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Prices and volumes of options: A simple theory of risk sharing when markets are incomplete *

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Abstract

We present a simple theory of business-cycle movements of option prices and volumes. This theory relies on time-varying heterogeneity between agents in their demand for insurance against aggregate risk. Formally, we build an infinite-horizon model where agents face an aggregate risk, but also different levels of idiosyncratic risk. We manage to characterize analytically a general equilibrium in which positive quantities of derivatives are traded. This allows us to explain the informational content of derivative volumes over the business cycle. We also carry out welfare analysis with respect to the introduction of options, which appears not to be Pareto-improving.

Keywords: Option Pricing, Open Interest, Incomplete Markets.

JEL codes: G1, G10, E44.

Résumé

Cet article présente un modèle simple de détermination du prix et du volume des options dans le cycle économique. La théorie présentée repose sur les variations de la demande d'assurance des agents économiques face aux risques agrégés. Nous élaborons un modèle à horizon infini où les agents font face à la fois à des idiosyncratiques non-assurables et à des chocs agrégés. Nous caractérisons analytiquement un équilibre général où des volumes positifs de titres financiers et de produits dérivés sont échangés. Ceci nous permet d'expliquer le contenu informationnel du volume de produit dérivé (option) échangé dans le cycle. Une analyse de bien-être est aussi réalisée par rapport à l'introduction des options. Nous trouvons que cette introduction n'est pas toujours Pareto-améliorante.

Mots-clés : Prix des options, Volume des options, Marchés incomplets.

Codes JEL : G1, G10, E44.

1 Introduction

Derivative assets are financial instruments allowing agents to exchange risks. The volume of derivatives exchanged contains information about the opportunities for risk trading, and thus about the heterogeneity between agents in their risk assessments. This volume is time-varying and depends on the volatility of the underlying asset price. Considering one of the most-frequently traded derivatives, options on the S&P 500 index, notably reveals that the traded volume increases with the volatility of the underlying asset, the S&P 500 index. More aggregate risk thus generates greater heterogeneity and modifies the willingness to bear risks. We develop a general-equilibrium model with incomplete markets, where this time-varying heterogeneity is endogenously derived and which is consistent with the previous basic fact. We provide general pricing equations when options are not redundant and use the model to analyze the welfare effect of introducing a derivative asset.

The model we present here departs from the seminal Black and Scholes (1973) setting, in which options are redundant and priced by replication. In this setup, option volumes are not determined and the exchange of insurance between agents does not play any particular role. We consider a deviation from the standard infinite-horizon economy *à la* Lucas, endowed with a single exogenous risky asset, where agents face heterogeneous uninsurable idiosyncratic risks and borrowing constraints. As a result of this last assumption, agents differ in their willingness to bear the aggregate risk on the top of the heterogeneous idiosyncratic risk. Incomplete markets for some risks thus provide the foundation for aggregate risk sharing and the exchange of insurance due to derivative assets. This motivation is appealing, since the assumption of market incompleteness has received empirical support, and has been introduced as a partial solution to many pricing puzzles, such as the equity premium (Constantinides and Duffie (1996)) and the risk-free rate (Huggett (1993)).

This foundation of the demand for derivative assets has previously been analyzed in the literature, notably by Franke, Stapleton, and Subrahmanyam (1998) in a two-date economy. Our contribution consists in introducing a tractable infinite-horizon setting where pricing formulae and portfolio compositions can easily be derived and where it is possible to carry out welfare analysis. We are able to do so because we present a new class of models, where closed form solutions can be obtained with both aggregate shocks and heterogeneity in the exposure to uninsurable idiosyncratic risks. In this equilibrium, the ex-post heterogeneity is reduced to a finite number of different agents' classes. We thus avoid the computation of an approximate equilibrium, which is possible only with a reduced number of assets (typically two) in the presence of aggregate shocks. On the contrary, our setup allows for the introduction of an arbitrarily large number of derivatives. More specifically, we model the idiosyncratic risk as an employment risk and make two assumptions to reduce heterogeneity. The first is that labor supply is sufficiently elastic (as in Scheinkman and

Weiss (1986) and Lagos and Wright (2005)) that the employed compensate for income fluctuations by working more. The second is that income when unemployed (unemployment benefit or home production) is such that unemployed agents choose to liquidate their entire portfolio when they move into unemployment and thus hit the credit constraint. In this setup, we introduce heterogeneity across agents by allowing for different severities of idiosyncratic risk and thus different types of agents. Although this framework is highly stylized, it does allow us to clearly identify the equilibrium exchange of risk under incomplete markets in a tractable model. We are notably able to provide new results concerning the interaction between option trading and time-varying aggregate risk.

The first part of the paper features two types of agents with two different levels of idiosyncratic risks who buy shares of a risky tree, both to smooth consumption and to self-insure against uninsurable income risk. We consider this setup with and without a single call option, which completes the market in this simple framework. We analyze how the availability of the derivative asset affects the portfolio structure of both agents. Agents facing the lowest level of idiosyncratic risk choose to bear a larger share of aggregate risk by purchasing call options. Since the call pays off only in the good state of the world, the low-risk agent chooses, when purchasing call options, to hold a riskier portfolio and thus to bear a larger share of the aggregate risk and to provide insurance to the high-risk agent, who has sold the option.

We prove that the volume of options traded increases with heterogeneity and exposure to idiosyncratic risk, and when the dividends of the risky asset become more volatile. This model is thus able to reproduce the correlation between asset volatility and the volume of traded options reported in the data. The intuition for these results is the following. An increase in the riskiness of the asset, which can be thought as a mean-preserving spread of dividends, implies that in some states of the world the payoff of the risky asset is low. Agents want to insure themselves against the conjunction of a bad idiosyncratic outcome and a bad aggregate shock, which means a low portfolio liquidation value, while being unemployed. As idiosyncratic risk differs across agents, those with a less severe idiosyncratic risk will be willing to bear a larger share of the aggregate risk than those with a more severe one. High-risk agents purchase a greater quantity of underlying assets and also sell call options (i.e. buy insurance) in order to smooth their portfolio payoffs across states of the world.

Second, we show that the introduction of the option increases the welfare of agents with more idiosyncratic risk and reduces the welfare of those facing less idiosyncratic risk. The former are more able to self-insure and bear a smaller share of the aggregate risk after the sale of options. The latter purchase options and hold riskier portfolios. They support a greater share of the aggregate risk and their welfare falls. The introduction of options is therefore not Pareto-improving, which illustrates the conclusions of Elul (1995) and Cass and Citanna (1998) on the impact of financial innovation on welfare.

The second version of the model generalizes the previous analysis. An arbitrarily high number of different agent types facing different levels of idiosyncratic risks are able to trade an arbitrarily large number of options derived from a single risky Lucas tree. The technical part consists in proving the equilibrium existence and in deriving the joint equations for portfolio compositions and prices. As an empirical illustration, we calibrate the model to reproduce roughly option volumes and prices in a simplified representation of the business cycle.

The remainder of the paper is organized as follows. The following section provides a literature review, and Section 3 presents the simple economy with only one risky asset. Section 4 introduces options to complete the market for the aggregate risk. Section 5 generalizes the equilibrium existence to an economy with an arbitrarily large number of agents and securities. The final section provides a simple numerical example.

2 Literature review

Our paper belongs first to the option-pricing literature allowing for heterogeneity and simultaneously deriving option prices and quantities. Three reasons justifying aggregate risk-sharing between agents can be found in the literature. The first is that agents have different preferences, notably with respect to risk aversion. As a consequence, they value differently the various aggregate risks (Bhamra and Uppal (2009)). The second is that agents have different beliefs or information regarding aggregate risk (Biais and Hillion (1994); Buraschi and Jiltsov (2006)). The third is that agents face different background risks. Their willingness to bear aggregate risk on the top of this idiosyncratic (or background) risk then differs (Franke, Stapleton, and Subrahmanyam (1998)). Our model is related to this third justification. Our contribution is to depart from a two-date model and propose an infinite-horizon economy, in which we analytically characterize prices and volumes and carry out welfare analysis.

There are a considerable number of papers in the microstructrure literature documenting the stylized fact that we reproduce, which is the positive relationship between underlying volatility and activity on the options market. Easley, O'Hara and Srivinas (1998) reveal a positive correlation between contemporaneous stock price changes and option volumes. Donders, Kouwenberg and Vorst (2000) find the same positive relationship using earnings announcements. Moreover, the survey by Karpoff (1987) notes a strong positive link between stock-price changes and trading volumes in the stock market. In addition, Anthony (1988) and Stephan and Whaley (1990) find a positive correlation between stock and option volumes. All of these results together confirm the relationship between stock prices and option volumes. An empirical result closer to our objectives and our model is the positive correlation between the option open interest and volatility of the S&P 500 found by Buraschi and Jiltsov (2006) using daily data on the S&P 500 index. There is a strong consensus about this positive relationship, but there remains a debate about the informational content of derivative volumes as a predictor of future stock prices (Easley, O'Hara and Srivinas (1998) and Pan and Poteshman (2006), amongst others) or future stock volatility (Ni, Pan and Poteshman (2008)), often over very short horizons (intraday or less than a week). Obviously, this aspect is far beyond the scope of our paper and we are only interested in option volumes over the business cycle.

Considering the theoretical literature on incomplete markets and asset pricing, the equilibrium definition used in our paper is related to a previous work in monetary theory by Algan, Challe and Ragot (Forthcoming). We generalize this work by introducing risky assets and by allowing for heterogeneity in the exposure to idiosyncratic risk, while still being able to obtain closed form solutions. This framework allows for actual trade between heterogeneous agents, which is necessary to analyze the properties of the traded volume of insurance. Our paper thus departs from no-trade equilibria models, which allow for analytical prices in an infinite-horizon economy with incomplete markets. Constantidines and Duffie (1996) propose such a framework to solve the equity premium puzzle. Krusell, Mukoyama, and Smith (2008) recently investigate asset prices in a no-trade equilibrium, in order to derive bond and equity prices simultaneously.

