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MORE ABOUT DIVISIBLE DESIGN GRAPHS

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More about divisible design graphs

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Abstract

Divisible design graphs (DDG for short) have been recently defined by Kharaghani, Meulenberg and the second author as a generalization of (v, k, λ)-graphs. In this paper we give some new constructions of DDGs, most of them using Hadamard matrices and (v, k, λ)-graphs. For three parameter sets we give a nonexistence proof. Furthermore, we find conditions for a DDG to be walk-regular. It follows that most of the known examples are walk-regular, but some are not. In case walk-regularity of a DDG is forced by the parameters, necessary conditions for walk-regularity lead to new nonexistence results for DDGs. We examine all feasible parameter sets for DDGs on at most 27 vertices, establish existence in all but one cases, and decide on existence of a walk-regular DDG in all cases.

Keywords: divisible design graph, divisible design, walk-regular graph, (v, k, λ)-graph, Hadamard matrix.
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1 Divisible design graphs

A graph Γ can be interpreted as a design by taking the vertices of Γ as points, and the neighborhoods of the vertices as blocks. Such a design is called a *neighborhood design* of Γ. The adjacency matrix of Γ is the incidence matrix of its neighborhood design.

A k-regular graph on v vertices with the property that any two distinct vertices have exactly λ common neighbors is called a (v, k, λ)-graph (see [7]). The neighborhood design of a (v, k, λ)-graph is a symmetric (v, k, λ)-design. Divisible design graphs have been introduced in [4] as a generalization of (v, k, λ)-graphs.

Definition 1.1 *A k-regular graph on v vertices is a* divisible design graph *(DDG for short) with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ whenever the vertex set can be partitioned into m classes of size n, such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors.*

Note that a DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is a (v, k, λ)-graph. In this case we call the DDG improper, otherwise it is proper.

An incidence structure with v points and the constant block size k is a (group) divisible design with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ whenever the point set can be partitioned into m classes of size n, such that two vertices from the same class have exactly

λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors. A divisible design D is said to be symmetric (or to have the dual property) if the dual of D is a divisible design with the same parameters as D.

It is clear that the neighborhood design of a DDG is a symmetric divisible design. Conversely, a symmetric divisible design having symmetric incidence matrix with zero diagonal is the neighborhood design of a DDG.

2 The adjacency and the quotient matrix

Here we briefly survey some relevant properties on DDGs. For necessary proofs and more details we refer to [4]. As usual, we denote by I_v and J_v the $v \times v$ identity and all-ones matrix, respectively. The all-ones matrix and vector with undetermined size are denoted by J and $\mathbf{1}$. Let us define $K_{(m,n)} = I_m \otimes J_n$. Then the adjacency matrix A of a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ satisfies:

$$AA^\top = A^2 = kI_v + \lambda_1(K_{(m,n)} - I_v) + \lambda_2(J_v - K_{(m,n)}). \tag{1}$$

From this it readily follows that A has the following eigenvalues

$$k,\ \theta_1 = \sqrt{k - \lambda_1},\ \theta_2 = -\theta_1,\ \theta_3 = \sqrt{k^2 - \lambda_2 v},\ \theta_4 = -\theta_3$$

with multiplicities $1,\ f_1,\ f_2,\ g_1,\ g_2$, respectively, where $f_1+f_2 = m(n-1)$, $g_1+g_2 = m-1$, and $k+(f_1-f_2)\theta_1+(g_1-g_2)\theta_3 = \operatorname{trace} A = 0$. Note that the parameters do not always determine the multiplicities. But they do as soon as one more equation holds. This is the case if θ_1 or θ_3 is not an integer, then $f_1 = f_2$ or $g_1 = g_2$, respectively. So in general, a proper DDG has five distinct eigenvalues. But some may coincide and multiplicities may be equal to 0. A connected DDG with at most three distinct eigenvalues is a (v, k, λ)-graph, therefore a connected proper DDG has four or five distinct eigenvalues.

