_i-a.e. $t_i \in T_i$. By Lemma A.12, this partial order is closed. Therefore, Lemma A.6 implies that $\mathbf{B}_i(s_{-i})$ is a complete

[49] Formally, the resulting metric space (M_i, δ_i) is the space of equivalence classes of functions in M_i that are equal μ_i almost everywhere – i.e., two functions are in the same equivalence class if the set on which they coincide contains a measurable subset having μ_i-measure one. Nevertheless, analogous to the standard treatment of $\mathcal{L}_p$ spaces, in the interest of notational simplicity we focus on the elements of the original space M_i rather than on the equivalence classes themselves.

[50] One cannot improve upon Lemma A.16 by proving, for example, that M_i, metrized by δ_i, is homeomorphic to a convex set. It need not be (e.g., see footnote 24).

[51] This is because if $Q_1, ..., Q_n$ are such that $\mu(Q_i \times T_{-i}) = \mu_i(Q_i) = 1$ for all i, then $\mu(\times_i Q_i) = \mu(\cap_i(Q_i \times T_{-i})) = 1$.

[52] For example, if $T_i = [0,1]^2$ and μ_i is absolutely continuous with respect to Lebesgue measure, we may take $\Phi_i(t_i) = (t_{i1} + t_{i2})/2$.

semilattice so that $\tilde{s}_i = \vee \mathbf{B}_i(s_{-i})$ is a well-defined member of $\mathbf{B}_i(s_{-i})$. Consequently for every $s_i \in \mathbf{B}_i(s_{-i})$, $\tilde{s}_i(t_i) \geq s_i(t_i)$ for μ_i-a.e. $t_i \in T_i$. By Lemma A.14, there exists $\bar{s}_i \in M_i$ such that $\bar{s}_i(t_i) = \tilde{s}_i(t_i)$ for μ_i-a.e. t_i – and hence $\bar{s}_i \in \mathbf{B}_i(s_{-i})$ – and such that $\bar{s}_i(t_i) \geq s_i(t_i)$ for every $t_i \in T_i$ and every s_i that is μ_i-a.e. less or equal to $\tilde{s}_i$ and therefore in particular for every $s_i \in \mathbf{B}_i(s_{-i})$.[53]

Define $h : [0,1] \times \mathbf{B}_i(s_{-i}) \to \mathbf{B}_i(s_{-i})$ as follows: For every $t_i \in T_i$,

$$h(\tau, s_i)(t_i) = \begin{cases} s_i(t_i), & \text{if } \Phi_i(t_i) \leq 1 - \tau \text{ and } \tau < 1 \\ \bar{s}_i(t_i), & \text{otherwise.} \end{cases} \tag{6.1}$$

Note that $h(0, s_i) = s_i$, $h(1, s_i) = \bar{s}_i$, and $h(\tau, s_i)(t_i)$ is always either $\bar{s}_i(t_i)$ or $s_i(t_i)$ and so is an interim best reply for μ_i almost every t_i. Moreover, $h(\tau, s_i)$ is monotone because Φ_i is monotone and $\bar{s}_i(t_i) \geq s_i(t_i)$ for all $t_i \in T_i$. Hence, $h(\tau, s_i) \in \mathbf{B}_i(s_{-i})$. Therefore, h will be a contraction for $\mathbf{B}_i(s_{-i})$ and $\mathbf{B}_i(s_{-i})$ will be contractible if $h(\tau, s_i)$ is continuous, which we establish next.[54]

Suppose $\tau_n \in [0,1]$ converges to τ and $s_i^n \in \mathbf{B}_i(s_{-i})$ converges to s_i, both as $n \to \infty$. By Lemma A.12, there is a measurable subset, D, of i's types such that $\mu_i(D) = 1$ and for all $t_i \in D$, $s_i^n(t_i) \to s_i(t_i)$. Consider any $t_i \in D$. There are three cases: (a) $\Phi_i(t_i) < 1 - \tau$, (b) $\Phi_i(t_i) > 1 - \tau$, and (c) $\Phi_i(t_i) = 1 - \tau$. In case (a), $\tau < 1$ and $\Phi_i(t_i) < 1 - \tau_n$ for n large enough and so $h(\tau_n, s_i^n)(t_i) = s_i^n(t_i) \to s_i(t_i) = h(\tau, s_i)$. In case (b), $\Phi_i(t_i) > 1 - \tau_n$ for n large enough and so for such large enough n, $h(\tau_n, s_i^n)(t_i) = \bar{s}_i(t_i) = h(\tau, s_i)(t_i)$. Because the remaining case (c) occurs only if t_i is in a set of types having μ_i-measure zero, we have shown that $h(\tau_n, s_i^n)(t_i) \to h(\tau, s_i)(t_i)$ for μ_i-a.e. t_i, which, by Lemma A.12 implies that $h(\tau_n, s_i^n) \to h(\tau, s_i)$, establishing the continuity of h.

Thus, for each player i, the correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is contractible-valued. Under the product topology, $\times_{i=1}^n \mathbf{B}_i$ is therefore contractible-valued as well.

Steps I-III establish that $\times_{i=1}^n \mathbf{B}_i$ satisfies the hypotheses of Theorem 2.1 and therefore possesses a fixed point. ■

Remark 8. *The proof of Theorem 4.2 mimics that of Theorem 4.1, but where each M_i is replaced with $M_i \cap C_i$, and where each correspondence $\mathbf{B}_i : M_{-i} \twoheadrightarrow M_i$ is replaced with*

[53] One might wonder why we do not take the more driect route of defining, for each $t_i \in T_i$, $\bar{s}_i(t_i) = \vee s_i(t_i)$, where the join is taken over all $s_i \in \mathbf{B}_i(s_{-i})$. It is because one must show using an argument such as that given here that $\bar{s}_i$ is in $\mathbf{B}_i(s_{-i})$, which is not obvious by virtue of the direct definition alone since each member of $\mathbf{B}_i(s_{-i})$ is an interim best reply only μ_i almost everywhere.

[54] With Φ_i defined as in footnote 52, Figure 6.1 provides snapshots of the resulting $h(\tau, s_i)$ as τ moves from zero to one. The axes are the two dimensions of the type vector (t_{i1}, t_{i2}), and the arrow within the figures depicts the direction in which the negatively-sloped line, $(t_{i1} + t_{i2})/2 = 1 - \tau$, moves as τ increases. For example, panel (a) shows that when $\tau = 0$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for all t_i in the unit square. On the other hand, panel (c) shows that when $\tau = 3/4$, $h(\tau, s_i)(t_i)$ is equal to $s_i(t_i)$ for t_i below the negatively-sloped line and equal to $\bar{s}_i(t_i)$ for t_i above it.

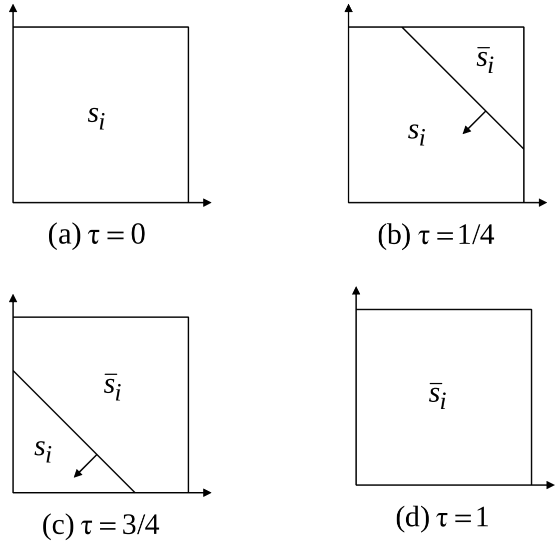

Figure 6.1: $h(\tau, s_i)$ as τ varies from 0 (panel (a)) to 1 (panel (d)) and the domain is the unit square.

the correspondence $\mathbf{B}_i^* : M_{-i} \cap C_{-i} \twoheadrightarrow M_i \cap C_i$ *defined by* $\mathbf{B}_i^*(s_{-i}) = \mathbf{B}_i(s_{-i}) \cap C_i$*. The proof goes through because the hypotheses of Theorem 4.2 imply that each* $M_i \cap C_i$ *is compact, nonempty, join-closed, piecewise-closed, and pointwise-limit-closed (and hence the proof that each* $M_i \cap C_i$ *is an absolute retract mimics the proof of Lemma A.16), and that each correspondence* $\mathbf{B}_i^*$ *is upper-hemicontinuous, nonempty-valued and contractible-valued (the contraction is once again defined by 6.1). The result then follows from Theorem 2.1.*

A. Appendix

To simplify the notation, we drop the subscript i from T_i, μ_i, and A_i throughout the appendix. Thus, in this appendix, T, μ, and A should be thought of as the type space, marginal distribution, and action space, respectively, of any one of the players, not as the joint type spaces, joint distribution, and joint action spaces of all the players. Of course, the theorems that follow are correct with either interpretation, but in the main text we apply the theorems below to the players individually rather than jointly and so the former interpretation is the more relevant. For convenience, we rewrite here without subscripts the assumptions from section 3.2 that will be used in this appendix.