3 A simple economy with one risky asset

We consider an economy with one risky asset, where two types of agents (types 1 and 2) face an uninsurable unemployment shock. The risky asset is a simple Lucas tree, whose mass V remains constant over time. The price of one unit of this tree at date t is denoted by P_t. The tree pays off a dividend y_t at each date t in consumption goods, the variations in which represent the sole aggregate risk in the economy. These dividends can take on only two values in $Y = \{y_G, y_B\}$, with $y_G \geq y_B$: G refers to the good state and B to the bad state. Dividends evolve as a two-state Markov chain, whose transition probabilities of moving from state k to state l $(k, l = G, B)$ are denoted by π_{kl}. To avoid the discussion of uninteresting cases, we make the following assumption, stating that aggregate states are persistent and do not fluctuate too often, which is consistent with data.[1]

Assumption A (Persistence) *Both aggregate states are persistent, i.e.* $\pi_{GG} + \pi_{BB} > 1$.

[1]For instance, Hamilton ((1994), Chapter 22) finds $\pi_{GG} + \pi_{BB} = 1.65$ at a quarterly frequency in the US economy.

In addition to this aggregate risk, all agents face an uninsurable unemployment risk: agents can be either employed or unemployed. When employed, type-1 agents face a probability α^1 of becoming unemployed, and thus a probability $1 - \alpha^1$ of remaining employed. When unemployed, these type-1 agents face a probability ρ^1 of remaining unemployed and a probability $1 - \rho^1$ of finding a new job. With obvious notations, type-2 agents face analogous probabilities of α^2 and ρ^2. Transition probabilities of moving into and out of unemployment are constant, so that we can deduce the long-run fraction η^i of type-i employed agents:

$$\text{for } i = 1, 2 \quad \eta^i = \frac{1 - \rho^i}{1 + \alpha^i - \rho^i}$$

The initial fraction of employed agents of type i is also assumed to be η^i to avoid transitory dynamics.

When employed, agents earn a constant hourly wage that we normalize to 1, and can freely adjust their labor supply. When unemployed, agents benefit from a constant level of "domestic production" that we denote by $\delta > 0$. Home production is low enough for the unemployed to be worse-off than the employed, which ensures that agents are willing to participate in the labor market. More formally, we make the following assumption.

Assumption B (Labor market participation) $u'(\delta) > 1$.

3.1 Preferences and the agents' program

In each period, agents enjoy utility from consumption and leisure. Preferences are separable over time, as well as in consumption and leisure. More precisely the instantaneous utility of any agent over consumption c and labor l is written as $u(c) - l$, where $u : \mathbb{R} \to \mathbb{R}$ is twice differentiable, increasing and concave. We follow Scheinkman and Weiss (1986) and Lagos and Wright (2005), among others, and assume that agents have a linear disutility of labor. Instantaneous preferences are discounted by a common factor of $\beta \in (0, 1)$, representing the time decay. We discuss below the implications of the linear disutility of labor.

We denote by $e_t^i \in \{0, 1\}$ the employment status of a type-i agent ($i = 1$ or 2), with $e_t^i = 1$ when employed and $e_t^i = 0$ otherwise. The program of a type-i agent consists in maximizing his intertemporal utility under a set of constraints, by choosing consumption, labor supply and asset demand, which are denoted respectively c_t^i, l_t^i and x_t^i. The operator $E_0[\cdot]$ is the unconditional expectation over the aggregate and idiosyncratic shocks.

$$\max_{c^i,l^i,x^i} E_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_t^i) - l_t^i\right) \tag{1}$$

$$\text{s.t.} \quad c_t^i + P_t\, x_t^i = e_t^i\, l_t^i + (1-e_t^i)\delta + (P_t + y_t)\, x_{t-1}^i \tag{2}$$

$$c_t^i,\, l_t^i \geq 0 \text{ and } x_t^i \geq 0 \tag{3}$$

$$\lim_{t\to\infty} \beta^t\, u'(c_t^i)\, x_t^i = 0 \tag{4}$$

The initial asset endowment is denoted by x_{-1}^i. Agents of type $i = 1, 2$ maximize their expected intertemporal utility (1) subject to a set of constraints (2)–(4). In constraint (2), total resources made up of labor income (or domestic production for the unemployed), asset dividends and asset-sale values are used to consume and to purchase assets. The second condition (3) states that both consumption and labor supply are positive, which will always be the case in all of the equilibria that we consider. The borrowing constraint stating that agents are prevented from short selling the tree appears in equation (3). The last equation (4) simply sets out the transversality condition that will hold in our equilibria.

Agents' risk-sharing is limited along three dimensions, since agents are only allowed to trade positive quantities of a single asset. First, the market for the aggregate risk is incomplete, because there is a single asset, while aggregate risk takes on two possible values. Second, individual unemployment risk is uninsurable, because there is no asset contingent on labor-force status. Finally, agents can only trade positive asset amounts, which limits their portfolio composition. This set of restrictions will be relaxed in the next section, where we complete the market for aggregate risk. These assumptions regarding idiosyncratic risk are common in the literature on liquidity constraints, which is also called the heterogeneous-agent literature (see for instance Krusell, Mukoyama, and Smith (2008)).

We define $\phi_t^i \geq 0$ as the Lagrange multiplier associated with the borrowing constraint (3). Writing the Lagrangian of the previous program allows us to derive the following first-order conditions:

$$\text{Labor-market condition:} \quad \begin{cases} u'(c_t^i) = 1 & \text{if} \quad e_t^i = 1 \\ l_t^i = 0 & \text{if} \quad e_t^i = 0 \end{cases} \tag{5}$$

$$\text{Consumption smoothing:} \quad P_t\, u'(c_t^i) = \beta\, E_t\left[u'(c_{t+1})\,(P_{t+1} + y_{t+1})\right] + \phi_t^i \tag{6}$$

The first-order condition (5) says that while an unemployed agent does not supply any labor, an employed agent works so as to set equal his marginal utility of consumption to the constant marginal disutility of labor, which is equal to 1. As a consequence, all employed agents choose the same level of consumption. The Euler equation (6) sets equal the cost of giving up one unit of consumption today to the expected gain ($E_t[\cdot]$ is the expectation over both the aggregate and idiosyncratic states, conditional on all of the available information at date t) from one additional unit of consumption tomorrow. When the Lagrange multiplier ϕ_t^i is strictly positive, the type i agent would like to borrow and to consume more, but his binding credit constraint prevents him from doing so.

3.2 Market clearing and equilibrium

The asset-market clearing condition simply sets equal asset supply, which is constant and equal to V, to overall asset demand. To aggregate individual demands, we define the probability measure $\Lambda_t^i : \mathbb{R} \times E \to \mathbb{R}$ describing the distribution of type-i agents as a function of their asset holdings and labor status. This probability is consistent with the history of both aggregate and idiosyncratic shocks. As an example, $\Lambda_t^2(x, 1)$ is the measure of employed agents of type 2 holding the quantity of assets x. The market-clearing condition is therefore written as follows:

$$\int_{(x^1,e)\in\mathbb{R}^+\times E} x^1 \, d\Lambda_t^1(x^1, e) + \int_{(x^2,e)\in\mathbb{R}^+\times E} x^2 \, d\Lambda_t^2(x^2, e) = V \tag{7}$$

Walras' Law implies that the goods market clears whenever the asset market clears. As a result, we can define an equilibrium in our economy as follows:

Definition 1 (Equilibrium) *For an initial distribution of asset holdings $\left(x_{-1}^i\right)_{i=1,2}$ and employment statuses $\left(\Lambda_0^i\right)_{i=1,2}$ for both agent types, an equilibrium consists in individual choices $\{c_t^i, l_t^i, x_t^i\}_{t=0,\dots,\infty}^{i=1,2}$ and a price sequence $(P_t)_{t=0,\dots,\infty}$, such that:*

1. *Given prices, individual strategies solve the agents' optimization program (1).*
2. *The distribution of Λ_t^i ($i = 1, 2$ and $t \geq 0$) is consistent with both aggregate and idiosyncratic state evolutions for both agent types.*
3. *The asset market clears (i.e. equation (7) holds).*

3.3 Reduced heterogeneity equilibrium

3.3.1 Assumptions and description

In standard economies featuring uninsurable idiosyncratic shocks and credit constraints, the equilibrium cannot be explicitly derived, since it involves an infinite distribution of agents with different individual histories. The usual strategy consists in computing approximate equilibria. In this paper, we derive an equilibrium where the heterogeneity in insurance demand can be computed with paper and pencil.

This equilibrium is based on two assumptions. The first has already been introduced and is the linearity of the disutility of labor. In this case, when employed, agents freely adjust their labor supply to attain a constant marginal utility of consumption, equal to 1 (equation (5)). All employed agents thus consume the same amount. This assumption is introduced to reduce the heterogeneity across employed agents.

Our second assumption is that the quantity V of the asset remains small enough such that even after selling off their entire portfolio, unemployed agents remain credit-constrained. In other words, we assume that the asset quantity is not sufficient for agents to overcome their credit constraint after falling into unemployment. More formally, this assumption states that the Lagrangian multiplier on the credit constraint (6) in the agent's program binds if and only if the agent is unemployed; the multiplier is slack when he is employed, as summarized in the following assumption:

Assumption C (Small asset volumes) *Unemployed agents are always credit-constrained, while employed agents always participate in the asset market:*

$$\forall i \in \{1,2\},\ \phi_t^i > 0 \ \Leftrightarrow e_t^i = 0$$

Of course, this assumption has to be checked in equilibrium. In Section 3.3.2 below, we show that this condition will hold as long as V is small enough. A second implication of this assumption is that all employed agents, whatever their type, purchase the asset. This is ensured by the assumption that the idiosyncratic risks of the two types of agents are not too different, or that α^1 is not too different from α^2.

In this equilibrium, it is easy to write the Euler equation (6) of employed agents of type i, separating aggregate from idiosyncratic states, which is written as:

$$P_t = \beta(1-\alpha^i)E_t\left[P_{t+1}+y_{t+1}\right] + \beta\,\alpha^i\,E_t\left[u'\left(\delta + (P_{t+1}+y_{t+1})\,x_t^i\right)(P_{t+1}+y_{t+1})\right] \tag{8}$$

The left-hand side is the opportunity cost of buying one unit of the asset at period t. The first term on the right-hand side is the valuation of the expected return from holding the asset if the agent remains employed (with probability $1-\alpha^i$). As the agent is employed, his marginal utility is 1. The second term on the right-hand side is the valuation of the same return when the agent is unemployed. In this case, his marginal utility is $u'\left(\delta + (P_{t+1}+y_{t+1})\,x_t^i\right)$, where x_t^i is the quantity of assets bought in period t, as the agent liquidates all financial holdings in order to consume.

