Maastricht University

Burak Can, Ton Storcken

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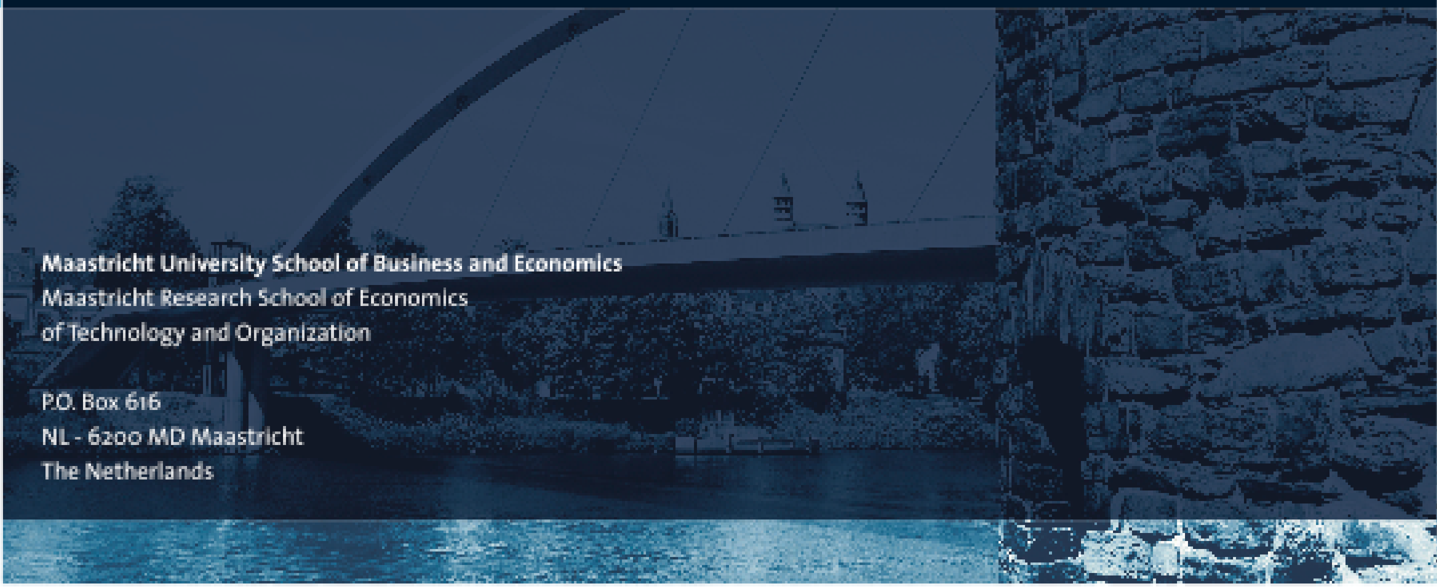

Maastricht University School of Business and Economics
Maastricht Research School of Economics
of Technology and Organization

P.O. Box 616
NL - 6200 MD Maastricht
The Netherlands

Impossibilities with Kemeny updating

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Abstract

Impossibility theorems for preference correspondences based on a new monotonicity concept are discussed. Here monotonicity means that if preferences update in such a way that they get closer to an outcome then at the new situation this outcome remains chosen. Strong monotonicity requires further that in those cases the outcome at the new profile is a subset of the outcome at the old profile. It is shown that only dictatorial preference correspondences are unanimous and strongly monotone.

1 Introduction

In social choice theory, it is well-known that monotonicity (Maskin, 1985) leads to some impossibility theorems. In fact Muller & Satterthwaite (1977) show that under citizen sovereignty[1], any monotone, single-valued social choice rule is dictatorial. Satterthwaite (1975) relates these impossibilites to the Arrow's impossibility theorem for welfare functions.