G.1 T is endowed with a sigma-algebra of subsets, $\mathcal{T}$, a measurable partial order, and a countably additive probability measure μ.

G.2 The probability measure μ is atomless.

G.3 There is a countable subset T^0 of T such that every set in $\mathcal{T}$ assigned positive probability by μ contains two points between which lies a point in T^0.

G.4 A is a compact metric space and a semilattice with a closed partial order.

G.5 Either (i) A is a convex subset of a locally convex linear topological space, and the partial order on A is convex, or (ii) A is a locally-complete metric semilattice.

A.1. Partially Ordered Measure Spaces

Preliminaries. We say that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a *partially ordered probability space* if G.1 holds, i.e., if $\mathcal{T}$ is a sigma-algebra of subsets of T, $\geq$ is a measurable partial order on T, and μ is a countably additive probability measure with domain $\mathcal{T}$. If, in addition, G.2 holds, we say that Ψ is a *partially ordered atomless probability space.*

If $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space, Lemma 7.6.1 of Cohn (1980) implies that the sets $\geq(t) = \{t' \in T : t' \geq t\}$ and $\leq(t) = \{t' \in T : t \geq t'\}$ are in $\mathcal{T}$ for each $t \in T$. Hence, for all $t, t' \in T$, the interval $[t, t'] = \{t'' \in T : t' \geq t'' \geq t\}$ is a member of $\mathcal{T}$, being the intersection of $\geq(t)$ and $\leq(t')$. In particular, the singleton set $\{t\}$, being a degenerate interval, is a member of $\mathcal{T}$ for every $t \in T$.

Lemma A.1. *Suppose that $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that $D \in \mathcal{T}$ has positive measure under μ.Then there are sequences, $\{t_n\}_{n=1}^{\infty}$ in T^0 and $\{t'_n\}_{n=1}^{\infty}$ in D, such that μ assigns positive measure to the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$ for every n.*

Proof. For each of the countably many t^0 in T^0, remove from D all members of $\geq(t^0)$ if $D\cap \geq(t^0)$ has μ-measure zero and remove from D all members of $\leq(t^0)$ if $D\cap \leq(t^0)$ has μ-measure zero. Having removed from D countably many subsets each with μ-measure zero, we are left with a set D' with the same positive measure as D. Applying G.3 to D', there exist t, t' in D' and $\tilde{t}_1$ in T^0 such that $t' \geq \tilde{t}_1 \geq t$. Hence, t' is a member of both D' and $\geq(\tilde{t}_1)$ implying that $\mu(D\cap \geq(\tilde{t}_1)) > 0$, and t is a member of both D' and $\leq(\tilde{t}_1)$ implying that $\mu(D\cap \leq(\tilde{t}_1)) > 0$.

Setting $D_0 = D$, we may inductively apply the same argument, for each $k \geq 1$, to the positive μ-measure set $D_k = D_{k-1}\cap \geq(\tilde{t}_k)$, yielding $\tilde{t}_{k+1} \in T^0$ such that $\mu(D_k\cap \geq(\tilde{t}_{k+1})) > 0$ and $\mu(D_k\cap \leq(\tilde{t}_{k+1})) > 0$.

Define the sequence $\{t_n\}_{n=1}^{\infty}$ in T^0 by setting $t_n = \tilde{t}_{3n-2}$ and define the sequence $\{t'_n\}_{n=1}^{\infty}$ in D by letting t'_n be any member of $D \cap [\tilde{t}_{3n-1}, \tilde{t}_{3n}]$. The latter set is always nonempty because for every $k \geq 1$,

$$\begin{aligned} \mu(D \cap [\tilde{t}_k, \tilde{t}_{k+1}]) &\geq \mu([D_{k-1}\cap \geq (\tilde{t}_k)]\cap \leq (\tilde{t}_{k+1})]) \\ &= \mu(D_k\cap \leq (\tilde{t}_{k+1})) \\ &> 0, \end{aligned} \tag{A.1}$$

where the first line follows because D contains D_{k-1} and the second line follows from the definition of D_k. Hence the two sequences, $\{t_n\}$ in T^0 and $\{t'_n\}$ in D, are well-defined.

Finally, for every $n \geq 1$, (A.1) implies $\mu([t_n, t'_n]) \geq \mu([\tilde{t}_{3n-2}, \tilde{t}_{3n-1}]) \geq \mu(D\cap[\tilde{t}_{3n-2}, \tilde{t}_{3n-1}]) > 0$ and $\mu([t'_n, t_{n+1}]) \geq \mu([\tilde{t}_{3n}, \tilde{t}_{3n+1}]) \geq \mu(D \cap [\tilde{t}_{3n}, \tilde{t}_{3n+1}]) > 0$, as desired. ■

Corollary A.2. *Under the hypotheses of Lemma A.1, if $\mu([a, b]) > 0$ then $\mu([a, t^*]) > 0$ and $\mu([t^*, b]) > 0$ for some $t^* \in T^0$.*

Proof. Let $D = [a, b]$ and obtain sequences $\{t_n\}$ in T^0 and $\{t'_n\}$ in $[a, b]$ satisfying the conclusion of Lemma A.1. Then letting $t^* = t_2 \in T^0$ for example, yields $\mu([a, t^*]) \geq \mu([t'_1, t_2]) > 0$

where the first inequality follows because $t_1' \in [a,b]$ implies $[a,t^*]$ contains $[t_1',t^*] = [t_1',t_2]$, and $\mu([t^*,b]) \geq \mu([t_2,t_2']) > 0$ where the first inequality follows because $t_2' \in [a,b]$ implies $[t^*,b]$ contains $[t^*,t_2'] = [t_2,t_2']$. ■

Lemma A.3. *If $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered atomless probability space satisfying G.3, then there is a monotone and measurable function $\Phi : T \to [0,1]$ such that $\mu(\Phi^{-1}(\alpha)) = 0$ for every $\alpha \in [0,1]$.*

Proof. Let $T^0 = \{t_1, t_2, ...\}$ be the countable subset of T in G.3. Define $\Phi : T \to [0,1]$ as follows:

$$\Phi(t) = \sum_{k=1}^{\infty} 2^{-k} \mathbf{1}_{\geq(t_k)}(t). \tag{A.2}$$

Clearly, Φ is monotone and measurable, being the pointwise convergent sum of monotone and measurable functions. It remains to show that $\mu(\Phi^{-1}(\alpha)) = 0$ for every $\alpha \in [0,1]$.

Suppose, by way of contradiction, that $\mu(\Phi^{-1}(\alpha)) > 0$. Because μ is atomless, $\mu(\Phi^{-1}(\alpha)\backslash T^0) = \mu(\Phi^{-1}(\alpha)) > 0$ and so applying G.3 to $\Phi^{-1}(\alpha)\backslash T^0$ yields t', t'' in $\Phi^{-1}(\alpha)\backslash T^0$ and $t_k \in T^0$ such that $t'' \geq t_k \geq t'$. But then $\alpha = \Phi(t'') \geq \Phi(t') + 2^{-k} > \Phi(t') = \alpha$, a contradiction. ■

A.2. Semilattices

The standard proofs of the next two lemmas are omitted.

Lemma A.4. *If G.4 holds, and a_n, b_n, c_n are sequences in A such that $a_n \leq b_n \leq c_n$ for every n and both a_n and c_n converge to a, then b_n converges to a.*

Lemma A.5. *If G.4 holds, then every nondecreasing sequence and every nonincreasing sequence in A converges.*

Lemma A.6. *If G.4 holds, then A is a complete semilattice.*

Proof. Let S be a nonempty subset of A. Because A is a compact metric space, S has a countable dense subset, $\{a_1, a_2, ...\}$. Let $a^* = \lim_n a_1 \vee ... \vee a_n$, where the limit exists by Lemma A.5. Suppose that $b \in A$ is an upper bound for S and let a be an arbitrary element of S. Then, some sequence, a_{n_k}, converges to a. Moreover, $a_{n_k} \leq a_1 \vee a_2 \vee ... \vee a_{n_k} \leq b$ for every k. Taking the limit as $k \to \infty$ yields $a \leq a^* \leq b$. Hence, $a^* = \vee S$. ■

A.3. The Space of Monotone Functions from T into A

In this subsection we introduce a metric, δ, under which the space $\mathcal{M}$ of monotone functions from T into A will be shown to be a compact metric space. Further, it will be shown that under suitable conditions, the metric space $(\mathcal{M}, \delta)$ is an absolute retract. Some preliminary results are required.

Recall that a property $P(t)$ is said to hold for μ-a.e. $t \in T$ if the set of $t \in T$ on which $P(t)$ holds contains a measurable subset having μ-measure one. We next introduce an important definition.