From the previous equation, we deduce that the equilibrium asset price and the quantity of assets chosen by employed agents only depends on their type and on the aggregate parameters, and not on their previous wealth. With a slight abuse of notation, we denote these respectively as P_t and x_t^i. Moreover, this price P_t and the asset quantities x_t^i depend only on the current aggregate state y_t, and not on the whole history of shocks. We simply denote these respectively by P_k and x_k^i for the type-i agent in the state of the world $k = G, B$.

For each type of agents $i = 1, 2$ there are four different classes of households in each period t: (i) Unemployed agents at date t, who were employed before. These agents, denoted eu, consume the amount $\delta + (P_t + y_t)\, x^i_{t-1}$; (ii) Unemployed agents, who were unemployed in the previous period. These agents, denoted uu, consume the amount δ; (iii) Employed agents, ee, who were employed before; and (iv) Employed agents, ue, who were unemployed before. Agents belonging to these last two classes consume the same amount $u'^{-1}(1)$ and buy the same quantity of assets x^i_t. However, they supply a different quantity of labor, as their beginning-of-period wealth is different.

3.3.2 Proof of the existence of the equilibrium

The previous equilibrium exists if: (i) unemployed agents are credit constrained; and (ii) both agent types trade in asset markets. The first sufficient conditions are therefore that the Euler equations of eu and uu agents of both types hold with strict inequality, reflecting that the asset is too expensive for them. Since the quantity of assets x^i_h held in state $h = B, G$ in the beginning of the period by the type-i agent eu is positive, it is sufficient to show that eu agents are credit-constrained, which implies that the uu are also. This is simply written in state $k = B, G$ as:

$$P_k\, u'(\delta + (P_k + y_k)\, x^i_h) > \beta \sum_{j=G,B} \pi_{k,j} u'(\delta)(P_j + y_j) + \pi_{k,j}\rho^i\, (1 - u'(\delta))\, (P_j + y_j) \tag{9}$$

The second condition for the equilibrium to exist is that the valuation of the asset by employed agents of both types is similar. Using equation (12), we deduce that in state $k = G, B$:

$$\alpha^1 \sum_{j=G,B} \pi_{k,j} \left(u'\left(\delta + (P_j + y_j)\, x^1_k\right) - 1\right) (P_j + y_j) = \alpha^2 \sum_{j=G,B} \pi_{k,j} \left(u'\left(\delta + (P_j + y_j)\, x^2_k\right) - 1\right) (P_j + y_j) \tag{10}$$

This equation states that both agent types value the asset similarly when falling into unemployment.

The proof of the existence of the equilibrium is set out in the Appendix. The following proposition summarizes the result.

Proposition 1 (Equilibrium existence) *We assume that in an homogeneous economy with a tree of mass zero and without aggregate shocks ($\alpha^1 = \alpha^2 = \alpha$, $y_G = y_B = y$, and $V = 0$) the following condition holds:*

$$1 < u'(\delta) < 1 + \frac{1-\beta}{\beta\, \alpha} \tag{11}$$

Then:

1. *A unique equilibrium exists in that economy.*

2. *A unique equilibrium also exists in the vicinity of this economy, i.e. when 1) the tree size V is not too large, 2) the heterogeneity in the idiosyncratic risk (α^1 and α^2 close to α) is limited, and 3) the aggregate uncertainty remains limited (y_B and y_G close to y).*

The intuition of the proposition is the following. Condition (11) is necessary for the equilibrium to exist when there is no aggregate risk and when all agents types have the same idiosyncratic risk. In this case, we can simply derive the price of the asset P and check that unemployed agents are credit constrained. Both of the inequalities in this condition are compatible since $\beta < 1$. We will assume that this condition holds in the remainder of the paper. The condition is all the less restrictive as the probability of falling into unemployment α is small. This condition also allows us to rule out sunspot equilibria and to be sure that the equilibrium we consider is unique. Contrary to Bowman and Faust (1997), options cannot play any role in our economy when the market is complete for the aggregate risk before the introduction of options.

In the case with risk, the intuition is simply that even an agent endowed with all available assets is unable to unwind his credit constraint when falling into unemployment in a good state of the world. The restriction on aggregate uncertainty ensures that the portfolio liquidation value is not too high in any state of the world. Finally, the agents' probabilities α^i $(i = 1, 2)$ of falling into unemployment must be not too different from each other, for both of them to wish to hold the asset in equilibrium.

3.4 Interaction between aggregate uncertainty and heterogeneity

As prices and quantities depend only on the aggregate state, the Euler equation (8) defining the asset price in state $k = B, G$ can be expressed as:

$$P_k = \beta \sum_{j=G,B} \pi_{k,j} \left(1 + \alpha^i \left(u' \left(\delta + (P_j + y_j)\, x_k^i\right) - 1\right)\right) (P_j + y_j),\ i = 1, 2,\ \ k = G, B \tag{12}$$

Eq. (12) summarizes four different equations, stating that the asset valuation is the same for both agent types $i = 1, 2$ in both states of the world $k = G, B$.

The market-clearing condition in each state of the world can be simply written as follows:

$$V = \eta^1\, x_k^1 + \eta^2\, x_k^2,\ k = G, B \tag{13}$$

The equilibrium now consists of the set of six endogenous variables $\left\{P_G, P_B, x_G^1, x_B^1, x_G^2, x_B^2\right\}$ defined by the six equations (12) and (13). It is simple to analyze the interaction between aggregate uncertainty and heterogeneity. Before presenting the main result of this section, we need to make an additional assumption about the shape of the utility function.

Assumption D (Utility function) *We assume that the following inequality holds in the equilibrium without aggregate risks ($y_B = y_G = y$):*

$$-x^1 \frac{u''(\delta + x^1 (P + y))}{u'(\delta + x^1 (P + y)) - 1} > -x^2 \frac{u''(\delta + x^2 (P + y))}{u'(\delta + x^2 (P + y)) - 1}$$

where P is the price and x^1 and x^2 are the asset quantities chosen by agents 1 and 2 in the economy without aggregate risk.

Solving for the price P and quantities x^1 and x^2, the previous inequality becomes a condition on the shape of the utility function. This condition always holds for standard utility classes, such as CRRA, CARA or quadratic utilities, and is thus quite general. This condition states that the type-1 agents who have the highest probability of falling into unemployment ($\alpha^1 > \alpha^2$) benefit relatively more from an increase in the return to the financial asset. To see this, note that the function of $P + y$ given by $u'(\delta + x^i (P + y)) - 1$ is the difference in marginal utilities between the employed and the unemployed. The condition states that the elasticity of this function with respect to $P + y$ is higher for type-1 than for type-2 agents, which means that the difference in marginal utilities between the employed and the unemployed is proportionally higher for 1 than for type-2 agents. The following proposition summarizes the results regarding the effect of aggregate shocks.

Proposition 2 (Aggregate shock effects) *In the vicinity of the equilibrium without uncertainty ($y_B = y_G = y$),*

- *(i) Agents who are more likely to fall into unemployment ($\alpha^1 > \alpha^2$) purchase in both states of the world a greater quantity of assets than the other agents: $x_k^1 > x_k^2 > 0$ for $k = B, G$.*
- *(ii) Type-1 agents hold fewer assets in the good than in the bad state of the world: $x_G^1 < x_B^1$, while the reverse holds for type-2 agents: $x_G^2 > x_B^2$.*

First, as the demand for insurance against the risk of falling into unemployment by type-1 agents is greater than that of type-2 agents, they hold more assets in all states of the world. Second, as the aggregate state is persistent, the average return to the asset is higher in the good than in the bad state of the world. As type-1 agents wish to own the asset to self-insure, they require fewer assets in good states to obtain the same expected value of self-insurance. The same is true for type-2 agents, but we can show that Assumption D implies that equilibrium quantities are driven by the behavior of type-1 agents, who are more sensitive to the return on the risky asset. In consequence, type-2 agents hold fewer assets in state B than in state G.

The next proposition summarizes the main results with respect to idiosyncratic risk.

Proposition 3 (Heterogeneity effects) *In the vicinity of the symmetric equilibrium ($\alpha^1 = \alpha^2$), a greater unemployment risk for type-1 agents (a larger α^1) has the following consequences:*

(i) The asset price increases in both states of the world, but more in the good than in the bad state.

(ii) The asset holdings of type-1 agents also increase in both states of the world, but more in the bad than in the good state.

The previous proposition considers the case where type-1 agents have a higher α^1 and thus a more severe risk than type-2 agents. As they face a greater probability of unemployment, these agents need to hedge better against that risk, which increases their asset demand. This greater demand increases asset prices in both states of the world, but not homogeneously. Since aggregate states are persistent, purchasing the asset in the bad state of the world is mainly purchasing a hedge against unemployment in the bad state, when the asset pays off less well and is less effective in helping to overcome credit constraints. Consequently, the asset price increases less in the bad than in the good state. A higher probability of unemployment therefore raises both prices, but also their variance.

Type-1 agents, who experience greater individual risk, need to self-insure more, and therefore demand a greater quantity of assets in both states of the world. However, from Assumption D, with the same quantity of assets they will be much better-insured in the good state of the world. As a result, they will purchase more assets in both states of the world, but more in the bad than in the good state. The variance of their holdings will therefore also increase.

We have presented in this section the results in the case of market incompleteness for the aggregate shock. In the next section, we introduce options and consider the exchange of insurance over the business cycle.

4 The economy with a risky asset and a derivative call option

In this section, we complete the credit market of the preceding economy by introducing a derivative asset, whose basis is the risky tree. More precisely, agents are allowed to trade a supplementary security, which is a call option with a maturity of one period and a strike contained between the asset price in the bad state of the world and that in the good state. The call is therefore designed in order to only payoff in the good state of the world. The market is complete for the aggregate risk, while it is still incomplete regarding the idiosyncratic risk, and credit constraints still limit agents' investment sets.

In addition to tree shares, which are in positive supply, agents trade options, which are in zero net supply. Since agents of different types have different risk appetites, they are willing to exchange insurance between themselves through the trade of positive quantities of options, in order to smooth their portfolio payoffs across states of the world.

We now turn to the formal description of the model.