The vertex partition from the definition of a DDG gives a partition (called the *canonical partition*) of the adjacency matrix

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \dots & A_{m,m} \end{bmatrix}.$$

Bose[1] proved that the canonical partition of a symmetric divisible design is a tactical decomposition (also called: equitable partition) of its incidence matrix A, which means that each block $A_{i,j}$ has constant row and column sum (see [4] for a short proof). This enables us to define the matrix $R = [r_{i,j}]$, where $r_{i,j}$ is the row (and column) sum of $A_{i,j}$. The matrix R is called the *quotient matrix* of A.

Theorem 2.1 *Consider a proper DDG with parameter set $(v, k, \lambda_1, \lambda_2, m, n)$ and eigenvalues k, $\pm\theta_1$, $\pm\theta_3$ with respective multiplicities 1, f_1, f_2, g_1, g_2. Then the canonical partition is a tactical decomposition. The quotient matrix $R = [r_{i,j}]$ is symmetric and satisfies*

$$R\mathbf{1} = k\mathbf{1}\ ,\ R^2 = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m.$$

The eigenvalues of R are k, $\theta_3 = \sqrt{k^2 - \lambda_2 v}$ and $\theta_4 = -\theta_3$ with multiplicities 1, g_1 and g_2, respectively. Moreover, $r_{i,i} \leq n - 1$, and $r_{i,i}$ is even if n is odd for $i = 1, \dots, m$.

Also the following result goes back to Bose [1]

Theorem 2.2 *Consider a proper DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ and quotient matrix R. Write $k = mt + k_0$ for some integers t and k_0 with $0 \le k_0 \le m-1$. Then the entries of R take exactly one, or two consecutive values if and only if*

$$k_0^2 - mk_0 - k^2 + km + \lambda_1 m(n-1) = 0 \ .$$

If this is the case then $R = tJ_m + N$, where N is the incidence matrix of a (possibly degenerate) (m, k_0, λ)-design.

3 New existence results

3.1 Constructions

The first construction generalizes Construction 4.8 from [4].

Theorem 3.1 *Let H be a regular graphical Hadamard matrix of order $4u^2$ with diagonal entries -1 and row sum $2u$ (u can be negative), and let A be the adjacency matrix of an (n, k', λ)-graph. Replace each entry -1 of H by A, and each $+1$ by $J_n - A$. Then we obtain the adjacency matrix of a DDG with parameters*

$$(4nu^2, 2nu^2 + u(n-2k'), 4\lambda u^2 + u(2u+1)(n-2k'), nu^2 + u(n-2k'), 4u^2, n).$$

Proof. Straightforward. □

Theorem 3.2 *If there exist a regular graphical Hadamard matrix of order $4u^2$ with row sum $2u$, and a Hadamard matrix of order $2u^2$, then there exists a DDG with parameters $(24u^2, 12u^2 - 2u, 4u^2 - 2u, 6u^2 - 2u, 12u^2, 2)$.*

Proof. Let H_0 be a regular graphical Hadamard matrix of order $4u^2$ with row sum $2u$, and let H_1 and H_2 be Hadamard matrices of order $4u^2$ and $2u^2$, respectively. (H_1 does not have to be related to H_0.) Define $D = \frac{1}{2}(H_0 + J_{4u^2})$ (then D is the adjacency matrix of a $(4u^2, 2u^2+u, u^2+u)$-graph), and $\overline{D} = D \otimes J_2$. Furthermore, let $\overline{H_1}$ be the matrix obtained from H_1 by replacing $+1$ by $[\, I_2 \,|\, I_2 \,]$, and each -1 by $[\, J_2 - I_2 \,|\, J_2 - I_2 \,]$, and let $\overline{H_2}$ be the matrix obtained from H_2 by replacing -1 and $+1$ by

$$\begin{bmatrix} J_2 - I_2 & I_2 \\ I_2 & J_2 - I_2 \end{bmatrix}, \text{ and } \begin{bmatrix} I_2 & J_2 - I_2 \\ J_2 - I_2 & I_2 \end{bmatrix},$$

respectively. Now it is straightforward to check that the matrix

$$A = \left[\begin{array}{c|cc} \overline{D} & \multicolumn{2}{c}{\overline{H_1}} \\ \hline \overline{H_1}^\top & \overline{D} & \overline{H_2} \\ & \overline{H_2}^\top & \overline{D} \end{array}\right]$$

satisfies

$$A^2 = (12u^2 - 2u)I_{24u^2} + (4u^2 - 2u)(K_{(12u^2,2)} - I_{24u^2}) + (6u^2 - 2u)(J_{24u^2} - K_{(12u^2,2)}).$$