Definition A.7. *Given a partially ordered probability space $\Psi = (T, \mathcal{T}, \mu, \geq)$ and a partially ordered metric space A, say that a monotone function $f : T \to A$ is Ψ* quasi-continuous *at $t \in T$ if there are sequences $\{t_n\}$ and $\{t_n'\}$ in T such that $\lim_n f(t_n) = \lim_n f(t_n') = f(t)$ and the intervals $[t_n, t]$ and $[t, t_n']$ have positive μ-measure for every n.*

Remark 9. *(i) The positive measure condition implies that the intervals are nonempty, i.e., that $t'_n \geq t \geq t_n$ for every n. (ii) Because we have not endowed T with a topology, neither $\{t_n\}$ nor $\{t'_n\}$ is required to converge. (iii) f is Ψ quasi-continuous at every atom t of μ because we may set $t_n = t'_n = t$ for all n.*

Lemma A.8. *Suppose that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3, that A satisfies G.4, and that $f : T \to A$ is measurable and monotone. Then the set of points at which f is Ψ quasi-continuous is measurable.*

Proof. Suppose that f is Ψ quasi-continuous at $t \in T$ and that the sequences $\{t_n\}$ and $\{t'_n\}$ satisfy the conditions in Definition A.7. Then, by Corollary A.2, for each n there exist $\tilde{t}_n, \tilde{t}'_n$ in T^0 such that the intervals $[t_n, \tilde{t}_n]$ $[\tilde{t}_n, t]$, $[t, \tilde{t}'_n]$, and $[\tilde{t}'_n, t'_n]$ each have positive μ-measure. In particular, $t_n \leq \tilde{t}_n \leq t$ implies $f(t_n) \leq f(\tilde{t}_n) \leq f(t)$ and $t \leq \tilde{t}'_n \leq t'_n$ implies $f(t) \leq f(\tilde{t}'_n) \leq f(t'_n)$. Consequently, by Lemma A.4, $\lim_n f(\tilde{t}_n) = \lim_n f(\tilde{t}'_n) = f(t)$. We conclude that the definition of Ψ quasi-continuity at any $t \in T$ would be unchanged if the sequences $\{t_n\}$ and $\{t'_n\}$ were required to be in T^0.

Let d be the metric on A and for every $t_1, t_2 \in T$ and every $n = 1, 2, ...,$ define

$$T^n_{t_1,t_2} = \left\{ t \in T : \mu([t_1, t]) > 0, \mu([t, t_2]) > 0, d(f(t_1), f(t)) < \frac{1}{n}, d(f(t_2), f(t)) < \frac{1}{n} \right\}.$$

Then according to the conclusion drawn in the preceding paragraph, the set of points at which f is Ψ quasi-continuous is,

$$\bigcap_{n \geq 1} \bigcup_{t_1,t_2 \in T^0} T^n_{t_1,t_2}.$$

Consequently, it suffices to show that each $T^n_{t_1,t_2}$ is measurable, and for this it suffices to show that, as functions of t, the functions $\mu([t_1, t])$, $\mu([t, t_2])$, $d(f(t_1), f(t))$, and $d(f(t_2), f(t))$ are measurable.

The functions $d(f(t_1), f(t))$ and $d(f(t_2), f(t))$ are measurable in t because the metric d is continuous in its arguments and f is measurable. For the measurability of $\mu([t_1, t])$, let $E = \{(t', t'') \in T \times T : t' \geq t''\} \cap (T \times \geq(t_1))$. Then E is in $\mathcal{T} \times \mathcal{T}$ by the measurability of $\geq$, and $[t_1, t] = E_t$ is the slice of E in which the first coordinate is t. Proposition 5.1.2 of Cohn (1980) states that $\mu(E_t)$ is measurable in t. A similar argument shows that $\mu([t, t_2])$ is measurable in t. ■

Lemma A.9. *Suppose that G.1, G.3 and G.4 hold, i.e., that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that A satisfies G.4. If $f : T \to A$ is measurable and monotone, then f is Ψ quasi-continuous at μ-a.e. $t \in T$.*

Proof. Let D denote the set of discontinuity points of f. By Lemma A.8, D is a member of $\mathcal{T}$. It suffices to show that $\mu(D) = 0$.

Define $T^n_{t_1,t_2}$ as in the proof of Lemma A.8 so that,

$$D = \bigcup_{n=1} \bigcap_{t_1,t_2 \in T^0} \left(T^n_{t_1,t_2}\right)^c,$$

and suppose, by way of contradiction, that $\mu(D) > 0$. Then, for some $N \geq 1$, $\mu(D_N) > 0$, where $D_N = \cap_{t_1,t_2 \in T^0} \left(T^N_{t_1,t_2}\right)^c$.

Let d denote the metric on A. Then for every $t \in D_N$ and every $t_1, t_2 \in T^0$ such that the intervals $[t_1, t]$ and $[t, t_2]$ have positive μ-measure, either,

$$d(f(t_1), f(t)) \geq \frac{1}{N} \text{ or } d(f(t_2), f(t)) \geq \frac{1}{N}. \tag{A.3}$$

By Lemma A.1, there are sequences, $\{t_n\}_{n=1}^{\infty}$ in T^0 and $\{t'_n\}_{n=1}^{\infty}$ in D_N, such that μ assigns positive measure to the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$ for every n. Consequently, for every n, (A.3) implies that either,

$$d(f(t_n), f(t'_n)) \geq \frac{1}{N} \text{ or } d(f(t_{n+1}), f(t'_n)) \geq \frac{1}{N}. \tag{A.4}$$

On the other hand, because for every n the intervals $[t_n, t'_n]$ and $[t'_n, t_{n+1}]$, having positive μ-measure, are nonempty, we have $t_1 \leq t'_1 \leq t_2 \leq t'_2 \leq ...$. Hence, the monotonicity of f implies that,

$$f(t_1) \leq f(t'_1) \leq f(t_2) \leq f(t'_2) \leq ...$$

is a monotone sequence of points in A and must therefore converge by Lemma A.5. But then both $d(f(t_n), f(t'_n))$ and $d(f(t_{n+1}), f(t'_n))$ converge to zero, contradicting (A.4), and so we conclude that $\mu(D) = 0$. ■

Lemma A.10. *(A Generalized Helly Selection Theorem). Suppose that G.1, G.3 and G.4 hold, i.e., that $\Psi = (T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space satisfying G.3 and that A satisfies G.4. If $f_n : T \to A$ is a sequence of monotone functions – not necessarily measurable – then there is a subsequence, f_{n_k}, and a measurable monotone function, $f : T \to A$, such that $f_{n_k}(t) \to_k f(t)$ for μ-a.e. $t \in T$.*

Proof. Let $T^0 = \{t_1, t_2, ...\}$ be the countable subset of T satisfying G.3. Choose a subsequence, f_{n_k}, of f_n such that, for every i, $\lim_k f_{n_k}(t_i)$ exists. Define $f(t_i) = \lim_k f_{n_k}(t_i)$ for every i, and extend f to all of T by defining $f(t) = \vee\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$.[55] By Lemma A.6, this is well defined because $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t\}$ is nonempty for each t since it contains any limit point of $f_{n_k}(t)$. Indeed, if $f_{n_{k_j}}(t) \to_j a$, then $a = \lim_j f_{n_{k_j}}(t) \leq \lim_j f_{n_{k_j}}(t_i) = f(t_i)$ for every $t_i \geq t$. Further, as required, the extension to T is monotone and leaves the values of f on $\{t_1, t_2, ...\}$ unchanged, where the latter follows because the monotonicity of f on $\{t_1, t_2, ...\}$ implies that $\{a \in A : a \leq f(t_i) \text{ for all } t_i \geq t_k\} = \{a \in A : a \leq f(t_k)\}$. To see that f is measurable, note first that $f(t) = \lim_m g_m(t)$, where $g_m(t) = \vee\{a \in A : a \leq f(t_i) \text{ for all } i = 1, ..., m \text{ such that } t_i \geq t\}$, and where the limit exists by Lemma A.5. Because the partial order on T is measurable, each g_m is a measurable simple function. Hence, f is measurable, being the pointwise limit of measurable functions.

Let f be Ψ quasi-continuous at $t \in T$. By Lemma A.9, it suffices to show that $f_{n_k}(t) \to f(t)$. So, suppose that $f_{n_{k_j}}(t) \to a \in A$ for some subsequence n_{k_j} of n_k. By the compactness of A, it suffices to show that $a = f(t)$.

Because f is Ψ quasi-continuous at $t \in T$, the argument in the first paragraph of the proof of Lemma A.8 implies that there exist sequences $\{t_{i_n}\}$ and $\{t'_{i_n}\}$ in T^0 such that $\lim_n f(t_{i_n}) = \lim_n f(t'_{i_n}) = f(t)$ and such that the intervals $[t_{i_n}, t]$ and $[t, t'_{i_n}]$ have positive μ-measure for every n. In particular, the intervals $[t_{i_n}, t]$ and $[t, t'_{i_n}]$ are always nonempty and

[55] Hence, $f(t) = \vee A$ if no $t_i \geq t$.

so $t_{i_n} \leq t \leq t'_{i_n}$, implying by the monotonicity of each f_{n_k} that,

$$f_{n_k}(t_{i_n}) \leq f_{n_k}(t) \leq f_{n_k}(t'_{i_n})$$

for every k and n. Because the partial order on A is closed, taking the limit first in k yields,

$$f(t_{i_n}) \leq a \leq f(t'_{i_n}),$$

and taking the limit next in n yields

$$f(t) \leq a \leq f(t),$$

from which we conclude that $a = f(t)$, as desired. ■

By setting $\{f_n\}$ in Lemma A.10 equal to a constant sequence, we obtain the following.