4.1 Description of the economy

Except for the introduction of options, the economy remains unchanged from its previous incarnation. Since prices are endogenous, it is not necessarily direct to find a strike K, contained between the equilibrium asset prices in both states of the world: $P_B < K < P_G$. The simplest solution consists in first introducing options in a homogeneous economy, where both agents face a similar unemployment risk. As they are perfectly symmetric, agents hold similar portfolios. As a result, introducing the option in zero net supply implies that the call is priced but not traded. Asset prices in both states of the world are different (see Proposition 2) and we can identify the strike between the two asset prices. In a second step, increasing the heterogeneity between agents changes asset prices in a continuous way, and we deduce that it remains possible to find such a strike.

We denote by s_t^i the quantity of calls that an agent of type i purchases at the price of Q_t in period t. The type-i agent's program consists in choosing consumption, labor supply, and asset and option demands in order to maximize intertemporal utility subject to a budget constraint (including option purchases and payoffs) and a credit constraint stating that the agents' financial wealth must remain positive. Denoting by E_0 the unconditional expectation over aggregate and idiosyncratic uncertainty, the program of a type-i agent is written as follows:

$$\max_{\{c_t, l_t, x_t, s_t\}_{t=0,1,\ldots}} E_0 \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^i\right) - l_t^i \right) \tag{14}$$

$$\text{s.t.} \quad c_t^i + P_t\, x_t^i + Q_t\, s_t^i = e_t^i\, l_t^i + (1 - e_t^i)\delta + (P_t + y_t)\, x_{t-1}^i + (P_t - K)^+\, s_{t-1}^i \tag{15}$$

$$P_t x_t^i + Q_t s_t^i \geq 0 \tag{16}$$

$$c_t^i,\ l_t^i \geq 0 \tag{17}$$

$$\lim_{t \to \infty} \beta^t\, u'(c_t^i)\, x_t^i = \lim_{t \to \infty} \beta^t\, u'(c_t^i)\, s_t^i = 0 \tag{18}$$

The agent's objective is to maximize intertemporal utility (14) under the set of constraints (15)–(18). Equation (15) is the budget constraint at time t and sets equal the agent's resources (labor income, or domestic production, and revenue from the sale of both financial securities) and expenditure (consumption and security purchases). Inequality (16) is the borrowing constraint of the agent, stating that financial wealth must remain positive. The other two constraints, the positivity of consumption and labor in (17) and the transversality conditions in (18) are technical and will always be satisfied in the equilibria that we will consider.

It is worth noting that condition (16) allows agents to "issue" options, at least up to a certain point. As a result of heterogeneity and zero net volume, there will be a positive exchange of options in equilibrium, and the option purchased by one agent has to be issued and sold by another agent. The quantity of traded options is also called the open interest and measures the activity in the options market.[2]

4.2 The equilibrium

4.2.1 Euler equations

As in the previous economy, we construct an equilibrium with four agent classes for each type, where classes depend on the present and past employment status of agents. Unemployed agents eu do not participate in the financial markets, liquidate their portfolios when they fall into unemployment, and remain credit constrained. Their financial wealth is supposed to be insufficient to overcome their borrowing constraints. Liquidating assets and options, their consumption in state $j = B, G$ (where the previous state is denoted $k = B, G$) is $\delta + (P_j + y_j)\, x^i_k + (P_j - K)^+\, s^i_k$. The long-run unemployed uu only consume their domestic production δ. Employed agents ue and ee of both types $i = 1, 2$ participate in both the asset and option markets. Since they are infinitely elastic in labor, all employed agents of the same type consume the same amount $u'^{\,-1}(1)$ and hold the same financial portfolio. These agents only differ with respect to their labor effort. As in the no-option equilibrium, prices and quantities depend only on the current aggregate state and not the whole history. We deduce the following Euler equations for the asset and the option, as well as the corresponding market-clearing condition ($k = B, G$ and $i = 1, 2$):

$$P_k = \beta \sum_{j=G,B} \pi_{k,j} \left(1 + \alpha^i \left(u'\left(\delta + (P_j + y_j)\, x^i_k + (P_j - K)^+\, s^i_k\right) - 1\right)\right)(P_j + y_j) \tag{19}$$

$$Q_k = \beta\, \pi_{k,G} \left(1 + \alpha^i \left(u'\left(\delta + (P_G + y_G)\, x^i_k + (P_G - K)^+\, s^i_k\right) - 1\right)\right)(P_G - K) \tag{20}$$

$$V = \eta^1\, x^1_k + \eta^2\, x^2_k \tag{21}$$

$$0 = \eta^1\, s^1_k + \eta^2\, s^2_k \tag{22}$$

The pricing kernel has a similar interpretation to that in the economy without options. A first component refers to consumption-smoothing between two employment periods. The second term reflects hedging against unemployment risk. The security is all the more valued as it helps to

[2] In our economy, traded volumes and open interest are exactly the same, since every option contract is traded exactly once over the period.

insure against this risk and as the agent's expected cost of unemployment increases. The cost of unemployment is measured by the (inframarginal) variation in marginal utility experienced by the agent when falling into unemployment, which is $u'\left(\delta + (P_j + y_j)\, x_k^i + (P_j - K)^+\, s_k^i\right) - 1$, where k and j denote respectively the previous and the current state. This expected cost is higher as (i) the probability of job loss rises, and (ii) the portfolio-liquidation value that the agent consumes when unemployed is smaller.

There are however two main differences from the previous economy. First, obviously, there is an additional security and therefore a supplementary Euler equation (20). Second, the option only pays off in the good state of the world. The Euler equation (20) thus only includes a term for the good state of the world. Moreover, options only matter in the liquidation component of the Euler equation in the good state of the world.

To sum up, an equilibrium in this economy consists of the set of four prices $\{P_k, Q_k\}_{k=B,G}$ and eight security quantities $\{x_k^i, s_k^i\}_{k=B,G}^{i=1,2}$, such that the twelve equations (19)–(22) ($k = B, G$ and $i = 1, 2$) hold.

4.2.2 Equilibrium existence conditions

The previous equilibrium with four agent categories for each type $i = 1, 2$ exists as long as (i) employed agents of both types participate in financial markets and (ii) unemployed agents are credit-constrained when moving into unemployment.

The first condition implies that the agents' valuation for both securities is the same. The equality of the expected costs of unemployment for both agents can therefore be written in each state of the world B and G, where $h = B, G$ denotes the previous state, as:

$$\alpha^1 \left(u'\left(\delta + (P_G + y_G)\, x_h^1 + (P_G - K)\, s_h^1\right) - 1\right) = \alpha^2 \left(u'\left(\delta + (P_G + y_G)\, x_h^2 + (P_G - K)\, s_h^2\right) - 1\right) \tag{23}$$

$$\alpha^1 \left(u'\left(\delta + (P_B + y_B)\, x_h^1\right) - 1\right) = \alpha^2 \left(u'\left(\delta + (P_B + y_B)\, x_h^2\right) - 1\right) \tag{24}$$

We will use these equations later to analyze the portfolio holdings of both agents.

The second condition for the equilibrium to exist implies that both asset and option prices are too high for unemployed agents to purchase them. Agents of type eu, who have just lost their jobs, hold a financial portfolio at the beginning of the period and are wealthier than the uu. If the former do not participate in financial markets because prices are too high, the latter will also be excluded. The following two inequalities, one for each security market, ensure that no unemployed agents trade any securities:

$$P_k\, u'(\delta + (P_k + y_k)\, x^i + (P_k - K)^+\, s^i) > \beta \sum_{j=G,B} \pi_{k,j} \left(u'(\delta) + \rho^i\, (1 - u'(\delta))\right) (P_j + y_j) \tag{25}$$

$$Q_k\, u'(\delta + (P_k + y_k)\, x^i + (P_k - K)^+\, s^i) > \beta \pi_{k,G} \left(u'(\delta) + \rho^i\, (1 - u'(\delta))\right) (P_G - K) \tag{26}$$

Using the expressions for the prices P and Q, we deduce two conditions from (25) and (26) which are very similar to (9) for the equilibrium without options. Consequently, unemployed agents will neither trade options, nor assets, as long as we remain in the vicinity of the equilibrium with small asset volumes and without heterogeneity or aggregate uncertainty.

We can therefore state a similar proposition to Proposition 1 in the previous section.

Proposition 4 (Equilibrium existence) *We suppose that condition (11) holds. A unique equilibrium with limited heterogeneity and option trading exists as long as the tree size V is not too large and heterogeneity and aggregate uncertainty remain limited.*

The proof strategy is exactly the same as in the no-option economy, and the formal proof is set out in the Appendix.

4.3 Interactions between aggregate uncertainty and heterogeneity

We now analyze the interaction between uncertainty and heterogeneity in two dimensions. We first discuss portfolio composition, and then how these holdings, as well as security prices, are affected by aggregate uncertainty. In the next section, we focus on the welfare consequences of the introduction of options.

We suppose that Assumption D still holds. The greater the asset quantity an agent holds, the more he will benefit from higher asset prices when falling into unemployment. The key equations determining agents' holdings are the two market-clearing conditions (21) and (22), as well as equations (23) and (24), which ensure that both agent types trade positive quantities of both securities. More precisely, equation (24) shows how agents choose their asset quantity in order to hedge themselves against unemployment in the bad state of the world. Equation (23) describes how agents choose their option quantity, so as to adjust their insurance against unemployment in the good state of the world. In a nutshell, the asset matters for hedging in the bad state, while the option matters in the good state.

The following proposition sums up the results regarding portfolio composition.

Proposition 5 (Agents' portfolios) *In the economy with a call option, agents' portfolios exhibit the following features:*

(i) *Asset and option quantities do not depend on the state of the world and are denoted respectively x^i and s^i for type $i = 1, 2$ agents.*

(ii) *Type-1 agents, who face a greater risk of unemployment, choose to hold a greater quantity of assets than do type-2 agents, $x^1 > x^2 > 0$, and the former agents sell options to the latter: $s^1 < 0 < s^2$.*

The intuition for the first result is the following. Since the option only pays off in the good state of the world, both agents need to value the option analogously when falling into unemployment in the good state of the world. As a result, the expected cost of unemployment for both agent types has to be the same in the good state of the world. The asset pays off in both states of the world but due to the preceding remark, agents need to value the asset similarly in the bad state of the world: the expected cost of unemployment in the bad state also has to be identical for both agent types. In addition to the market-clearing conditions, we can show that these equalities between the expected costs of unemployment in each state of the world imply that portfolio-liquidation values do not depend on the state of the world, and nor do the security quantities. This is the consequence of the complete-market assumption.