In a recent study, Can & Storcken (2011) introduce "update monotonicity" for preference rules, i.e. social welfare functions/correspondences[2] and shows the existence of non-trivial welfare rules that satisfy this monotonicity condition. In this companion paper, we analyze a similar monotonicity condition, *Kemeny update monotonicity*, which requires that a rule preserves the outcome assigned to a profile whenever the profile is changed such that each individual preference is "closer" to the initial outcome. The closeness is measured, in particular, by the Kemeny distance (Kemeny & Snell, 1960). A stronger version of this distance based monotonicity is also analyzed.

Our study leads to three impossibility theorems regarding these monotonicity conditions. Despite the fact that our findings are impossibility theorems for welfare functions/correspondences, the proofs that we introduce differ substantially from the standard Arrovian framework. The proofs are essentially based on maximal conflicts, i.e., societies that are polarized into two completely

[1] Citizen sovereignty means for any alternative there exists a profile wherein that alternative is chosen. This condition is weaker than unanimity and Pareto optimality.

[2] A correspondence is a rule which is possibly not single-valued and assigns set-valued outcomes to profiles.

opposite preference. We first shortly introduce the model, and the properties. Thereafter, we provide the lemmas which will be used to show three impossibility theorems later on concerning the Kemeny update monotonicity.

2 Model

2.1 Notation

Given a set of alternatives A, let L denote the set of all linear orders, i.e., complete, transitive and antisymmetric binary relations, over A. Let N be the set of agents and let L^N denote the set of all preference profiles p, which assign to every agent i in N a preference $p(i)$ in L. Given a profile p and any subset of agents $S \subseteq N$, the set of all reported preferences from S at p is denoted by $p(S) = \{p(i) : i \in S\}$. Given $R \in L$ we denote the inverse linear order as $\neg R = \{(y, x) : (x, y) \in R\}$. For R in L and a coalition $S \subseteq N$, a profile $p = (R^S, \neg R^{N-S})$, where $p(S) = R$ and $p(N - S) = \neg R$, is called a *maximal conflict* between S and $N - S$ on R and $\neg R$. A *preference correspondence* or *rule* is a function φ from the set of preference profiles L^N to the power set of L, i.e. $\mathcal{P}(L)$, such that for all profiles p in L^N the outcome $\varphi(p)$ is a non-empty subset of L. For R^1 and R^2 in L, the (Kemeny) distance between these two linear orders is defined by

$$\delta(R^1, R^2) = \#[(R^1 - R^2) \cup (R^2 - R^1)]/2.$$

Two linear orders R and R' are said to be *elementary changes* if $\delta(R, R') = 1$. A sequence of linear orders $(R^0, R^1, ..., R^k)$ is called a *path* from R^0 to R^k if for all $i = 1$ to k, R^{i-1} and R^i are elementary changes. A linear order R^3 is said to be *between* linear orders R^1 and R^2 if $\delta(R^1, R^2) = \delta(R^1, R^3) + \delta(R^3, R^2)$. That is if there is a shortest path via R^3 from R^1 to R^2. A subset V of L is *convex* if for any R^1 and R^2 in V, all linear orders between R^1 and R^2 are also in V.

2.2 Properties of preference rules

Unanimity: A rule φ is *unanimous* if $\varphi(R^N) = \{R\}$ for all linear orders R.
Dictatoriality: A rule φ is *weakly dictatorial* if there is a unique agent i, the dictator, such that $p(i) \in \varphi(p)$ for all preference profiles p.
Convexity: A rule φ is convex valued if $\varphi(p)$ is a convex set for all preference profiles p.
Single valuedness:A rule φ is *single valued* if $\varphi(p)$ is a singleton for all p in L^N.
(**Kemeny-update) monotonicity:** A rule φ is *Kemeny-update monotone* if for all R in $\varphi(p)$ and all preference profiles q we have that $R \in \varphi(q)$, whenever

$$\delta(q(i), R) \leq \delta(p(i), R) \text{ for all } i \text{ in } N.$$

A rule φ is *strongly Kemeny-update monotone* if for all R in $\varphi(p)$ and all preference profiles q we have that $R \in \varphi(q) \subseteq \varphi(p)$, whenever

$$\delta(q(i), R) \leq \delta(p(i), R) \text{ for all } i \text{ in } N.$$

3 Results

From here on we refer to the (strong) Kemeny-update monotonicity simply as (strong) monotonicity. Below are some implications of monotonicity together with other conditions introduced above.