Lemma A.11. *Under G.1, G.3 and G.4, every monotone function from T into A is μ almost everywhere equal to a measurable monotone function.*

We now introduce a metric on $\mathcal{M}$, the space of monotone functions from T into A. Denote the metric on A by d and assume without loss that $d(a, b) \leq 1$ for all $a, b \in A$. Define the metric, δ, on $\mathcal{M}$ by

$$\delta(f, g) = \int_T d(f(t), g(t))d\mu(t),$$

which is well-defined by Lemma A.11.

Formally, the resulting metric space $(\mathcal{M}, \delta)$ is the space of equivalence classes of monotone functions that are equal μ almost everywhere – i.e., two functions are in the same equivalence class if there is a measurable subset of T having μ-measure one on which they coincide. Nevertheless, and analogous to the standard treatment of $\mathcal{L}_p$ spaces, we focus on the elements of the original space $\mathcal{M}$ rather than on the equivalence classes themselves.

Lemma A.12. *Under G.1, G.3 and G.4, $\delta(f_k, f) \to 0$ if and only if $d(f_k(t), f(t)) \to 0$ for μ-a.e. $t \in T$.*

Proof. (only if) Suppose that $\delta(f_k, f) \to 0$. By Lemma A.9, it suffices to show that $f_k(t) \to f(t)$ for all Ψ quasi-continuity points, t, of f.

Let t_0 be a Ψ quasi-continuity point of f. Because A is compact, it suffices to show that an arbitrary convergent subsequence, $f_{k_j}(t_0)$, of $f_k(t_0)$ converges to $f(t_0)$. So, suppose that $f_{k_j}(t_0)$ converges to $a \in A$. By Lemma A.10, there is a further subsequence, $f_{k'_j}$ of f_{k_j} and a monotone measurable function, $g : T \to A$ such that $f_{k'_j}(t) \to g(t)$ for μ a.e. t in T. Because d is bounded, the dominated convergence theorem implies that $\delta(f_{k'_j}, g) \to 0$. But $\delta(f_{k'_j}, f) \to 0$ then implies that $\delta(f, g) = 0$ and so $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T.

Because t_0 is a Ψ quasi-continuity point of f, there are sequences $\{t_n\}_{n=1}^{\infty}$ and $\{t'_n\}_{n=1}^{\infty}$ in T such that $\lim_n f(t_n) = \lim_n f(t'_n) = f(t_0)$ and the intervals $[t_n, t_0]$ and $[t_0, t'_n]$ have positive μ-measure for every $n \geq 1$.

Consequently, because $f_{k'_j}(t) \to f(t)$ for μ a.e. t in T and because the intervals $[t_n, t_0]$ and $[t_0, t'_n]$ have positive μ-measure, for every n there exist $\tilde{t}_n$ and $\tilde{t}'_n$ such that $t_n \leq \tilde{t}_n \leq t_0 \leq \tilde{t}'_n \leq t'_n$, $f_{k'_j}(\tilde{t}_n) \to_j f(\tilde{t}_n)$ and $f_{k'_j}(\tilde{t}'_n) \to_j f(\tilde{t}'_n)$. Consequently, $f_{k'_j}(\tilde{t}_n) \leq f_{k'_j}(t_0) \leq f_{k'_j}(\tilde{t}'_n)$, and taking the limit as $j \to \infty$ yields $f(\tilde{t}_n) \leq a \leq f(\tilde{t}'_n)$, so that $f(t_n) \leq f(\tilde{t}_n) \leq a \leq f(\tilde{t}'_n) \leq$

$f(t_n')$ and therefore $f(t_n) \leq a \leq f(t_n')$. Taking the limit of the latter inequality as $n \to \infty$ yields $f(t_0) \leq a \leq f(t_0)$, so that $a = f(t_0)$, as desired.

(if) To complete the proof, suppose that $f_k(t)$ converges to $f(t)$ for μ-a.e. $t \in T$. Then, because d is bounded, the dominated convergence theorem implies that $\delta(f_k, f) \to 0$. ■

Combining Lemmas A.10 and A.12 we obtain the following.

Lemma A.13. *Under G.1, G.3 and G.4, the metric space $(\mathcal{M}, \delta)$ is compact.*

Lemma A.14. *Suppose that G.1, G.3 and G.4 hold and $f : T \to A$ is monotone. If for every $t \in T$, $\bar{f}(t) = \vee g(t)$, where the join is taken over all monotone $g : T \to A$ s.t. $g(t) \leq f(t)$ for μ-a.e. $t \in T$, then $\bar{f} : T \to A$ is monotone and $\bar{f}(t) = f(t)$ for μ-a.e. $t \in T$.*[56]

Proof. Note that $\bar{f}(t)$ is well-defined for each $t \in T$ by Lemma A.6, and $\bar{f}$ is monotone, being the pointwise join of monotone functions. It remains only to show that $\bar{f}(t) = f(t)$ for μ-a.e. $t \in T$.

Suppose first that f is measurable. Let C denote the measurable (by Lemma A.8) set of Ψ quasi-continuity points of f, and let L_f denote the set of monotone $g : T \to A$ such that $g(t) \leq f(t)$ for μ-a.e. $t \in T$. By Lemma A.9, $\mu(C) = 1$.

We claim that $f(t) \geq g(t)$ for every $t \in C$ and every $g \in L_f$. To see this, fix $g \in L_f$ and let D be a measurable set with μ-measure one such that $g(t) \leq f(t)$ for every $t \in D$. Consider $t \in C$. Because t is a Ψ quasi-continuity point of f, there are sequences $\{t_n\}$ and $\{t_n'\}$ in T such that $\lim_n f(t_n) = \lim_n f(t_n') = f(t)$ and such that the intervals $[t_n, t]$ and $[t, t_n']$ have positive μ-measure for every n. Therefore, in particular, the set $D \cap [t, t_n']$ has positive μ-measure for every n. Consequently, for every n we may choose $\tilde{t}_n \in D \cap [t, t_n']$, and therefore $f(t_n') \geq f(\tilde{t}_n) \geq g(\tilde{t}_n) \geq g(t)$, for all n. In particular, $f(t_n') \geq g(t)$ for all n, so that $f(t) = \lim_n f(t_n') \geq g(t)$, proving the claim.

Consequently, $f(t) \geq \vee_{g \in L_f} g(t)$ for every $t \in C$. Hence, because f itself is a member of L_f, $f(t) = \vee_{g \in L_f} g(t) = \bar{f}(t)$ for every $t \in C$ and therefore for μ-a.e. $t \in T$.

If f is not measurable, then by Lemma A.11, we may repeat the argument replacing f with a measurable and monotone $\tilde{f} : T \to A$ that is μ-almost-everywhere equal to f, concluding that $\tilde{f}(t) = \vee_{g \in L_{\tilde{f}}} g(t)$ for μ-a.e. $t \in T$. But $L_f = L_{\tilde{f}}$ then implies that for μ-a.e. $t \in T$, $f(t) = \tilde{f}(t) = \vee_{g \in L_{\tilde{f}}} g(t) = \vee_{g \in L_f} g(t) = \bar{f}(t)$. ■

Lemma A.15. *Assume G.1, G.3 and G.4. Suppose that the join operator on A is continuous and that $\Phi : T \to [0, 1]$ is a monotone and measurable function such that $\mu(\Phi^{-1}(c)) = 0$ for every $c \in [0, 1]$. Define $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by defining for every $t \in T$,*

$$h(\tau, f, g)(t) = \begin{cases} f(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau < 1/2 \\ g(t), & \text{if } \Phi(t) \leq |1 - 2\tau| \text{ and } \tau \geq 1/2 \\ f(t) \vee g(t), & \text{if } \Phi(t) > |1 - 2\tau| \end{cases} \tag{A.5}$$

Then $h : [0, 1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is continuous.

Proof. Suppose that $(\tau_k, f_k, g_k) \to (\tau, f, g) \in [0, 1] \times \mathcal{M} \times \mathcal{M}$. By Lemma A.12, there is a μ-measure one subset, D, of T such that $f_k(t) \to f(t)$ and $g_k(t) \to g(t)$ for every $t \in D$. There are three cases: $\tau = 1/2$, $\tau > 1/2$ and $\tau < 1/2$.

[56] It can be further shown that, for all $t \in T$, $\bar{f}(t) = \vee\{a \in A : a \leq f(t')$ for all $t' \geq t$ s.t. $t' \in T$ is a Ψ quasi-continuity point of $f\}$. But we will not need this result.

Suppose that $\tau < 1/2$. For each $t \in D$ such that $\Phi(t) < |1-2\tau|$, we have $\Phi(t) < |1-2\tau_k|$ for all k large enough. Hence, $h(\tau_k, f_k, g_k)(t) = f_k(t)$ for all k large enough, and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \to f(t) = h(\tau, f, g)(t)$. Similarly, for each $t \in D$ such that $\Phi(t) > |1-2\tau|$, $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(\tau, f, g)(t)$, where the limit follows because $\vee$ is continuous. Because $\mu(\{t \in T : \Phi(t) = |1-2\tau|\}) = 0$, we have therefore shown that if $\tau < 1/2$, then $h(\tau_k, f_k, g_k)(t) \to h(\tau, f, g)(t)$ for μ a.e. $t \in T$ and so, by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$.