The intuition for the second result in Proposition 5 is the following. Type-1 agents, who are the more likely to fall into unemployment, purchase a greater quantity of assets to self-insure against the risk of falling into unemployment in the bad state of the world. Without options, type-1 agents have insured against unemployment in the bad state of the world via the purchase of assets. In the good state of the world, the asset becomes a better insurance device (because of its higher sale price and higher dividend), which benefits type-1 agents more who hold a greater quantity of assets (from Assumption D). Type-1 agents hence hold portfolios that pay off too much in the good state of the world: They therefore sell options, in order to reduce their portfolio liquidation value in the good state of the world. On the other side, type-2 agents purchase a smaller quantity of assets and are not sufficiently insured against unemployment in the good state of the world (Assumption D again). The type-2 agent additionally needs to purchase options to improve his hedging in the good state of the world, and to reduce the expected cost of unemployment in that state of the world. It is worth noting that type-2 agents purchase a positive quantity of assets, at least when the heterogeneity is not too large.[3] Both agent types therefore optimally choose not to "issue" tree shares, which are the unique insurance against unemployment in the bad state.

From a financial point of view, type-1 agents choose a sort of delta hedging strategy in equilibrium, in the sense that they optimally choose a portfolio composition which is less affected by variations in the underlying asset price than is their portfolio without options.

This result can be compared with the results in Franke, Stapleton, and Subrahmanyam (1998), who prove that *agents with little or no uninsurable risk* have a concave sharing rule, which means that they *sell call options* to high-risk agents in the good state of the world. We find in our setup that in fact *low-risk agents buy call options* and have a convex sharing rule. The difference in the results stems from the difference in the nature of the heterogeneity in the background risk. While

[3]More precisely, we must have at the uncertain equilibrium: $\alpha^1 u'(\delta + (P+y)V) < \alpha^2 u'(\delta)$, which holds as long as heterogeneity remains limited.

we consider that agents face different probabilities of falling into unemployment and earning the same amount δ, Franke, Stapleton, and Subrahmanyam suppose that agents face a background risk with the same probability distributions but different magnitudes. In our setup, this would imply that the I agents types face the same probability α of leaving employment, but differs in the levels of home production δ^i $(1 \leq i \leq I)$. The major consequence is that in that case all agents' types have equal marginal utilities when unemployed in each state of the world, while this is not the case in our economy. Agents can only equalize their expected cost of falling into unemployment, which implies that our sharing rule does not only depend on agents' behavior relative to prudence and risk aversion. Using the HARA utility specification as in Franke, Stapleton, and Subrahmanyam (1998) would therefore not modify our sharing rules.

We now explain how aggregate uncertainty may affect security holdings and prices. This is the main theoretical result concerning the volume of options over the business cycle.

Proposition 6 (Aggregate uncertainty effects) *In the vicinity of the uncertain equilibrium ($y_G = y_B$), an increase in aggregate uncertainty has the following consequences:*

(i) *The asset price rises in the good state and falls in the bad state, while the average price remains unchanged.*

(ii) *The option price increases in both states, but more in the good than the bad state.*

(iii) *The more volatile are dividends, the larger is the option traded volume.*

The intuition for these results is the following. We consider a mean-preserving spread in dividends. In that case, aggregate uncertainty increases: the bad state becomes relatively worse, while the good state becomes better. As a result, the quality of securities as an insurance device changes. While the quality of assets in the bad state of the world deteriorates (due to aggregate state persistence, these will pay off badly with a higher probability), the quality of assets in the good state of the world and the quality of options in both states increase. The spread between prices in both states increases with uncertainty, even though the average price of the asset does not change (result *(i)*). The option price increases in both states, but more in the good than in the bad state. As the price of the underlying asset increases in the good state of the world, the option payoff, which is $P_G - K$ increases in the good state (recall that the option pays off 0 in the bad state of the world). As a consequence, the price of the option increases. Moreover, as aggregate states are persistent, this increase is higher in the good state of the world (result *(ii)*).

This mean spread of dividends also impacts on portfolio compositions. Due to a smaller dividend in the bad state, type-1 agents, who are more likely to fall into unemployment, would like to purchase a greater quantity of assets to obtain sufficient hedging. On the other hand, a rise in the dividend in the good state of the world provides them with an incentive to buy fewer assets. The overall effect remains positive since their first concern is insurance in the bad state of the world: more volatile dividends yield greater asset purchases by type-1 agents. These type-1 agents consequently sell a greater quantity of options in order to reduce their insurance in the good state of the world, while type-2 agents purchase these options in order to increase their hedging in the good state of the world.

More volatile dividends thus yield more volatile asset prices, more volatile (and on average higher) option prices, and a greater quantity of traded options (result *(iii)*). Our model is therefore able to reproduce the stylized fact that the open interest of options, i.e. the number of open contracts, which is similar to $|s^1|$ in our economy, rises as asset prices become more volatile. This fact is notably reported in Buraschi and Jiltsov (2006).

The degree of market incompleteness also has consequences for portfolio composition, as was the case in the no-option economy. The results are given in the next proposition.

Proposition 7 (Heterogeneity effects) *In the vicinity of the symmetric equilibrium ($\alpha^1 = \alpha^2$), greater unemployment risk for type-1 agents (i.e. higher α^1) has the following consequences:*

(i) *The prices of both securities increase in both states of the world, but more in the good than in the bad state.*

(ii) *Type-*1 *agents purchase more assets and sell more options.*

Greater heterogeneity therefore raises the volume of traded options.

We now consider an increase in the unemployment risk for type-1 agents (i.e. higher α^1) in the vicinity of the symmetric equilibrium ($\alpha^1 = \alpha^2$). Type-1 agents experience a greater probability of falling into unemployment, which increases their expected cost of unemployment. Agents therefore express a greater demand for self-insurance, which yields higher prices for both securities. The price increase is greater in the good state of the world, when both securities are a better hedge against unemployment, due to aggregate state persistence. To sum up, a greater risk of unemployment increases the average price of securities, but also price dispersion (result *(i)*). The increase in the demand for insurance by type-1 agents translates into a greater demand for the asset for insurance in the bad state of the world and a greater quantity of options sold for insurance in the good state of the world, as the option only provides greater insurance in the good state of the world.

As estimated in Storesletten, Telmer and Yaron (2004), the idiosyncratic risk should in fact be both highly persistent and countercyclical. Effects we prove in Propositions 6 and 7 therefore add to each other and reinforce the fact that options are more traded in downturns.

4.4 The effects of option introduction

We analyze the effects of option introduction, starting from an economy without aggregate risk, and progressively increase in risk via a mean-preserving spread in dividends. Introducing the option changes portfolio payoffs ex post in each state of the world. In the economy without options, agents are only able to buy one single asset to hedge themselves against the risk of unemployment in an "average" state of the world. With the option completing the market, type-1 agents facing a greater idiosyncratic risk insure themselves more in the bad state of the world and less in the good state of the world, where their unemployment cost falls more than that of type-2 agents because they hold a greater quantity of assets. As a result, in the economy with options, their portfolio will pay off more in the bad state of the world, but less in the good state. They are better able to smooth their portfolio payoffs across states of the world. It should also be noted that expected portfolio payoffs (tomorrow, contingent in being in a given state today) are similar in both economies. In fact, the asset quantities in the economy without options are chosen so as to replicate these expected payoffs.

We now turn to the impact of the introduction of options on ex ante welfare. We again analyze the impact in the vicinity of the equilibrium without aggregate shocks.

Proposition 8 (Impact of options on welfare) *The introduction of options redistributes welfare from agents facing a lower risk of unemployment to agents facing a higher risk.*

In the first order, welfare is affected through (i) the change in portfolio composition, keeping prices the same as those without the aggregate shocks, and through (ii) changes in prices, keeping quantities the same as those without aggregate shocks.

Since asset price is not affected in the first order by the introduction of options, the ex-ante welfare remains unchanged in the vicinity of the equilibrium without options. Therefore, only the quantity channel matters for the effect of option introduction on the welfare. Since both assets are in fixed supply, the portfolio composition of both agent types change in opposite directions, and so do ex-post welfares. One type of agents benefits from option introduction, while the other suffers from it. Agents who are more likely to fall into unemployment can now better self-insure with options: their portfolio pays off more in the bad state, but less in the good state. They are better able to smooth out their portfolio liquidation values. As such, they benefit from option introduction, which provides them with better self-insurance. On the contrary, the other agent type suffers from option introduction, which reduces their insurance ex post. Option introduction produces the redistribution of welfare from agents facing a lower risk of unemployment to those facing a higher risk.

Options are therefore not Pareto-improving, which illustrates Elul (1995) and Cass and Citanna (1998), who find that the introduction of a new asset is not always Pareto-improving. In our case, the result stems from changes in the volatility of agents' portfolios.[4]

We have described in detail the economy with two agent types and two aggregate states. We now generalize our equilibrium to an economy with an arbitrarily large number of agent types, aggregate states, and call options.

5 Generalization

We have so far restricted our attention to a simple economy where options are traded. This was in fact the simplest economy, in which options can be traded: the aggregate shock takes on only two values and heterogeneity is limited to two agent types. The introduction of a single option completes the market for the aggregate risk. The simplicity and the tractability of pricing equations in this framework allowed us to derive analytical results regarding the interaction of aggregate uncertainty and heterogeneity, and its impact on security prices and agents' portfolios.

We can however study equilibria in a more general setup, where options are traded. This allows us to derive pricing equations of derivatives varying along the business-cycle. A simple numerical example is then provided along these lines. We extend the previous economy in three dimensions: general number of aggregate states, general number of agents of different types and time-varying transition probabilities.

Aggregate states and dividends. First, we suppose that there are $n \geq 1$ different values for the aggregate state. Dividends will thus take values denoted $\{y_1, \ldots, y_n\}$. The dividend again evolves as a first-order Markov chain, and the probability of transition from state k to state l ($1 \leq k, l \leq n$) is denoted by $\pi_{k,l}$, with obviously $\sum_{l=1}^{n} \pi_{k,l} = 1$.