Therefore A is the adjacency matrix of a DDG with the required parameters. □

There exists a Hadamard matrix of order 2, a graphical Hadamard matrix of order 4 with row sum 2, and also one with row sum -2. Hence:

Corollary 3.3 *There exists DDGs with parameters* $(24, 10, 2, 4, 12, 2)$ *and* $(24, 14, 6, 8, 12, 2)$.

We also found one more sporadic DDG:

Theorem 3.4 *There exists a DDG with parameters* $(24, 10, 3, 4, 8, 3)$.

Proof. The following matrix

$$\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
0&0&0&1&1&0&1&1&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0&0\\
0&0&0&1&0&1&1&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&1&0&1&1&0&1&0&0&1&0&0&1&0&0&1&0\\
\hline
1&1&0&0&0&0&1&1&0&1&1&0&0&1&0&0&1&0&0&0&1&0&0&1\\
1&0&1&0&0&0&1&0&1&1&0&1&1&0&0&1&0&0&0&1&0&0&1&0\\
0&1&1&0&0&0&0&1&1&0&1&1&0&0&1&0&0&1&1&0&0&1&0&0\\
\hline
1&1&0&1&1&0&0&0&0&0&1&1&1&0&0&0&0&1&0&0&1&0&1&0\\
1&0&1&1&0&1&0&0&0&1&1&0&0&0&1&0&1&0&0&1&0&1&0&0\\
0&1&1&0&1&1&0&0&0&1&0&1&0&1&0&1&0&0&1&0&0&0&0&1\\
\hline
1&1&0&1&1&0&0&1&1&0&0&0&0&0&1&1&0&0&0&1&0&0&0&1\\
1&0&1&1&0&1&1&1&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1&0\\
0&1&1&0&1&1&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&1&0&0\\
\hline
1&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1&1&1&1&0&1&0&1\\
0&0&1&1&0&0&0&0&1&0&1&0&0&0&0&1&1&0&1&0&1&0&1&1\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&1&1&0\\
\hline
1&0&0&0&1&0&0&0&1&1&0&0&0&1&1&0&0&0&1&0&1&1&1&0\\
0&0&1&1&0&0&0&1&0&0&0&1&1&1&0&0&0&0&0&1&1&1&0&1\\
0&1&0&0&0&1&1&0&0&0&1&0&1&0&1&0&0&0&1&1&0&0&1&1\\
\hline
1&0&0&0&0&1&0&0&1&0&1&0&1&1&0&1&0&1&0&0&0&1&0&1\\
0&0&1&0&1&0&0&1&0&1&0&0&1&0&1&0&1&1&0&0&0&0&1&1\\
0&1&0&1&0&0&1&0&0&0&0&1&0&1&1&1&1&0&0&0&0&1&1&0\\
\hline
1&0&0&0&0&1&0&1&0&0&0&1&1&0&1&1&1&0&1&0&1&0&0&0\\
0&0&1&0&1&0&1&0&0&0&1&0&0&1&1&1&0&1&0&1&1&0&0&0\\
0&1&0&1&0&0&0&0&1&1&0&0&1&1&0&0&1&1&1&1&0&0&0&0
\end{array}\right]$$

is the adjacency matrix of a DDG with parameters $(24, 10, 3, 4, 8, 3)$. $\square$

3.2 Nonexistence

Theorem 3.5 *There exists no DDG with parameters* $(25, 12, 8, 5, 5, 5)$.