Because the case $\tau > 1/2$ is similar to $\tau < 1/2$, we consider only the remaining case in which $\tau = 1/2$. In this case, $|1-2\tau_k| \to 0$. Consequently, for any $t \in T$ such that $\Phi(t) > 0$, we have $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t)$ for k large enough and so $h(\tau_k, f_k, g_k)(t) = f_k(t) \vee g_k(t) \to f(t) \vee g(t) = h(1/2, f, g)(t)$. Hence, because $\mu(\{t \in T : \Phi(t) = 0\}) = 0$, we have shown that $h(\tau_k, f_k, g_k)(t) \to h(1/2, f, g)(t)$ for μ a.e. $t \in T$, and so again by Lemma A.12, $h(\tau_k, f_k, g_k) \to h(\tau, f, g)$. ■

Lemma A.16. *Under G.1-G.5, the metric space $(\mathcal{M}, \delta)$ is an absolute retract.*

Proof. Define $h : [0,1] \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by $h(\tau, s, s')(t) = \tau s(t) + (1-\tau)s'(t)$ for all $t \in T$ if G.5(i) holds, and by (A.5) if G.5(ii) holds, where the monotone function $\Phi(\cdot)$ appearing in (A.5) is defined by (A.2). Note that h maps into $\mathcal{M}$ in case G.5(i) holds because A is convex (which itself follows because the partial order on A is convex). We claim that, in each case, h is continuous. Indeed, if G.5(ii) holds, the continuity of h follows from Lemmas A.3 and A.15. If G.5(i) holds and the sequence $(\tau_n, s_n, s'_n) \in [0,1] \times \mathcal{M} \times \mathcal{M}$ converges to (τ, s, s'), then by Lemma A.12, $s_n(t) \to s(t)$ and $s'_n(t) \to s'(t)$ for μ-a.e. $t \in T$. Hence, because A is a convex topological space, $\tau_n s_n(t) + (1-\tau_n)s'_n(t) \to \tau s(t) + (1-\tau)s'(t)$ for μ-a.e. $t \in T$. But then Lemma A.12 implies $\tau_n s_n + (1-\tau_n)s'_n \to \tau s + (1-\tau)s'$, as desired.

One consequence of the continuity of h is that for any $g \in \mathcal{M}$, $h(\cdot, \cdot, g)$ is a contraction for $\mathcal{M}$ so that $(\mathcal{M}, \delta)$ is contractible. Hence, by Borsuk (1966, IV (9.1)) and Dugundji (1965), it suffices to show that for each $f' \in \mathcal{M}$ and each neighborhood U of f', there is a neighborhood V of f' and contained in U such that the sets V^n, $n \geq 1$, defined inductively by $V^1 = h([0,1], V, V)$, $V^{n+1} = h([0,1], V, V^n)$, are all contained in U.

We shall establish this by way of contradiction. Specifically, let us suppose to the contrary that for some neighborhood U of $f' \in \mathcal{M}$ there is no open set V containing f' and contained in U such that all the V^n as defined above are contained in U. In particular, for each $k = 1, 2, ...$, taking V to be $B_{1/k}(f')$, the $1/k$ ball around f', there exists n_k such that some $g_k \in V^{n_k}$ is not in U. We derive a contradiction separately for each of the two cases, G.5(i) and G.5(ii).

Case I. Suppose G.5(i) holds. For each n, $V^{n+1} \subset coV$, so that for every $k = 1, 2, ..., g_k \in V^{n_k} \subset coB_{1/k}(f')$. Hence, for each k there exist $f_1^k, ..., f_{n_k}^k$ in $B_{1/k}(f')$ and nonnegative weights $\lambda_1^k, ..., \lambda_{n_k}^k$ summing to one such that $g_k = \sum_{j=1}^{n_k} \lambda_j^k f_j^k \notin U$. Hence, $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ for μ-a.e. $t \in T$ and so for all t in some measurable set E having μ-measure one. Moreover, the sequence $f_1^1, ..., f_{n_1}^1, f_1^2, ..., f_{n_2}^2, ...$ converges to f'. Consequently, by Lemma A.12 the sequence $f_1^1(t), ..., f_{n_1}^1(t), f_1^2(t), ..., f_{n_2}^2(t), ...$ converges to $f'(t)$ for μ-a.e. $t \in T$ and so for all t in some measurable set D having μ-measure one. But then for each $t \in D \cap E$ and every convex neighborhood W_t of $f'(t)$, each of $f_1^k(t), ..., f_{n_k}^k(t)$ is in W_t for all k large enough, and therefore $g_k(t) = \sum_{j=1}^{n_k} \lambda_j^k f_j^k(t)$ is in W_t for k large enough as well. But this implies, by the local convexity of A, that $g_k(t) \to f'(t)$ for every $t \in D \cap E$ and hence for μ-a.e. $t \in T$. Lemma A.12 then implies that $g_k \to f'$, contradicting that no g_k is in U.

Case II. Suppose G.5(ii) holds. As a matter of notation, for $f, g \in \mathcal{M}$, write $f \leq g$ if $f(t) \leq g(t)$ for μ-a.e. $t \in T$. Also, for any sequence of monotone functions $f_1, f_2, ...,$ in $\mathcal{M}$, denote by $f_1 \vee f_2 \vee ...$ the monotone function taking the value $\lim_n[f_1(t) \vee f_2(t) \vee ... \vee f_n(t)]$ for each t in T. This is well-defined by Lemma A.5.

If $g \in V^1$, then $g = h(\tau, f_0, f_1)$ for some $\tau \in [0, 1]$ and some $f_0, f_1 \in V$. Hence, by the definition of h, we have $g \leq f_0 \vee f_1$ and either $f_0 \leq g$ or $f_1 \leq g$. We may choose the indices so that $f_0 \leq g \leq f_0 \vee f_1$. Inductively, it can similarly be seen that if $g \in V^n$, then there exist $f_0, f_1, ..., f_n \in V$ such that

$$f_0 \leq g \leq f_0 \vee ... \vee f_n. \tag{A.6}$$

Hence, for each $k = 1, 2, .. ,$ $g_k \in V^{n_k}$ and (A.6) imply that there exist $f_0^k, ..., f_{n_k}^k \in V = B_{1/k}(f')$ such that

$$f_0^k \leq g_k \leq f_0^k \vee ... \vee f_{n_k}^k. \tag{A.7}$$

Consider the sequence $f_0^1, ..., f_{n_1}^1, f_0^2, ..., f_{n_2}^2, ...$. Because f_j^k is in $B_{1/k}(f')$, this sequence converges to f'. Let us reindex this sequence as $f_1, f_2, ...$. Hence, $f_j \to f'$.

Because for every n the set $\{f_n, f_{n+1}, ...\}$ contains the set $\{f_0^k, ..., f_{n_k}^k\}$ whenever k is large enough, we have

$$f_0^k \vee ... \vee f_{n_k}^k \leq \vee_{j \geq n} f_j,$$

for every n and all large enough k. Combined with (A.7), this implies that

$$f_0^k \leq g_k \leq \vee_{j \geq n} f_j \tag{A.8}$$

for every n and all large enough k.

Now, $f_0^k \to f'$ as $k \to \infty$. Hence, by Lemma A.12, $f_0^k(t) \to f'(t)$ for μ-a.e. $t \in T$. Consequently, if for μ-a.e. $t \in T$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$, then (A.8) and Lemma A.4 would imply that $g_k(t) \to f'(t)$ for μ-a.e. $t \in T$. Then, Lemma A.12 would imply that $g_k \to f'$ contradicting that no g_k is in U, and completing the proof.

It therefore remains only to establish that for μ a.e. $t \in T$, $\vee_{j \geq n} f_j(t) \to f'(t)$ as $n \to \infty$. But, by Lemma A.18, because A is locally complete this will follow if $f_j(t) \to_j f'(t)$ for μ a.e. t, which follows from Lemma A.12 because $f_j \to f'$. ■

A.4. Locally Complete Metric Semilattices

Lemma A.17. *If A is a compact upper-bound-convex subset of Euclidean space and a semilattice under the coordinatewise partial order, then A is a metric semilattice, i.e., $\vee$ is continuous.*

Proof. Suppose that $a_n \to a$, $b_n \to b$, $a \vee b = c$, and $a_n \vee b_n \to d$, where all of these points are in A. We must show that $c = d$. Because $a_n \leq a_n \vee b_n$, taking limits implies $a \leq d$. Similarly, $b \leq d$, so that $c = a \vee b \leq d$. Thus, it remains only to show that $c \geq d$.