Agents' types and transition probabilities. Second, we suppose that there are $I \geq 2$ types of agents. There is a unit mass of each type of worker $1 \leq i \leq I$. Each agent type faces different unemployment risk though different transition probabilities between employment and unemployment, which additionally vary along the business cycle. We note α_k^i, with $i = 1, \ldots, I$, $k = 1 \ldots n$ the beginning-of-period probability for a type i worker to lose his job and fall into unemployment, when the current aggregate state is k. The number of type-i employed agents in the state of the world k is supposed to be a free parameter of the economy and is denoted η_k^i. This assumption regarding the number of employed agents rules out transitory dynamics and implies that both un–

[4]The aggregate ex-ante welfare would be impacted when departing from the vicinity of the equilibrium without aggregate shocks. However, as the ex-post welfare is a continuous functions of model parameters, a reasonable departure from that equilibrium would not modify our result stating that option introduction is not Pareto improving.

and employment rates jump instantaneously to their new values in each state of the world. As a consequence, simple flow accounting uniquely pins down the probability ρ^i_{sk} to leave unemployment for each type of agents, which depends on both the current and the past states of the world, respectively k and s in $\{1,\ldots,n\}$. The current η^i_k type-i employed agents in state k were either one of the η^i_s employed in the previous state s remained employed with probability α^i_k or one of the $1-\eta^i_s$ unemployed leaving unemployment with probability $1-\rho^i_{sk}$. More formally, the flow equality expresses as follows ($s,k=1,\ldots,n$ and $i=1,\ldots,I$):

$$\eta^i_k = \eta^i_s\left(1-\alpha^i_k\right) + \left(1-\rho^i_{sk}\right)\left(1-\eta^i_s\right) \tag{27}$$

Productivities. We finally suppose that the productivity w^i_k depends on both the agent's type $i=1,\ldots,I$ and the current aggregate state $k=1,\ldots,n$.

We now begin with analyzing the existence of a limited-heterogeneity equilibrium without options. We afterwards introduce $L\geq 1$ call options with one-period maturity and strike of K_l ($l=1,\ldots,L$), such that the market is not necessary complete for the aggregate risk after the introduction of options. We discuss in this setup the necessary conditions for options to be traded.

5.1 The economy without options

As in the simple two-state economy, we conjecture the existence of a limited-heterogeneity equilibrium, where employed agents of all types trade the asset, while unemployed agents are kept out of financial markets. For each type, we again suppose that there are four classes of agents, depending on past and current employment statuses.

The conjectured equilibrium is characterized by the $n\,I$ Euler equations of the employed agents (one per agent per aggregate state), the n market-clearing conditions (one in each state) and the $n\,I$ Euler inequalities which keep eu agents (and a fortiori the poorest uu) out of the financial market. In state $k=1,\ldots,n$, all employed agents of the same type $i=1,\ldots,I$ consume the same amount $u'(1/w^i_k)$ and purchase the same quantity of assets x^i_k. We denote by P_k the price of one unit of tree in state k. Unemployed agents consume their domestic production δ and possibly the sale value of their financial portfolio. The equations and inequalities characterizing the equilibrium are ($k,h=1,\ldots,n$ and $i=1,\ldots,I$):

$$P_k = \beta\sum_{j=1}^{n}\pi_{k,j}\frac{w^i_k}{w^i_j}\left(1+\alpha^i_j\left(w^i_j\,u'(\delta+(P_j+y_j)x^i_k)-1\right)\right)(P_j+y_j) \tag{28}$$

$$V = \sum_{i=1}^{I}\eta^i_k\,x^i_k \tag{29}$$

$$P_k\,u'(\delta+(P_k+y_k)x^i_h) > \beta\sum_{j=1}^{n}\pi_{k,j}\left(\rho^i_{kj}\frac{1}{w^i_j}+(1-\rho^i_{kj})u'(\delta)\right) \tag{30}$$

The Euler equations (28) states that employed agents of all types trade the asset, while Eq. (30) insure that unemployed ones are kept out of the financial market. Eq. (29) are the market equilibrium conditions.

We define as follows the limited-heterogeneity equilibrium in this framework.

Definition 2 *For given labor market conditions* $\{\eta_k^i, \alpha_k^i, \rho_{sk}^i, w_k^i\}_{s,k=1,\ldots,n,i=1,\ldots,I}$ *and asset properties* $\{V, y_k\}_{k=1,\ldots,n}$*, a limited heterogeneity equilibrium without options is a set of prices and quantities* $\{P_k, x_k^i\}_{k=1,\ldots,n,i=1,\ldots,I}$ *such that equalities (28)-(29) and inequalities (30) are fulfilled.*

The previous definition does not prove that a limited-heterogeneity equilibrium exists, even if the construction of such an equilibrium by continuity is actually straightforward. If we first assume that all agents are symmetrical and face constant labor market conditions: $\alpha_k^i = \alpha$, $\eta_k^i = \eta$, and $w_k^i = 1$, and that there is no aggregate risk $y_k = y$, we face back the conditions of Proposition 1, which insures us that a unique equilibrium exists as soon as condition (11) holds. Then increasing continuously the difference in aggregate outcome y_k and in labor market conditions $\alpha_k^i, \eta_k^i, \rho_{sk}^i$ one can construct limited-heterogeneity equilibria in this more complex setup.[5]

5.2 Economy with options

We now introduce L non-redundant call options maturing in one period into the previous economy. We proceed in three steps.

First, to introduce non-redundant options, we need to be sure that asset prices differ in some states of the world. To make this clear, we consider the economy without options, with no idiosyncratic shocks and with the dividend as the single source of aggregate risk. The price vector $P = (P_1, \ldots, P_n)$ is written in this case as a linear function of the dividend vector $y = (y_1, \ldots, y_n)$, such that $P = (I_n - \beta\, \pi)^{-1}\, y$, where $\pi = (\pi_{kl})_{1 \le k,l \le n}$ is the matrix of transition probabilities, and I_n the $n \times n$ identity matrix. As such, once we depart from the no aggregate shock equilibrium (while remaining in its vicinity), asset prices differ in at least two states of the world and are different from each other, except on a set of zero measure.[6]

Second, the continuity result in the no-option economy implies that prices remain different, even when we allow for idiosyncratic shocks and for another source of aggregate uncertainty, but where agents are homogeneous.

Finally, we are now able to introduce options, such that at most one strikes lies between two different asset prices. So, if there are h different prices indexed by k_i ranked, $P_{k_1} < \ldots < P_{k_h}$, it

[5]Using the implicit function theorem and at the cost of heavy algebra, one can prove that equilibrium prices and quantities are continuous functions of model parameters. The proof is similar to the ones in Sections A and D of the Appendix.

[6]More precisely, this set is the finite union of hyperplanes in $\mathbb{R}^n$.

is possible to introduce $h-1$ different calls, whose strikes are as follows $P_{k_1} < K_{k_1} < P_{k_2} < \ldots < P_{k_{h-1}} < K_{k_{h-1}} < P_{k_h}$. In this economy, options are priced, but not traded.

We now need to prove that introducing heterogeneity does not modify equilibrium existence. As in the no-option economy, the conjectured equilibrium is such that: (i) employed agents of all types participate in the $L+1$ financial markets, and (ii) unemployed agents do not trade any securities. In state $k = 1, \ldots, n$, an employed agent of type $i = 1, \ldots I$ purchases the quantity $s^i_{k,l}$ of options with strike K_l ($l = 1, \ldots, L$) at a price of $Q_{k,l}$.

With options, the equilibrium is characterized by $n\,I(L+1)$ Euler equations ($L+1$ financial securities priced in n states of the world where I agent types participate), $n(L+1)$ market-clearing conditions ($L+1$ markets) and $n\,I(L+1)$ inequalities characterizing the equilibrium ($k, h = 1, \ldots, n$; $m = 1, \ldots, L$ and $i = 1, \ldots, I$):

$$P_k = \beta \sum_{j=1}^{n} \pi_{k,j} \frac{w^i_k}{w^i_j} \left(1 + \alpha^i_j \left(w^i_j\, u'(\delta + (P_j + y_j)x^i_k + \sum_{l=1}^{L}(P_j - K_l)^+ s^i_{k,l}) - 1\right)\right)(P_j + y_j) \tag{31}$$

$$Q_{k,m} = \beta \sum_{j=1}^{n} \pi_{k,j} \frac{w^i_k}{w^i_j} \left(1 + \alpha^i_j \left(w^i_j\, u'(\delta + (P_j + y_j)x^i_k + \sum_{l=1}^{L}(P_j - K_l)^+ s^i_{k,l}) - 1\right)\right)(P_j - K_m)^+ \tag{32}$$

$$V = \sum_{i=1}^{I} \eta^i_k\, x^i_k \tag{33}$$

$$0 = \sum_{i=1}^{I} \eta^i_k\, s^i_{k,m} \tag{34}$$

$$P_k > \beta \frac{1}{u'(\delta + (P_k + y_k)x^i_h + \sum_{l=1}^{L}(P_k - K_l)^+ s^i_{h,l})} \sum_{j=1}^{n} \pi_{k,j} \left(\rho^i_{kj} \frac{1}{w^i_j} + (1 - \rho^i_{kj})u'(\delta)\right) \tag{35}$$

$$Q_{k,m} > \beta \frac{1}{u'(\delta + (P_k + y_k)x^i_h + \sum_{l=1}^{L}(P_k - K_l)^+ s^i_{h,l})} \sum_{j=1}^{n} \pi_{k,j} \left(\rho^i_{kj} \frac{1}{w^i_j} + (1 - \rho^i_{kj})u'(\delta)\right) \tag{36}$$

Equations (31) and (32) state that agents of all types trade both the asset and options with various strikes, while (35) and (36) insure that unemployed agents are kept out of financial markets. Equalities (33) and (34) are market equilibrium conditions for the asset and options.

The definition of the equilibrium in the economy with options is now the following.

Definition 3 *For given labor market conditions $\{\eta^i_k, \alpha^i_k, \rho^i_{sk}, w^i_k\}_{s,k=1,\ldots,n,i=1,\ldots,I}$ and asset properties $\{V, y_k\}_{k=1,\ldots,n}$, a limited heterogeneity equilibrium with options is a set of prices and quantities $\{P_k, Q_{k,m}, x^i_k, s^i_{k,m}\}_{k=1,\ldots,n,i=1,\ldots,I,m=1,\ldots,L}$ such that Euler equations (31)-(34) and Euler inequalities (35)-(36) are fulfilled.*

Once again, using the constructive strategy depicted in the previous section, one can easily exhibit limited-heterogeneity equilibria with options traded.