Proof. Our aim is to show that no quotient matrix exists for the given parameter set. If R is such a the quotient matrix, then $R\mathbf{1} = 12\,\mathbf{1}$ and $RR^\top = 19I_5 + 25J_5$. Therefore, each row sum of R equals 12, the inner product of any row of R with itself equals 44. It follows that the entries of a row of R can only take values from the multiset $\{5, 4, 1, 1, 1\}$ or $\{5, 3, 3, 1, 0\}$. Since R is symmetric, R has a diagonal entry equal to 5, which is impossible by Theorem 2.1. $\square$

Theorem 3.6 *There exists no DDG with parameters* $(27, 6, 3, 1, 9, 3)$.

Proof. The quotient matrix R of such a DDG satisfies $R\mathbf{1} = 6\,\mathbf{1}$ and $R^2 = 9I_9 + 3J_9$. Therefore, each row sum of R equals 6, the inner product of any row of R with itself equals 12, and any two distinct rows of R have inner product 9. It follows that the entries of a row of R can take values from the multiset $\{2, 2, 2, 0, 0, 0, 0, 0, 0\}$ or $\{3, 1, 1, 1, 0, 0, 0, 0, 0\}$. A row with the entries from the first multiset can not have inner product 9 with any other row, therefore all entries of R must have the entries the second multiset. Since R is symmetric, at least one diagonal entry equals 3, which contradicts Theorem 2.1. $\square$

Theorem 3.7 *There exists no DDG with parameters* $(26, 9, 0, 3, 13, 2)$.

Proof. According to Theorem 2.2, the quotient matrix R of such a DDG is the incidence matrix of a 2-$(13, 9, 6)$ design. This means that $N = J_{13} - R$ is the symmetric incidence matrix of the projective plane of order 3. It is well know that (up to simultaneous reordering of rows and columns) N is unique. Define $S = [\,\mathbf{1}\ I_3\,]$, then

$$N + I_{13} = \begin{bmatrix} 2I_4 & O & S^\top & J-S^\top \\ O & J_3 & I_3 & I_3 \\ S & I_3 & I_3 & I_3 \\ J-S & I_3 & I_3 & I_3 \end{bmatrix}.$$

Since $2I_4 = O \pmod 2$, it follows that

$$2\text{-rank}(N + I_{13}) \geq 2\text{-rank}\, S + 2\text{-rank}\, J_3 + 2\text{-rank}\, S^\top = 7.$$

Because $N = J_{13} - R$, we have $2\text{-rank}(R + I_{13}) \geq 6$.

Next we consider a signed version $Q = [q_{i,j}]$ of R, defined by $q_{i,j} = 1$ if the canonical partition block $A_{i,j} = I_2$, $q_{i,j} = -1$ if $A_{i,j} = J_2 - I_2$, and $q_{i,j} = 0$ if $A_{i,j} = O$. Clearly Q is symmetric and trace $Q = -9$ (all nonzero diagonal elements of Q are -1). Because $\lambda_2 = 3$ we have $\sum_j A_{i,j} A_{j,\ell} = 3J_2$, provided $i \neq \ell$. This implies that any two distinct rows of Q are orthogonal, and hence $Q^2 = 9I_{13}$. Therefore the eigenvalues of Q are ± 3 with multiplicities 5 and 8 respectively. Hence $\text{rank}(Q + 3I_{13}) = 5$, but $R + I_{13} = Q + 3I_{13} \pmod 2$, so $6 \leq 2\text{-rank}(R + I_{13}) = 2\text{-rank}(Q + 3I_{13}) \leq \text{rank}(Q + 3I_{13}) = 5$, contradiction. □

We remark that a divisible design with parameters $(26, 9, 0, 3, 13, 2)$ and a symmetric incidence matrix does exist (points and blocks are the nonzero vectors in $GF(3)^3$; incident if the inner product equals 1). It is the zero diagonal (reflected in: trace $Q = -9$) that causes a contradiction.

In the next section we obtain some more nonexistence results using the concept of walk-regularity.

4 Walk-regular graphs

4.1 Introduction

A graph is *walk-regular*, whenever for every $\ell \geq 2$ the number of closed walks of length ℓ at a vertex x is independent of the choice of x (see [3]). Note that walk-regularity implies regularity (take $\ell = 2$). Examples of walk-regular graphs are strongly regular graphs, and vertex-transitive graphs, but there is more. It turns out that many (but not all) known DDGs are walk-regular. In this section we investigate this phenomenon. First we quote a well-known characterization and its corollary (see for example [2], Chapter 15).