Let $\bar{a} = \vee A$ denote the largest element of A, which is well defined by Lemma A.6. By the upper-bound-convexity of A, $\varepsilon\bar{a} + (1-\varepsilon)c \in A$ for every $\varepsilon \in [0, 1]$. Because the coordinatewise partial order is closed, it suffices to show that $\varepsilon\bar{a} + (1 - \varepsilon)c \geq d$ for every $\varepsilon > 0$ sufficiently small. So, fix $\varepsilon \in (0, 1)$ and consider the kth coordinate, c_k, of c. If for some n, $a_{kn} > c_k$, then because $\bar{a}_k \geq a_{kn}$ we have $\bar{a}_k > c_k$ and therefore $\varepsilon\bar{a}_k + (1 - \varepsilon)c_k > c_k$. Consequently, because $a_{kn} \to_n a_k \leq c_k$, we have $\varepsilon\bar{a}_k + (1-\varepsilon)c_k > a_{kn}$ for all n sufficiently large. On the other hand, suppose that $a_{kn} \leq c_k$ for all n. Then because $\bar{a}_k \geq c_k$ we have $\varepsilon\bar{a}_k + (1 - \varepsilon)c_k \geq a_{kn}$ for all n. So, in either case $\varepsilon\bar{a}_k + (1 - \varepsilon)c_k \geq a_{kn}$ for all n sufficiently large. Therefore, because k is arbitrary, $\varepsilon\bar{a} + (1 - \varepsilon)c \geq a_n$ for all n sufficiently large. Similarly, $\varepsilon\bar{a} + (1 - \varepsilon)c \geq b_n$ for all

n sufficiently large. Therefore, because $\varepsilon\bar{a} + (1-\varepsilon)c \in A$, $\varepsilon\bar{a} + (1-\varepsilon)c \geq a_n \vee b_n$ for all n sufficiently large. Taking limits in n gives $\varepsilon\bar{a} + (1-\varepsilon)c \geq d$. ■

Lemma A.18. *If G.3 holds, then A is locally complete if and only if for every $a \in A$ and every sequence a_n converging to a, $\lim_n(\vee_{k\geq n} a_k) = a$.*

Proof. We first demonstrate the "only if" direction. Suppose that A is locally complete, that U is a neighborhood of $a \in A$, and that $a_n \to a$. By local completeness, there is a neighborhood W of a contained in U such that every subset of W has a least upper bound in U. In particular, because for n large enough $\{a_n, a_{n+1}, ...\}$ is a subset of W, the least upper bound of $\{a_n, a_{n+1}, ...\}$, namely $\vee_{k\geq n} a_k$, is in U for n large enough. Since U was arbitrary, this implies $\lim_n(\vee_{k\geq n} a_k) = a$.

We now turn to the "if" direction. Fix any $a \in A$, and let $B_{1/n}(a)$ denote the open ball around a with radius $1/n$. For each n, $\vee B_{1/n}(a)$ is well-defined by Lemma A.6. Moreover, because $\vee B_{1/n}(a)$ is nonincreasing in n, $\lim_n \vee B_{1/n}(a)$ exists by Lemma A.5. We first argue that $\lim_n \vee B_{1/n}(a) = a$. For each n, construct as in the proof of Lemma A.6 a sequence $\{a_{n,m}\}$ of points in $B_{1/n}(a)$ such that $\lim_m(a_{n,1} \vee ... \vee a_{n,m}) = \vee B_{1/n}(a)$. We may therefore choose m_n sufficiently large so that the distance between $a_{n,1} \vee ... \vee a_{n,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$. Consider now the sequence $\{a_{1,1}, ..., a_{1,m_1}, a_{2,1}, ..., a_{2,m_2}, a_{3,1}, ..., a_{3,m_3}, ...\}$. Because $a_{n,m}$ is in $B_{1/n}(a)$, this sequence converges to a. Consequently, by hypothesis,

$$\lim_n(a_{n,1} \vee ... \vee a_{n,m_n} \vee a_{(n+1),1} \vee ... \vee a_{(n+1),m_{(n+1)}} \vee ...) = a.$$

But because every $a_{k,j}$ in the join in parentheses on the left-hand side above (denote this join by b_n) is in $B_{1/n}(a)$, we have

$$a_{n,1} \vee ... \vee a_{n,m_n} \leq b_n \leq \vee B_{1/n}(a).$$

Therefore, because for every n the distance between $a_{n,1} \vee ... \vee a_{n,m_n}$ and $\vee B_{1/n}(a)$ is less than $1/n$, Lemma A.4 implies that $\lim_n \vee B_{1/n}(a) = \lim_n b_n$. But since $\lim_n b_n = a$, we have $\lim_n \vee B_{1/n}(a) = a$. Next, for each n, let S_n be an arbitrary nonempty subset of $B_{1/n}(a)$, and choose any $s_n \in S_n$. Then $s_n \leq \vee S_n \leq \vee B_{1/n}(a)$. Because $s_n \in B_{1/n}(a)$, Lemma A.4 implies that $\lim_n \vee S_n = a$. Consequently, for every neighborhood U of a, there exists n large enough such that $\vee S$ (well-defined by Lemma A.6) is in U for every subset S of $B_{1/n}(a)$. Since a was arbitrary, A is locally complete. ■

Lemma A.19. *Every compact Euclidean metric semilattice is locally complete.*

Proof. Suppose that $a_n \to a$ with every a_n and a in the semilattice, which we assume to be a subset of $\mathbb{R}^K$. By Lemma A.18, it suffices to show that $\lim_n(\vee_{k\geq n} a_k) = a$. By Lemma A.5, $\lim_n(\vee_{k\geq n} a_k)$ exists and is equal to $\lim_n \lim_m(a_n \vee ... \vee a_m)$ since $a_n \vee ... \vee a_m$ is nondecreasing in m, and $\lim_m(a_n \vee ... \vee a_m)$ is nonincreasing in n. For each dimension $k = 1, ..., K$, let $a^k_{n,m}$ denote the first among $a_n, a_{n+1}, ..., a_m$ with the largest kth coordinate. Hence, $a_n \vee ... \vee a_m = a^1_{n,m} \vee ... \vee a^K_{n,m}$, where the right-hand side consists of K terms. Because $a_n \to a$, $\lim_m a^k_{n,m}$ exists for each k and n, and $\lim_n \lim_m a^k_{n,m} = a$ for each k. Consequently, $\lim_n \lim_m(a_n \vee ... \vee a_m) = \lim_n \lim_m(a^1_{n,m} \vee ... \vee a^K_{n,m}) = (\lim_n \lim_m a^1_{n,m}) \vee ... \vee (\lim_n \lim_m a^K_{n,m}) = a \vee ... \vee a = a$, as desired. ■

Lemma A.20. *If G.4 holds and for all $a \in A$, every neighborhood of a contains a' such that $b' \leq a'$ for all b' close enough to a, then A is locally complete.*

Proof. Suppose that $a_n \to a$. By Lemma A.18, it suffices to show that $\lim_n (\vee_{k \geq n} a_k) = a$. For every n and m, $a_m \leq a_m \vee a_{m+1} \vee ... \vee a_{m+n}$, and so taking the limit first as $n \to \infty$ and then as $m \to \infty$ gives $a \leq \lim_m \vee_{k \geq m} a_k$, where the limit in n exists by Lemma A.5 because the sequence is monotone. Hence, it suffices to show that $\lim_m \vee_{k \geq m} a_k \leq a$.

Let U be a neighborhood of a and let a' be chosen as in the statement of the lemma. Then, because $a_m \to a$, $a_m \leq a'$ for all m large enough. Consequently, for m large enough and for all n, $a_m \vee a_{m+1} \vee ... \vee a_{m+n} \leq a'$. Taking the limit first in n and then in m yields $\lim_m \vee_{k \geq m} a_k \leq a'$. Because for every neighborhood U of a this holds for some a' in U, $\lim_m \vee_{k \geq m} a_k \leq a$, as desired. ■

A.5. Assumption G.3

Say that two points in a partially ordered metric space are *strictly ordered* if they are contained in disjoint open sets and every member of one set is greater or equal to every member of the other. The following lemma provides a sufficient condition for G.3 to hold when T happens to be a separable metric space.

Lemma A.21. *Suppose that $(T, \mathcal{T}, \mu, \geq)$ is a partially ordered probability space, that T is a separable metric space and that $\mathcal{T}$ contains the open sets. Then G.3 holds if every atomless set having positive μ-measure contains two strictly ordered points.*

Proof. Let T^0 be the union of a countable dense subset of T and the countable set of atoms of μ, and suppose that $D \in \mathcal{T}$ has positive μ-measure. We must show that $t_1 \geq t^0 \geq t_2$ for some $t_1, t_2 \in D$ and some $t^0 \in T^0$.

If D contains an atom, t^0, of μ, then we may set $t_1 = t_2 = t^0$ and we are done. Hence, we may assume that D is atomless.

Without loss, we may assume that $\mu(D \cap U) > 0$ for every open set U whose intersection with D is nonempty.[57] Because $\mu(D) > 0$, there exist $t_1', t_2 \in D$ and open sets U_1' containing t_1' and U_2 containing t_2 such that every member of U_1' is greater or equal to every member of U_2, which we shall write as $U_1' \geq U_2$.