Lemma 1 *Let φ be monotone and unanimous.*
a) Let $p \in L^N$. Then $\varphi(p) \subseteq p(N)$.
b) Let φ be, in addition, convex valued. Let $p \in L^N$ be such that for all i and j in N, $\delta(p(i), p(j)) \neq 1$. Then $\#\varphi(p) = 1$.

Proof. a) To the contrary let $R \in \varphi(p) - p(N)$. Then there are $\overline{R} \in L$ such that $\delta(R, \overline{R}) = 1$ and profiles q in L^N such that $q(i) = \overline{R}$ for all i in N. As $R \in \varphi(p) - p(N)$ it follows that $\delta(R, p(i)) \geq 1$ for all $i \in N$. Monotonicity therefore implies $R \in \varphi(q)$. Unanimity implies $\varphi(q) = \{\overline{R}\}$. This yields the contradiction $R = \overline{R}$. b) Assume further that φ is convex valued and $\delta(p(i), p(j)) \neq 1$ for all agents i and j in N. Then, between two different reported preferences $p(i)$ and $p(j)$, there are linear orders which are not in $p(N)$. As $\varphi(p)$ is convex and $\varphi(p) \subseteq p(N)$ it follows that $\varphi(p)$ can contain at most one element. ■

Coalition $S \subseteq N$ is *winning* (*on maximal conflicts*) *at* rule φ if $\varphi(p) = \{R\}$ for all $R \in L$ and all $p = (R^S, \neg R^{N-S})$. Let $\mathcal{W}$ denote the set of coalitions which are winning at φ.

Lemma 2 *Let φ be monotone and unanimous. Let $S \subseteq N$. Then either $S \in \mathcal{W}$ or $N - S \in \mathcal{W}$ in each of the following two cases:*
a) If φ is convex valued,
b) If φ is strongly monotone.

Proof. Let R_1 and R_2 be an elementary change. By Lemma 1 $\varphi(R_i^S, \neg R_i^{N-S}) \subseteq \{R_i, \neg R_i\}$ for $i \in \{1, 2\}$. Without loss of generality, let $R_1 \in \varphi(R_1^S, \neg R_1^{N-S})$. Because all maximal conflicts are connected by distance one preference deviations, it is sufficient to prove that $\neg R_2 \notin \varphi(R_2^S, \neg R_2^{N-S})$. Note that $\delta(R_1, \neg R_2) \geq 2$. Therefore a) in case φ is convex valued, by Lemma 1 and monotonicity it follows from $R_1 \in \varphi(R_1^S, \neg R_1^{N-S})$ that $\varphi(R_1^S, \neg R_2^{N-S}) = \{R_1\}$. Hence, $\neg R_2 \notin \varphi(R_1^S, \neg R_2^{N-S})$. Similarly, b) in case φ is strongly monotone, the latter follows because $\neg R_2 \notin \varphi(R_1^S, \neg R_1^{N-S})$. Now for both cases a) and b) $\neg R_2 \notin \varphi(R_1^S, \neg R_2^{N-S})$, then monotonicity yields $\neg R_2 \notin \varphi(R_2^S, \neg R_2^{N-S})$. ■

Lemma 3 *Let φ be strongly monotone and unanimous. Then φ is single valued.*

Proof. Let p be a preference profile and $R \in \varphi(p)$. It is sufficient to prove that $\varphi(p) = \{R\}$. Let $S = \{i \in N : p(i) = R\}$. Take $q = (R^S, \widehat{R}^{N-S})$ and $r = (R^S, \overline{R}^{N-S})$, where $\delta\left(R, \overline{R}\right) = 1$, $\delta(R, \widehat{R}) = 1$, and $R \neq \widehat{R}$. By Lemma 1 $\overline{R}$ is not in $\varphi(q)$. As monotonicity implies that R is in both $\varphi(q)$ and $\varphi(r)$ and therewith strong monotonicity implies that $\varphi(q) \subseteq \varphi(r)$. It follows that $\varphi(r) = \{R\}$ and $\overline{R}$ is not in $\varphi(r)$. So, monotonicity implies that $\overline{R}$ is not in $\varphi(\neg\overline{R}^S, \overline{R}^{N-S})$. In view of Lemma 2 S wins on maximal conflicts. But then strong monotonicity implies $\varphi(p) = \{R\}$. ■