Lemma 4.1 *Let Γ be a connected graph whose adjacency matrix A has r distinct eigenvalues. Then Γ is walk-regular if and only if A^ℓ has constant diagonal for $2 \leq \ell \leq r - 2$.*

For a graph Γ with adjacency matrix A, the average number of triangles through a vertex equals $\frac{1}{2n}\text{trace}A^3$. Suppose Γ is walk-regular. Then this number must be an integer. More generally, $\frac{1}{2n}\text{trace}A^\ell$ is an integer if ℓ is odd, and $\frac{1}{n}\text{trace}A^\ell$ is an integer if n is even. If Γ is regular, A^2 has constant diagonal, hence:

Corollary 4.2 *A connected regular graph with at most four distinct eigenvalues is walk-regular.*

4.2 Walk-regular DDGs

A DDG has at most five distinct eigenvalues, but in many cases it has only four, which makes the DDG walk-regular, provided it is connected. In a disconnected DDG $\lambda_2 = 0$ and each component is an (n, k, λ)-graph, or the incidence graph of a symmetric (n, k, λ)-design (see [4], Proposition 4.7). Such a graph is not walk-regular if both types of components are present, and walk-regular otherwise. A DDG with five distinct eigenvalues can also be walk-regular. To decide on this the following characterization is useful.

Theorem 4.3 *A proper DDG is walk-regular if and only if the quotient matrix R has constant diagonal.*

Proof. By the above remark, we only need to consider connected DDGs. By Lemma 4.1, if suffices to show that R has constant diagonal if and only if A^3 has constant diagonal. Formula 1 implies

$$A^3 = (k - \lambda_1)A + (\lambda_1 - \lambda_2)AK_{(m,n)} + \lambda_2 kJ_v.$$

From the fact that the canonical partition is a tactical decomposition with quotient matrix R, it follows that $AK_{(m,n)} = K_{(m,n)}A = R \otimes J_n$. Therefore $A^3 - (\lambda_1 - \lambda_2)(R \otimes J_n)$ has constant diagonal, which proves our claim. □

In particular, if a DDG is walk-regular, then $\operatorname{trace} R = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}$ is divisible by m and divisible by $2m$ if n is odd.

Theorem 4.4 *A DDG for which $k - \lambda_1$ is not a square, is walk-regular.*

Proof. If $k - \lambda_1$ is not a square, $f_1 = f_2$ and hence $\operatorname{trace} R = \operatorname{trace} A = 0$. Therefore R has zero diagonal. □

With the above mentioned necessary conditions for walk-regularity, Corollary 4.2 and Theorem 4.4 lead to nonexistence results for DDGs. For example:

Theorem 4.5 *If for a DDG $k - \lambda_1$ is not a square, then $k\lambda_2$ is even.*

Proof. From $f_1 = f_2$ it follows that that $0 = \operatorname{trace} A = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}$, and $\operatorname{trace} A^3 = k^3 + (g_1 - g_2)(k^2 - \lambda_2 v)^{\frac{3}{2}}$. Hence $\operatorname{trace} A^3 = vk\lambda_2$. By Theorem 4.4 the graph is walk-regular. Therefore each vertex is in exactly $(\operatorname{trace} A^3)/2v = k\lambda_2/2$ triangles. □

Corollary 4.6 *There exists no DDG with parameters* $(24, 9, 4, 3, 6, 4)$, $(24, 9, 6, 3, 12, 2)$ *and* $(24, 15, 12, 9, 12, 2)$.