As it is defined by a finite number of equations, an equilibrium with incomplete markets for both aggregate and idiosyncratic risks can easily be simulated, as we do in the next section.

The other point of interest of this general case is to allow us to identify the conditions under which positive volumes of options are traded in our equilibrium. The first condition is that some agents in the economy face binding credit constraints and would like to self-insure. In the case of an unconstrained economy, security volumes are undetermined in equilibrium, and options do not play any particular role. The second condition is that agents face aggregate uncertainty in addition to their idiosyncratic risk, such that the market for the aggregate risk remains incomplete, at least before the introduction of options. Without uncertainty, the market is complete with a single asset and the redundant option is priced by portfolio replication, as in the seminal paper of Black and Scholes (1973). There is no role for option volumes in such a framework. The last condition is that agents have heterogeneous risk appetites, such that they are willing to exchange insurance with each other. Otherwise, agents hold similar portfolios and there is no role for options, which only transfer risk from one agent to another. Heterogeneity in risk appetites mau come either from different probabilities of falling into unemployment, as in the previous section, or from different productivity levels.

The following proposition summarizes this result.

Proposition 9 *Option volumes are undetermined in our economy when one of the following conditions hold:*

(i) *Unemployed agents do not face binding credit constraints.*

(ii) *The aggregate risk vanishes* $y_k = y_h$*, for all* $k, h = 1, \ldots, n$*.*

(iii) *The idiosyncratic risk is null* $\alpha_k^i = 0$ *for* $i = 1, \ldots, I$ *and* $k = 1, \ldots, n$*.*

(iv) *There is no heterogeneity among agents* $\alpha_k^i = \alpha_k^j$*, and* $w_k^i = w_k^j$*, for* $i, j = 1, \ldots, I$*, and* $k = 1, \ldots, n$*.*

6 Numerical example

As a simple numerical example without any quantitative ambitions, we calibrate the model in the case of two types of agents with an aggregate risk taking on only two values and with time-varying idiosyncratic risk. To make notations more transparent, we denote values concerning agents with a greater idiosyncratic risk by the superscript H, while we use the superscript L for the ones concerning the agents with a lower risk. This illustrative calibration exercise is carried out to show that the model can roughly match the variations over the business cycle of prices of both the underlying asset and the option and of the volume of options.

We start by defining the good and bad aggregate states. We use a Hamilton procedure to estimate a two-state Markov chain on the S&P 500 index variation at a monthly frequency during the period 1999-M1 to 2008-M12 for which we could obtain monthly data for options on the S&P 500 index.[7] We obtain two regimes, with four associated transition probabilities. The transition matrix between the two aggregate shocks is the following:

$$T = \begin{bmatrix} 0.9452 & 0.0548 \\ 0.2381 & 0.7619 \end{bmatrix}$$

In other words, the probability of staying in the good state is 0.95 when we are in the good state, whereas the probability of staying in the bad state when we are in the bad state is 0.76. The average duration of bad states is thus shorter than the duration of good states, which is a standard outcome in this type of estimation (Hamilton (1994)).

For the S&P 500, we compute in the good and bad states of the world the average level $\widetilde{P}$, the average dividend $\widetilde{y}$, the average option price $\widetilde{Q}$, and the average open interest $\widetilde{s}$ (after removing a deterministic trend). The subscript B denotes the bad state (low price and low dividend in the S&P 500), while G denotes the good one. As a consequence, $\widetilde{P}_B$ is the average deviation from trend of S&P 500 in the bad state of the world. The tilde above the variables indicates that these values are estimated from the data. We summarize the data by providing the ratio of variables in the bad and good states of the world. The results are as follows:

$\widetilde{P}_G/\widetilde{P}_B$	$\widetilde{y}_G/\widetilde{y}_B$	$\widetilde{Q}_G/\widetilde{Q}_B$	$\widetilde{s}_G/\widetilde{s}_B$
1.26	1.18	1.19	0.95

Table 1: Data summary

Table 1 shows that dividends and prices in the S&P 500 are both higher in the good state of the world, which is what defines a good state of the world in our economy. The price of the option is higher in the good state of the world and the volume of options traded is greater in the bad state of the world. This is not surprising, as the volatility of asset prices is higher in bad states.

We now provide a simple calibration to show that the model is able to reproduce the values given in the previous table. We first use the previous estimation for the transition matrix of the aggregate risk: $\pi = T$. Second, we assume that utility is CRRA, such that marginal utility is $u'(c) = c^{-\sigma}$ ($\sigma \geq 0$). The preference parameter σ is the relative risk aversion, β is the discount factor, δ is the income of the unemployed, and V is the volume of the risky asset which has to be small for the equilibrium to exist. In addition and for simplicity, we consider constant employment

[7] More specifically, we consider deviations from a deterministic trend of the S&P 500. Data on the S&P500 stem from Robert Shiller's website. Data on options are taken from the Chicago Board Options Exchange.

share for both types of agents $\eta^1 = \eta^2 = \eta$ (i.e. independent from the state of the world). Table 2 summarizes the calibration of constant parameters.

σ	β	δ	V	η
2	0.9	0.5	0.1	0.7

Table 2: Calibration of constant parameters

The constant parameters are set equal to reasonable values. The volume of assets V and the share of employed agents η are chosen such that the equilibrium conditions (35)-(36) are fulfilled. The values of the time-varying parameters is sumed up in Table 3.

	y	α^H (%)	α^L (%)	w	K
State B	0.0107	2.51	1.62	1	0.95
State G	0.0126	6.52	5.81	1	1.18

Table 3: Calibration of time-varying parameters

The dividend process values are deduced from the S&P 500 empirical estimation; the probabilities of falling into unemployment for both types of agents α^H and α^L are calibrated to match the ratios of interest. The wage of each worker when employed is assumed to be constant and equal to 1. Option strikes are arbitrarily chosen between the two prices of the risky asset in both states of the world, as we do not have the data on the traded volumes for each strike. We can check that assumptions A-D of the paper are satisfied in this economy. The result of this model is given in the table below, which also recalls data values:

	P_G/P_B	y_G/y_B	Q_G/Q_B	s_G^H/s_B^H
Model	1.24	1.18	1.19	0.95
Data	1.26	1.18	1.19	0.95

Table 4: Main model results

Table 4 shows that the model is roughly able to match the data, although the variance of the price of the asset is a little too high in this example. To obtain intuitions regarding equilibrium allocations, Table 5 highlights the equilibrium portfolio compositions.

First, high-risk agents sell the call option to low-risk agents ($s^L = -s^H > 0$). As a result, low-risk agents sell insurance relative to the aggregate risk. The high-risk agents hold a greater amount of risky asset in both states of the world compared to low-risk agents. The former also

	s^H	x^H	x^L
State B	-0.032	0.115	0.034
State G	-0.031	0.086	0.064

Table 5: Portfolio allocation derived by the model

hold a greater amount of assets in the bad state of the world because they care more about self-insurance in the bad state. These results illustrate the mechanisms that we described in the previous theoretical sections.

7 Conclusion

We present an equilibrium where infinitely-lived agents face both aggregate risk and uninsurable idiosyncratic risk. Due to heterogeneity in the exposure to idiosyncratic risk, agents assess the aggregate risk differently. This creates opportunities to exchange risk through derivative trading. The uninsurable risk is modeled as an unemployment risk, which allows us to obtain a tractable framework, as in Scheinkman and Weiss (1986) or Lagos and Wright (2005), among others.

We compute simple pricing equations for derivative assets in this framework, where they are not priced by replication. We also derive implications for the volume of derivatives exchanged over the business cycle, which are consistent with the data. We finally analyze the welfare implications of the introduction of derivatives and show that they are not Pareto-improving. We conclude with a simple numerical example. One direction for future research would be to find additional simplifying assumptions to confront our theory of derivative trading more directly with data.

Appendix

A Proof of Proposition 1

In order to formally prove the equilibrium existence, we first note that the equilibrium exists when the asset tree is null and when there is no heterogeneity and no aggregate risk (Section A.1). Then, we show using the implicit-function theorem that endogenous variables of the economy are continuous with respect to V, dividends y_k ($k = B, G$) and probabilities α^i, $i = 1, 2$ (Section A.2).

A.1 Symmetric economy

Zero volume. We consider a symmetric economy without aggregate risk: $y_B = y_G = y > 0$ and zero volume. The asset price P is given by $P = \frac{\beta\left(1+\alpha(u'(\delta)-1)\right)}{1-\beta\left(1+\alpha(u'(\delta)-1)\right)}$. The condition for the equilibrium price to be positive is $\beta\left(1+\alpha(u'(\delta)-1)\right) < 1$.

For unemployed agents not to take part into financial market, we need to have $P\,u'\,(\delta) > \beta\,(\rho + (1-\rho)\,u'(\delta))\,(P + y)$. Substituting for the price P, this condition is equivalent to $u'\,(\delta) > 1$ (Assumption B). It is easy to see that both conditions hold when condition (11) holds.

Positive supply. When the asset is in positive supply, $V > 0$, the price satisfies the relationship:

$$P - \beta\left(1 + \alpha(u'\left(\delta + (P+y)\,\frac{V}{\eta^1+\eta^2}\right) - 1)\right)(P+y) = 0 \tag{37}$$

The condition for this equilibrium to exist (i.e. excluding unemployed from financial markets) is:

$$P\,u'\left(\delta + (P+y)\,\frac{V}{\eta^1+\eta^2}\right) > \beta\,\left(\rho + (1-\rho)\,u'(\delta)\right)\,(P+y) \tag{38}$$

We express (37) as $G\,(P,V) = 0$, where G is continuous and differentiable in V. The derivative relative to P in $V = 0$ is $G_P\,(P,0) = 1 - \beta\,(1 + \alpha(u'\,(\delta) - 1)) > 0$. By the implicit-function theorem, (37) defines P as a continuous function of V around $V = 0$. As condition (38) is fulfilled at $V = 0$, there exists by continuity of P a neighborhood $W_1\,(0) \subset \mathbb{R}^+$ such that condition (38) holds. Define $V^* > 0$ as a point of $W_1\,(0)$ and P^* as the asset price for this volume. In this equilibrium, the quantity of assets held by each agent is $x^* = \frac{V^*}{\eta^1+\eta^2} > 0$.