Lemma 4 *Let φ be monotone, unanimous and convex valued. Let $S, T \in \mathcal{W}$. Then $S \cap T \in \mathcal{W}$.*

Proof. Let R_1, R_2 and R_3 be three linear orders in L such that $2 \leq \delta(R_1, R_2) \leq \delta(R_1, R_3) \leq \delta(R_2, R_3)$ and take $\{X, Y, Z\} = \{S \cap T, N - S, S - T\}$. Take profile $p = (R_1^X, R_2^Y, R_3^Z)$. Suppose $S \cap T \notin \mathcal{W}$. Then monotonicity and Lemma 2 imply that $X \cup Y$, $X \cup Z$ and $Y \cup Z$ are in $\mathcal{W}$. Lemma 1 yields that $\varphi(p) = \{R_t\}$ for some $t \in \{1, 2, 3\}$. Suppose $\varphi(p) = \{R_1\}$. Then Lemma 1 and monotonicity imply that $\varphi(R_1^X, R_2^{Y \cup Z}) = \{R_1\}$. So, $R_2 \notin \varphi(R_1^X, R_2^{Y \cup Z})$ and therefore monotonicity yields that $R_2 \notin \varphi(R_2^{Y \cup Z}, \neg R_2^X)$. So, Lemma 1 implies $\varphi(R_2^{Y \cup Z}, \neg R_2^X) = \{\neg R_2\}$ contradicting $Y \cup Z$ is in $\mathcal{W}$. So, $\varphi(p) \neq \{R_1\}$. Similarly it follows that $\varphi(p) \neq \{R_2\}$ and $\varphi(p) \neq \{R_3\}$. This however contradicts $\varphi(p) = \{R_t\}$ for some t in $\{1, 2, 3\}$. So, $S \cap T \in \mathcal{W}$. ■

Theorem 1 *A unanimous, convex valued and monotone rule is weakly dictatorial.*

Proof. Let $R = p(j) \notin \varphi(p)$ for some $j \in N$ and preference profile p. Then monotonicity implies that $R \notin \varphi(R^{\{j\}}, \neg R^{N-\{j\}})$. So, Lemma 2 $N - \{j\} \in \mathcal{W}$. As $\emptyset \notin \mathcal{W}$, in view of Lemma 4, it follows that for some agent i at all preference profiles q we have that $q(i) \in \varphi(q)$. Because rule φ is convex valued there is at most one such agent. Therefore, φ is weakly dictatorial ■

Theorem 2 *A single valued, unanimous and monotone rule is dictatorial.*

Proof. Note that a single valued rule is convex valued. So, by Theorem 1 the rule is weakly dictatorial. Let i be such that $p(i) \in \varphi(p)$ for all preference profiles p. Then because of single valuedness $\varphi(p) = \{p(i)\}$ for all preference profiles and φ is dictatorial with dictator i. ■

Theorem 3 *A strongly monotone and unanimous rule is dictatorial.*

Proof. Follows from Theorem 2 and Lemma 3. ■

Example 1 *Monotone but not strongly monotone rules*

Fix a linear order R in L. For arbitrary preference profiles p define rule $\varphi_R(p)$ by $\varphi_R(p) = p(N) - \{R\}$ if $p(N) \neq \{R\}$ and $\varphi_R(p) = \{R\}$ if $p(N) = \{R\}$. It is straight forward to show that φ_R is monotone, unanimous and that it is neither convex valued nor strongly monotone. At rule φ_R agents play the same role therefore it is not weakly dictatorial and hence also not dictatorial.

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