Because $D \cap U_1'$ is nonempty – it contains t_1' – $\mu(D \cap U_1') > 0$. Consequently, there exist $t_1, t_1'' \in D \cap U_1'$ and open sets U_1 containing t_1 and U_1'' containing t_1'' such that $U_1 \geq U_1''$. Hence, $U_1 \cap U_1' \geq U_1'' \cap U_1' \geq U_2$. Therefore, because the open set $U_1'' \cap U_1'$ is nonempty – it contains t_1'' – it contains some t^0 in the dense set T^0. Hence, $t_1 \geq t^0 \geq t_2$, because $t_1 \in U_1 \cap U_1'$ and $t_2 \in U_2$. Noting that t_1 and t_2 are members of D completes the proof. ■

A.6. Proofs from Section 5

Proof of Corollary 5.2. Consider the uniform-price auction but where unit-bids can be any nonnegative real number. Because marginal values are between zero and one, without loss we may restrict attention to unit-bids in $[0, 1]$. The resulting game is discontinuous. Remark 3.1 of Reny (1999) establishes that if this game is better-reply secure, then the limit of a convergent sequence of pure strategy ε-equilibria, as ε tends to zero, is a pure strategy equilibrium. Hence, in view of Lemma A.13, it suffices to show that the auction game is

[57] Otherwise, replace D with $D \cap V^c$, where V is the largest open set whose intersection with D has μ-measure zero. To see that V is well-defined, let $\{U_i\}$ be a countable base of open sets. Then V is the union of all the U_i satisfying $\mu(U_i \cap D) = 0$.

better-reply secure (when players employ monotone pure strategies) and that it possesses, for every $\varepsilon > 0$, an ε-equilibrium in monotone pure strategies.

An argument analogous to that given in the first paragraph on p. 1046 of Reny (1999) shows that the uniform-price auction game with unit-bid space $[0, 1]$ is better-reply secure when bidders employ monotone pure strategies. Fix $\varepsilon > 0$. By Proposition 5.1, for each $k = 1, 2, ...,$ there is a monotone pure strategy equilibrium, b^k, of the uniform-price auction when unit-bids are restricted to the finite set $\{0, 1/k, 2/k, ..., k/k\}$. It suffices to show that for all k sufficiently large, b^k is an ε-equilibrium of the game in which unit-bids can be chosen from $[0, 1]$.

Fix player i. Let D denote the set of nonincreasing bid vectors in $[0, 1]^m$. It suffices to show that for all k sufficiently large and all monotone pure strategies $b : T_i \to D$ for player i, there is a monotone pure strategy $b' : T_i \to D \cap \{0, 1/k, 2/k, ..., k/k\}^m$ that yields player i utility within ε of $b(\cdot)$ uniformly in the others' strategies. By weak dominance, it suffices to consider monotone pure strategies $b : T_i \to D$ for player i such that each unit-bid, $b_j(t_i)$, is in $[0, t_{ij}]$ for every $t_i = (t_{i1}, ..., t_{im}) \in T_i$. So, let $b(\cdot)$ be such a monotone pure strategy and let $b' : T_i \to D \cap \{0, 1/k, 2/k, ..., k/k\}^m$ be such that for every $t_i \in T_i$, $b'_j(t_i)$ is the smallest member of $\{0, 1/k, ..., k/k\}$ greater or equal to $b_j(t_i)$. Hence, $b'(\cdot)$ is monotone and $b'_1(t_i) \geq ... \geq b'_m(t_i)$ every $t_i \in T_i$, so that $b'(\cdot)$ is a feasible monotone pure strategy. If bidder i employs $b'(\cdot)$ instead of $b(\cdot)$, then regardless of his type and for any strategies the others might employ and for each $j = 1, ..., m$, bidder i will win a jth unit whenever $b(\cdot)$ would have won a jth unit although the price might be higher because his bid vector is higher, and he may win a jth unit when $b(\cdot)$ would not have. The increase in the price caused by the at most $1/k$ increase in each of his unit-bids can be no greater than $1/k$, and because $b_j(t_i) \leq t_{ij}$ for every $t_i \in T_i$, the ex-post surplus lost on each additional unit won from employing $b'(\cdot)$ instead of $b(\cdot)$ can be no greater than $1/k$. Hence, the total ex-post loss in surplus as a result of the strategy change can be no greater than $2m/k$, which can be made arbitrarily small for k sufficiently large, regardless of the others' strategies. Hence, i's expected utility loss from employing $b'(\cdot)$ instead of $b(\cdot)$ is, for k large enough, less than ε, and this holds uniformly in the others' strategies. ■

Proof of Corollary 5.5. Analogous to the proof of Corollary 5.2 above. ■

Proof of Lemma 5.3. Fix monotone pure strategies for all players but i. For the remainder of this proof, we omit most subscripts i to keep the notation manageable. Let $v(b, t)$ denote bidder i's expected payoff from employing the bid vector $b = (b_1, ..., b_m)$ when his type vector is $t = (t_1, ..., t_m)$. Then, letting $P_k(b_k)$ denote the probability that bidder i wins at least k units – which depends only on his kth unit-bid b_k – we have, where 1_k is an m-vector of k ones followed by $m - k$ zeros,

$$\begin{aligned} v(b,t) &= u(0) + \sum_{k=1}^{m} P_k(b_k)\left(u((t-b)\cdot 1_k) - u((t-b)\cdot 1_{k-1})\right) \\ &= \frac{1}{r}\sum_{k=1}^{m} e^{r(b_1+...+b_{k-1})} P_k(b_k)\left(1 - e^{-r(t_k-b_k)}\right) e^{-r(t_1+...+t_{k-1})}, \end{aligned}$$

where $u(x) = \frac{1-e^{-rx}}{r}$ is bidder i's utility function with constant absolute risk aversion parameter $r \geq 0$, where it is understood that $u(x) = x$ when $r = 0$. Note that the dependence of r on i has been suppressed.

From now on we shall proceed as if $r > 0$ because all of the formulae employed here have well-defined limits as r tends to zero that correspond to the risk neutral case $u(x) = x$.

Letting $w_k(b_k, t) = \frac{1}{r} P_k(b_k) \left(1 - e^{-r(t_k - b_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})}$, we may write,

$$v(b, t) = \sum_{k=1}^{m} e^{r(b_1 + \ldots + b_{k-1})} w_k(b_k, t).$$

As shown in (5.2) from subsection 5.1 (and setting $\bar{p} = \underline{p} = 0$ there), for each $k = 2, ..., m$,

$$u(t_1 + \ldots + t_k) - u(t_1 + \ldots + t_{k-1}) = \frac{1}{r}(1 - e^{-rt_k}) e^{-r(t_1 + \ldots + t_{k-1})}, \tag{A.9}$$

is nondecreasing in t according to the partial order $\geq_i$ defined in (5.1). Henceforth, we shall employ the partial order $\geq_i$ on i's type space. We next demonstrate the following facts.

(i) $w_k(b_k, t)$ is nondecreasing in t, and

(ii) $w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t)$ is nondecreasing in t for all $\bar{b}_k \geq \underline{b}_k$,

To see (i), write,

$$\begin{aligned} w_k(b_k, t) &= \frac{1}{r} P_k(b_k) \left(1 - e^{-r(t_k - b_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &= \frac{1}{r} P_k(b_k) \left(1 - e^{-rt_k}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &\quad + \frac{1}{r} P_k(b_k) \left(e^{rb_k} - 1\right) \left(-e^{-r(t_1 + \ldots + t_k)}\right). \end{aligned}$$

The first term in the sum is nondecreasing in t according to $\geq_i$ by (A.9) and the second term, being nondecreasing in the coordinatewise partial order is, a fortiori, nondecreasing in t according to $\geq_i$.

Turning to (ii), if $P_k(\underline{b}_k) = 0$ then $w_k(\underline{b}_k, t) = 0$ and (ii) follows from (i). So, assume $P_k(\underline{b}_k) > 0$. Then,

$$\begin{aligned} w_k(\bar{b}_k, t) - w_k(\underline{b}_k, t) &= \frac{1}{r} P_k(\bar{b}_k) \left(1 - e^{-r(t_k - \bar{b}_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &\quad - \frac{1}{r} P_k(\underline{b}_k) \left(1 - e^{-r(t_k - \underline{b}_k)}\right) e^{-r(t_1 + \ldots + t_{k-1})} \\ &= \left(\frac{P_k(\bar{b}_k)}{P_k(\underline{b}_k)} - 1\right) w_k(\underline{b}_k, t) \\ &\quad + \frac{1}{r} P_k(\bar{b}_k) \left(e^{r\bar{b}_k} - e^{r\underline{b}_k}\right) \left(-e^{-r(t_1 + \ldots + t_k)}\right). \end{aligned}$$

The first term in the sum is nondecreasing in t according to $\geq_i$ by (i) and the second term, being nondecreasing in the coordinatewise partial order is, a fortiori, nondecreasing in t according to $\geq_i$. This proves (ii).