Theorem 4.7 *Suppose Γ is a DDG for which $k^2 - \lambda_2 v$ is not a square. If $m = 3$, or m does not divide k, then Γ is not walk-regular.*

Proof. Assume Γ is walk-regular. Since $k^2 - \lambda_2 v$ is not a square, $\operatorname{trace} R = k$. Hence m divides k. If $m = 3$, there is only one possible quotient matrix with row sum k and constant diagonal: $R = \frac{1}{3}kJ_3$. But then R has an eigenvalue 0, so $k^2 - \lambda_2 v = 0$, which is a contradiction. □

For example a DDG with parameters $(24u^2, 8u^2 + 2u4u^2 + 2u, 2u^2 + u, 3, 8u^2)$ is not walk-regular, because $m = 3$ and $k^2 - \lambda_2 v = 4u^2(4u^2 + 2u + 1)$ which is not a square. The DDGs of Construction 4.9 from [4] have these parameters, and so are examples of connected proper DDGs that are not walk-regular. The next theorem gives more examples.

Theorem 4.8 *If n is even and $n \geq 6$, then there exists a connected proper DDG with parameters $(4n, n{+}2, n{-}2, 2, 4, n)$ and one with parameters $(4n, 3n{-}2, 3n{-}6, 2n{-}2, 4, n)$ which is not walk regular.*

Proof. Let Z denote the reverse identity matrix of order n (that is $(Z)_{i,j} = 1$ if $i + j = n + 1$, and $(Z)_{i,j} = 0$ otherwise). Then Z is symmetric with zero diagonal and $Z^2 = I_n$. Define $\overline{Z} = J_n - Z$, $I = I_n$ and $\overline{I} = J_n - I_n$. Then

$$\begin{bmatrix} \overline{I} & I & I & I \\ I & \overline{I} & I & I \\ I & I & Z & \overline{Z} \\ I & I & \overline{Z} & Z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Z & \overline{Z} & \overline{I} & \overline{I} \\ \overline{Z} & Z & \overline{I} & \overline{I} \\ \overline{I} & \overline{I} & \overline{I} & I \\ \overline{I} & \overline{I} & I & \overline{I} \end{bmatrix}$$

are adjacency matrices of DDGs with the required parameters. Obviously the diagonal entries of each quotient matrix take the values 1 and $n - 1$, so the graph is not walk-regular when $n > 2$. Moreover, $n \geq 6$ implies $\lambda_1 > \lambda_2 > 0$, therefore the DDG is connected and proper. □

4.3 Small parameters

For many parameter sets the above conditions imply that a DDG with those parameters is walk-regular. These include most parameter sets with at most 27 vertices which are presented in Table 1. For all parameter sets on at most 27 vertices we are able to decide if there exist a walk-regular DDG or not, and for all but one parameter sets we are able to decide if there exits a DDG that is not walk-regular. Most cases follow directly from one of the results above. Only a few cases need a closer examination.

Proposition 4.9 *Any DDG with one of the following parameters sets is walk-regular.* $(12, 5, 1, 2, 4, 3)$, $(12, 7, 3, 4, 4, 3)$, $(15, 4, 0, 1, 5, 3)$, $(20, 7, 3, 2, 4, 5)$, $(20, 13, 9, 8, 4, 5)$.

Proof. Assume there exist a DDG with parameters $(12, 5, 1, 2, 4, 3)$ that is not walk-regular. Then the eigenvalues are 5, 2, -2, 1 and -1. The respective multiplicities f_1, f_2, g_1 and g_2 are all positive (otherwise the graph would be walk-regular). This gives only one possibility $(f_1, f_2, g_1, g_2) = (3, 5, 1, 2)$. Hence trace $R = 4$. By Theorem 2.2 the entries of the quotient matrix R can take only two values: 1 and 2. Thus all diagonal entries are equal to 1, contradiction.

Assume there exist a DDG with parameters $(12, 7, 3, 4, 4, 3)$ that is not walk-regular. Again there is only one possible spectrum with five distinct eigenvalues, being: 7, 2, -2, 1, -1 with multiplicities 1, 2, 6, 2 and 1. Hence trace $R = 8$. The diagonal entries of R are at most $n - 1 = 2$, so they must all be equal to 2, contradiction.

Assume there exist a DDG with parameters $(15, 4, 0, 1, 5, 3)$ that is not walk-regular. The only possibility for five distinct eigenvalues is 4, 2, -2, 1 and -1 with multiplicities 1, 4, 6, 2 and 2, respectively. Hence trace $R = 4$. By Theorem 2.2 the entries of the quotient matrix R can take only two values: 0 and 1. Thus four diagonal entries of R are equal to one, which is impossible because n is odd.

Assume there exist a DDG with parameters $(20, 7, 3, 2, 4, 5)$ that is not walk-regular. The only possible spectrum with five distinct eigenvalues is 7, 2, -2, 3, -3 with multiplicities 1, 7, 9, 1, 2. Hence trace $R = 4$. Since n is odd, the diagonal entries of R are even. This leads to only one feasible quotient matrix

$$R = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 0 & 3 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & 3 & 3 & 0 \end{bmatrix}.$$

However it turns out to be impossible to built a DDG with this R (see [6] for details).

Assume there exist a DDG with parameters $(20, 13, 9, 8, 4, 5)$ that is not walk-regular. The only possible spectrum with five distinct eigenvalues is 13, 2, -2, 3, -3 with multiplicities 1, 4, 12, 2, 1. Hence $\operatorname{trace} R = 16$. The diagonal entries of R are at most 4, so they are all equal to 4, contradiction. □

Finally we deal with the parameter set $(20, 9, 0, 4, 10, 2)$. A walk-regular example with these parameters exists (see [4]); it is the distance-regular Johnson graph $J(6, 3)$.

Theorem 4.10 *There exist a DDG with parameter set* $(20, 9, 0, 4, 10, 2)$*, which is not walk-regular.*

Proof. Consider the matrix

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & -1 & 0 & -1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 0 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Replace each -1 by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, each 1 by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and each 0 by $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. It is easily checked that the matrix thus obtained is the adjacency matrix of a DDG with the above parameters. Clearly the quotient matrix doesn't have constant diagonal, so the graph is not walk-regular. □

The above two theorems are already given in Master's thesis of M.A. Meulenberg[6]. The parameters set of the above theorem is the smallest one for which there exists a connected DDG that is not walk-regular[1].

5 Proper DDGs with at most 27 vertices

In [4] the authors generated all putative parameters sets $(v, k, \lambda_1, \lambda_2, m, n)$ for proper DDGs on at most 27 vertices. Except for the sets corresponding to the straightforward cases $\lambda_1 = k$, $\lambda_2 = 0$ and $\lambda_2 = 2k - v$, they obtain fifty parameter sets and for each set they tried to decide on existence or nonexistence of DDGs. In ten cases existence remained undecided. In this paper we have resolved nine of these ten cases. Moreover, we decided for which parameter sets there exits a walk-regular DDG and a non-walk-regular DDG. We copied this information in Table 1, included the new results, but deleted the parameter sets for which no DDG exits (proved in [4], Section 3.2, or Corollary 4.6). In the table multiplicities of the eigenvalues (denoted as exponents) are only given if they are determined by the parameters; the column 'WR' indicates if there exists a walk-regular DDG, and the column 'notWR' indicates existence of a DDG which is not walk-regular.

[1] This DDG was presented for the first time at the conference IPM 20–Combinatorics 2009 in Tehran as a present for the 20th anniversary of the IPM.