Suppose now that the vector of bids b is optimal for bidder i when his type vector is t, and that b' is optimal when his type is $t' \geq_i t$. We must argue that $b \vee b'$ is optimal when his type is t'. If $b_k \leq b'_k$ for all k, then $b \vee b' = b'$ and we are done. Hence, we may assume that there is a maximal set of consecutive coordinates of b that are strictly greater than those of b'. That is, there exist coordinates j and l with $j \leq l$ such that $b_k > b'_k$ for $k = j, ..., l$ and $b_{j-1} \leq b'_{j-1}$ and $b_{l+1} \leq b'_{l+1}$, where the first of the last two inequalities is ignored if $j = 1$ and the second is ignored if $l = m$.

Let $\hat{b}$ be the bid vector obtained from b by replacing its coordinates j through l with the coordinates j through l of b'. Because b is optimal at t and $\hat{b}$ is nonincreasing and therefore feasible, $v(b,t) - v(\hat{b},t)$ is nonnegative. Dividing $v(b,t) - v(\hat{b},t)$ by $e^{r(b_1+...+b_j)}$, this implies

$$\begin{aligned} 0 \quad \leq \quad & w_j(b_j,t) - w_j(b'_j,t) + \sum_{k=j+1}^{l} e^{r(b_j+...+b_{k-1})} \left(w_k(b_k,t) - w_k(b'_k,t)\right) \\ & + \left(e^{r(b_j+...+b_l)} - e^{r(b'_j+...+b'_l)}\right) \left[w_{l+1}(b_{l+1},t) + e^{rb_{l+1}} w_{l+2}(b_{l+2},t) + ... + e^{r(b_{l+1}+...+b_{m-1})} w_m(b_m,t)\right] \end{aligned}$$

Consequently, for $t' \geq_i t$, (i) and (ii) imply,

$$\begin{aligned} 0 \quad \leq \quad & w_j(b_j,t') - w_j(b'_j,t') + \sum_{k=j+1}^{l} e^{r(b_j+...+b_{k-1})} \left(w_k(b_k,t') - w_k(b'_k,t')\right) \\ & + \left(e^{r(b_j+...+b_l)} - e^{r(b'_j+...+b'_l)}\right) \left[w_{l+1}(b_{l+1},t') + e^{rb_{l+1}} w_{l+2}(b_{l+2},t') + ... + e^{r(b_{l+1}+...+b_{m-1})} w_m(b_m,t')\right] \end{aligned} \tag{A.10}$$

Focusing on the second term in square brackets in (A.10), we claim that

$$\begin{aligned} & w_{l+1}(b_{l+1},t') + e^{rb_{l+1}} w_{l+2}(b_{l+2},t') + ... + e^{r(b_{l+1}+...+b_{m-1})} w_m(b_m,t') \\ \leq \quad & w_{l+1}(b'_{l+1},t') + e^{rb'_{l+1}} w_{l+2}(b'_{l+2},t') + ... + e^{r(b'_{l+1}+...+b'_{m-1})} w_m(b'_m,t') \end{aligned} \tag{A.11}$$

To see this, note that because $b_{l+1} \leq b'_{l+1}$, the bid vector b'' obtained from b' by replacing its coordinates $l+1$ through m with the coordinates $l+1$ through m of b is a feasible (i.e., nonincreasing) bid vector. Consequently, because b' is optimal at t' we must have $0 \leq v(b',t') - v(b'',t')$. But this difference in utilities is precisely the difference between the right-hand and left-hand sides of (A.11) multiplied by $e^{r(b_1+...+b_l)}$, thereby establishing (A.11).

Thus, we may conclude, after making use of (A.11) in (A.10) that,

$$\begin{aligned} 0 \quad \leq \quad & w_j(b_j,t') - w_j(b'_j,t') + \sum_{k=j+1}^{l} e^{r(b_j+...+b_{k-1})} \left(w_k(b_k,t') - w_k(b'_k,t')\right) \\ & + \left(e^{r(b_j+...+b_l)} - e^{r(b'_j+...+b'_l)}\right) \left[w_{l+1}(b'_{l+1},t') + e^{rb'_{l+1}} w_{l+2}(b'_{l+2},t') + ... + e^{r(b'_{l+1}+...+b'_{m-1})} w_m(b'_m,t')\right] \\ = \quad & \frac{v(\tilde{b},t') - v(b',t')}{e^{r(b'_1+...+b'_{j-1})}}, \end{aligned}$$

where $\tilde{b}$ is the nonincreasing and therefore feasible bid vector obtained from b' by replacing its coordinates j through l with the coordinates j through l of b. Hence, $\tilde{b}$ is optimal at t' because $v(\tilde{b}, t') \geq v(b', t')$ and b' is optimal at t'.

Thus, we have shown that whenever $j, ..., l$ is a maximal set of consecutive coordinates such that $b_k > b'_k$ for all $k = j, ..., l$, replacing in b' the unit-bids $b'_j, ..., b'_l$ with the coordinate-by-coordinate larger unit bids $b_j, ..., b_l$ results in a bid vector that is optimal at t'. Applying this result finitely often leads to the conclusion that $b \vee b'$ is optimal at t', as desired. ■

Lemma A.22. *Consider the price competition game from subsection 5.3. Under the partial orders on types $\geq_i$ defined there for each firm i, each firm possesses a monotone pure strategy best reply when the other firms employ monotone pure strategies.*

Proof. Suppose that all firms $j \neq i$ employ monotone pure strategies according to $\geq_j$ defined in subsection 5.3. Therefore, in particular, $p_j(c_j, x_j)$ is nondecreasing in c_j for each x_j, and (5.6) applies. For the remainder of this proof, we omit most subscripts i to keep the notation manageable.

Because firm i's interim payoff function is continuous in his price for each of his types and because his action space, $[0, 1]$, is totally ordered and compact, firm i possesses a largest best reply, $\hat{p}(c, x)$, for each of his types $(c, x) \in [0, 1]^2$. We will show that $\hat{p}(\cdot)$ is monotone according to $\geq_i$.

Let $\bar{t} = (\bar{c}, \bar{x})$, $\underline{t} = (\underline{c}, \underline{x})$ in $[0, 1]^2$ be two types of firm i, and suppose that $\bar{t} \geq_i \underline{t}$. Hence, $\bar{c} \geq \underline{c}$ and $\bar{x} - \underline{x} = \beta(\bar{c} - \underline{c})$ for some $\beta \in [0, \alpha_i]$. Let $\bar{p} = \hat{p}(\bar{c}, \bar{x})$, $\underline{p} = \hat{p}(\underline{c}, \underline{x})$, and $t^\lambda = (1 - \lambda)\underline{t} + \lambda\bar{t}$ for $\lambda \in [0, 1]$. We wish to show that $\bar{p} \geq \underline{p}$.

By the fundamental theorem of calculus,

$$v_i(\underline{p}, t^\lambda) - v_i(p', t^\lambda) = \int_{p'}^{\underline{p}} \frac{\partial v_i(p, t^\lambda)}{\partial p} dp,$$

so that

$$\begin{aligned}
\frac{\partial \left[v_i(\underline{p}, t^\lambda) - v_i(p', t^\lambda)\right]}{\partial \lambda} &= \int_{p'}^{\underline{p}} \frac{\partial^2 v_i(p, t^\lambda)}{\partial \lambda \partial p} dp \\
&= \int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p, t^\lambda)}{\partial c \partial p}(\bar{c} - \underline{c}) + \frac{\partial^2 v_i(p, t^\lambda)}{\partial x \partial p}(\bar{x} - \underline{x})\right] dp \\
&= (\bar{c} - \underline{c}) \int_{p'}^{\underline{p}} \left[\frac{\partial^2 v_i(p, t^\lambda)}{\partial c \partial p} + \beta \frac{\partial^2 v_i(p, t^\lambda)}{\partial x \partial p}\right] dp \\
&\geq 0,
\end{aligned}$$

where the inequality follows by (5.6) if $\underline{p} \geq p' \geq \bar{c}$. Therefore, $v_i(\underline{p}, \bar{t}) - v_i(p', \bar{t}) \geq v_i(\underline{p}, \underline{t}) - v_i(p', \underline{t}) \geq 0$, where the first inequality follows because $t^0 = \underline{t}$, $t^1 = \bar{t}$, and the second because $\underline{p}$ is a best reply at $\underline{t}$. Therefore, we have shown the following: If $\underline{p} \geq \bar{c}$, then

$$v_i(\underline{p}, \bar{t}) - v_i(p', \bar{t}) \geq 0, \text{ for all } p' \in [\bar{c}, \underline{p}].$$

Hence, if $\underline{p} \geq \bar{c}$, then $\hat{p}(\bar{t}) = \bar{p} \geq \underline{p} = \hat{p}(\underline{t})$ because $\hat{p}(\bar{t})$ is the largest best reply at $\bar{t}$ and because no best reply at $\bar{t} = (\bar{c}, \bar{x})$ is below $\bar{c}$. On the other hand, if $\underline{p} < \bar{c}$, then $\bar{p} = \hat{p}(\bar{t}) \geq \bar{c} > \underline{p} = \hat{p}(\underline{t})$, where the first inequality again follows because no best reply at $\bar{t}$ is below $\bar{c}$. We conclude that $\bar{p} \geq \underline{p}$, as desired. ■

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