v	k	λ_1	λ_2	m	n	$\theta_1^{f_1}$	$\theta_2^{f_2}$	$\theta_3^{g_1}$	$\theta_4^{g_2}$	WR	notWR	reference
8	4	0	2	4	2	2^1	-2^3	0^3	-	yes	no	[4], 4.2
10	5	4	2	5	2	-	-1^5	$\sqrt{5}^2$	$-\sqrt{5}^2$	yes	no	[4], 4.2
12	5	0	2	6	2	$\sqrt{5}^3$	$-\sqrt{5}^3$	-	-1^5	yes	no	[4], 4.2
12	5	1	2	4	3	2	-2	1	-1	yes	no	[4], 4.9
12	6	2	3	3	4	2^3	-2^6	0^2	-	yes	no	[4], 4.2
12	7	3	4	4	3	2	-2	1	-1	yes	no	[4], 4.9
15	4	0	1	5	3	2	-2	1	-1	yes	no	[4], 4.9
18	9	6	4	6	3	$\sqrt{3}^6$	$-\sqrt{3}^6$	3^1	-3^4	yes	no	[4], 4.4
18	9	8	4	9	2	-	-1^9	3^4	-3^4	yes	no	[4], 4.2
20	7	3	2	4	5	2	-2	3	-3	yes	no	[4], 4.9
20	7	6	2	10	2	-	-1^{10}	3^5	-3^4	yes	no	[4], 4.2
20	9	0	4	10	2	3	-3	1	-1	yes	yes	[4], 4.10
20	13	9	8	4	5	2	-2	3	-3	yes	no	[4], 4.9
20	13	12	8	10	2	-	-1^{10}	3^4	-3^5	yes	no	[4], 4.2
24	6	2	1	3	8	2^9	-2^{12}	$\sqrt{12}^1$	$-\sqrt{12}^1$	no	yes	4.7, [4]
24	7	0	2	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	-	-1^7	yes	no	[4], 4.2
24	8	4	2	4	6	2	-2	4	-4	yes	yes	[4], 4.8
24	10	2	4	12	2	$\sqrt{8}^6$	$-\sqrt{8}^6$	2^3	-2^8	yes	no	3.3, 4.4
24	10	3	4	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	2^1	-2^6	yes	no	3.4, 4.4
24	10	6	3	3	8	2^8	-2^{13}	$\sqrt{28}^1$	$-\sqrt{28}^1$	no	yes	4.7, [4]
24	14	6	8	12	2	$\sqrt{8}^6$	$-\sqrt{8}^6$	2^2	-2^9	yes	no	3.3, 4.4
24	14	7	8	8	3	$\sqrt{7}^8$	$-\sqrt{7}^8$	-	-2^7	yes	no	[4], 4.2
24	16	12	10	4	6	2	-2	4	-4	yes	yes	[4], 4.8
26	13	12	6	13	2	-	-1^{13}	$\sqrt{13}^6$	$-\sqrt{13}^6$	yes	no	[4], 4.2
27	16	12	9	9	3	2^5	-2^{13}	$\sqrt{13}^4$	$-\sqrt{13}^4$	no		4.7, –
27	18	9	12	9	3	3^6	-3^{12}	0^8	-	yes	no	[4], 4.2

Table 1: Feasible parameters for proper DDGs with $v \le 27$, $0 < \lambda_2 < 2k - v$, $\lambda_1 < k$.

6 Miscellaneous results

In this section we present two observations on DDGs that we consider worthwhile to be mentioned.

6.1 Codes spanned by quotient matrices

It is proved in [5] that orbit matrices of block designs can be used as generator matrices of self-orthogonal codes. In a similar way we show that quotient matrices of divisible design graphs generate self-orthogonal codes over certain finite fields.

Theorem 6.1 *Let R be the quotient matrix of a proper divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$. If p is a prime dividing $n\lambda_2$ and k, then R generates a self-orthogonal code of length m over GF(p).*

Proof. The quotient matrix R satisfies $R^2 = RR^T = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m$. The fact that $v = mn$ completes the proof. □

Theorem 6.2 *Let R be the quotient matrix of a proper divisible design graph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$. If p is a prime dividing n and $k - \lambda_1$, then R generates a self-orthogonal code of length m over GF(p).*

Proof. Taking row sums on both sides of Equation 1 yields $k^2 = k + \lambda_1(n-1) + \lambda_2 n(m-1)$, hence $k^2 - \lambda_2 v + \lambda_2 n = n\lambda_1 + (k - \lambda_1)$. □

6.2 Automorphisms

Theorem 6.3 gives a simple observation about an automorphism of a DDG with $n = 2$.

Theorem 6.3 *If* Γ *is a proper DDG with parameters* $(v, k, \lambda_1, \lambda_2, m, 2)$*, then* Γ *admits an automorphism of order two acting in* $\frac{v}{2}$ *orbits of length two.*

Proof. The canonical partition divides the adjacency matrix in $\frac{v}{2} \times \frac{v}{2}$ blocks of dimension 2×2. There are four possibilities for these 2×2 blocks:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Every block matrix consisting of these blocks admits an automorphism of order two. □

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