A.2 General case

We define $X = (y_B, y_G, V, \alpha^1, \alpha^2) \in (\mathbb{R}^+)^5$ as the vector of parameters and $Z = (P_B, P_G, x_B^1, x_G^1, x_B^2, x_G^2) \in (\mathbb{R}^+)^6$ as the vector of endogenous variables. We define the vector of equations F of length 6 as ($i = 1,2$ and $k = 1,2$):

$$F_1\,(Z,X) = P_B - \beta \sum_{j=B,G} \pi_{B,j}\left(1 + \alpha^1\,u'\left(\delta + (P_j + y_j)\,x_B^1\right)\right)(P_j + y_j)$$

$$F_2\,(Z,X) = P_G - \beta \sum_{j=B,G} \pi_{G,j}\left(1 + \alpha^1\,u'\left(\delta + (P_j + y_j)\,x_G^1\right)\right)(P_j + y_j)$$

$$F_3\,(Z,X) = P_B - \beta \sum_{j=B,G} \pi_{B,j}\left(1 + \alpha^2\,u'\left(\delta + (P_j + y_j)\,x_B^2\right)\right)(P_j + y_j)$$

$$F_4\,(Z,X) = P_G - \beta \sum_{j=B,G} \pi_{G,j}\left(1 + \alpha^2\,u'\left(\delta + (P_j + y_j)\,x_G^2\right)\right)(P_j + y_j)$$

$$F_5\,(Z,X) = \eta^1\,x_B^1 + \eta^2\,x_B^2 - V \qquad F_6\,(Z,X) = \eta^1\,x_G^1 + \eta^2\,x_G^2 - V$$

The vector F stacks pricing functions for both agent types and the market equilibrium equation. For a given set of parameters X, the equilibrium is defined as $F\,(Z,X) = 0$.

- We know that there exists an equilibrium for $X^* = (y_B, y_G, V^*, \alpha^1, \alpha^2)$, where the unemployed do not trade the asset; this equilibrium is defined by $Z^* = (P^*, P^*, x^*, x^*, x^*, x^*)$.
- We now show that the Jacobian $\Delta = \left(\frac{\partial F_i}{\partial z_j}\,(Z^*, X^*)\right)_{i,j=1,\ldots,6}$ of F relative to Z is invertible.
  - In the vicinity of the symmetric equilibrium, the Jabobian Δ has this simple shape:

$$\Delta = \begin{bmatrix} A & C & 0_{2\times2} \\ A & 0_{2\times2} & C \\ 0_{2\times2} & 1_{2\times2} & 1_{2\times2} \end{bmatrix}$$

    $C = -\beta\,\alpha\,u''\,((P^*+y)\,x + \delta)\,(P^*+y)^2\,diag(1,1) > 0$ is a 2×2 diagonal matrix. A is a 2×2 matrix such that $A_{k,j} = 1_{k=j} - \beta\pi_{k,j}\,M$, with $M = 1 + \alpha(u'((P^*+y)\,x^* + \delta) - 1 + x^*u''((P^*+y)\,x + \delta)(P^*+y))$. The matrix A can be written as $A = I_2 - \beta\,M\,T$ (I_2 is the 2×2 identity matrix and $T = (\pi_{k,l})_{k,l=B,G}$) and is invertible. Condition (11) implies that $0 < \beta\,M < 1$. It is easy to check that the matrix $AA = w\left(\sum_{k=0}^{\infty}(\varphi T)^k\right)$ exists and satisfies $AA.A = I_2$.

– We now prove that Δ is invertible. Let $X = (X_0, X_1, X_2) \in (\mathbb{R})^6$. $X \in \ker\Delta$ implies that for $j = 1, 2$, $A\,X_0 + C\,X_j = 0_2$ and $X_1 + X_2 = 0_2$. Summing the first two equations over j yields $2AX_0 + C(X_1 + X_2) = 2AX_0 = 0_2$ and $X_0 = 0_2$ (A is invertible). Therefore, for $i = 1, 2$, $C\,X_i = 0_n$ and $X_i = 0_2$ (C is diagonal with strictly positive elements). We conclude that Δ is invertible.

– As the Jacobian of F with respect to Z is invertible at the point (Z^*, X^*), the implicit-function theorem proves that there exists a function $\widetilde{F}$ such that $Z = \widetilde{F}(X)$, for X close to X^*. In consequence, our equilibrium exists in the vicinity of (Z^*, X^*).

B Proof of Proposition 2

We study the evolution of prices and quantities around the riskless equilibrium $y_G = y_B = y$, where the asset price P, and asset holdings x^1 and x^2 are defined as:

$$P = \beta\left(1 + \alpha^i\left(u'\left(\delta + (P+y)\,x^i\right) - 1\right)\right)(P+y) \tag{39}$$

$$\text{with: } \alpha^1\ \left(u'\left(\delta + (P+y)\,x^1\right) - 1\right) = \alpha^2\ \left(u'\left(\delta + (P+y)\,x^2\right) - 1\right) \tag{40}$$

$$\text{and: } \eta^1\,x^1 + \eta^2\,x^2 = V \tag{41}$$

B.1 Quantities

We derive the market participation condition (10) relative to y_l ($l = B, G$) in the vicinity of the riskless equilibrium. After simplification using (40) and the derivative of (41), we get:

$$\eta^1\,(P+y)\frac{\partial x_k^1}{\partial y_l} = -\,\eta^2\,(P+y)\frac{\partial x_k^2}{\partial y_l} = -\,\kappa \sum_{j=B,G} \pi_{k,j}\left(\frac{\partial P_j}{\partial y_l} + 1_{j=l}\right) \tag{42}$$

$$\text{with}\quad \kappa = \frac{-\,x^1\,\frac{u''\left(\delta+(P+y)\,x^1\right)}{u'\left(\delta+(P+y)\,x^1\right)-1} + x^2\,\frac{u''\left(\delta+(P+y)\,x^2\right)}{u'\left(\delta+(P+y)\,x^2\right)-1}}{-\,\frac{1}{\eta^1}\,\frac{u''\left(\delta+(P+y)\,x^1\right)}{u'\left(\delta+(P+y)\,x^1\right)-1} - \frac{1}{\eta^2}\,\frac{u''\left(\delta+(P+y)\,x^2\right)}{u'\left(\delta+(P+y)\,x^2\right)-1}} > 0 \tag{43}$$

The function κ is supposed to be strictly positive (Assumption D). We can prove that it holds for standard utility classes (CARA, CRRA and quadratic).

B.2 Prices

We differentiate the price equation (12) with respect to y_l, $l = B, G$ in the vicinity of the riskless equilibrium.

$$\frac{\partial P_k}{\partial y_l} = \beta\,\widehat{M} \sum_{j=B,G} \pi_{k,j}\left(\frac{\partial P_j}{\partial y_l} + 1_{j=l}\right)$$

$$\widehat{M} = 1 + \alpha^i\left(u'\left(\delta + (P+y)\,x^i\right) - 1\right)\left(1 - \frac{V}{\eta^1\,\eta^2}\,\frac{u''\left(\delta + (P+y)\,x^1\right)}{u'\left(\delta + (P+y)\,x^1\right) - 1}\,\frac{u''\left(\delta + (P+y)\,x^2\right)}{u'\left(\delta + (P+y)\,x^2\right) - 1}\right) \tag{44}$$

$\widehat{M}$ is the modified pricing kernel for the valuation of one additional unit of dividend tomorrow.

After some manipulations, we deduce that:

$$\begin{bmatrix} \frac{\partial P_G}{\partial y_l} \\ \frac{\partial P_B}{\partial y_l} \end{bmatrix} = \frac{\beta\widehat{M}}{(1-\beta\widehat{M})(1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M})} \begin{bmatrix} \left(\pi_{GG} - \beta\widehat{M}(\pi_{GG}+\pi_{BB}-1)\right)1_{l=G} + (1-\pi_{GG})\,1_{l=B} \\ (1-\pi_{BB})\,1_{l=G} + \left(\pi_{BB} - \beta\widehat{M}(\pi_{GG}+\pi_{BB}-1)\right)1_{l=B} \end{bmatrix} > 0 \tag{45}$$

B.3 A useful lemma before going further

Lemma 10 *Let $\Phi : Y \to \mathbb{R}$ be a continuous and differentiable function depending on y_G and y_B. We denote by $V[y]$ ($E[y]$) the variance (mean) of the dividend process y. The impact of a mean-preserving spread of dividends can be written as:*

$$\left.\frac{\partial \Phi}{\partial V[y]}\right|_{E[y]=constant} = \frac{1}{2(y_G - y_B)} \left[\frac{2-\pi_{GG}-\pi_{BB}}{1-\pi_{BB}}\frac{\partial \Phi}{\partial y_G} - \frac{2-\pi_{GG}-\pi_{BB}}{1-\pi_{GG}}\frac{\partial \Phi}{\partial y_B}\right]$$

Proof. We write y_G and y_B as functions of $E[y]$ and $V[y]$ ($q = \frac{1-\pi_{BB}}{2-\pi_{GG}-\pi_{BB}}$):

$$y_G = E[y] + (1-q)\sqrt{\frac{V[y]}{q(1-q)}} \qquad \frac{\partial y_G}{\partial V[y]} = (1-q)\frac{1}{2\sqrt{q(1-q)V[y]}}$$

$$y_B = E[y] - q\sqrt{\frac{V[y]}{q(1-q)}} \qquad \frac{\partial y_B}{\partial V[y]} = -q\frac{1}{2\sqrt{q(1-q)V[y]}}$$

The derivative of Φ relative to $V[y]$ is:

$$\frac{\partial \Phi}{\partial V[y]} = \frac{\partial \Phi}{\partial y_G}\frac{\partial y_G}{\partial V[y]} + \frac{\partial \Phi}{\partial y_B}\frac{\partial y_B}{\partial V[y]} = \frac{2-\pi_{GG}-\pi_{BB}}{2(1-\pi_{GG})(1-\pi_{BB})(y_G-y_B)}\left[(1-\pi_{GG})\frac{\partial \Phi}{\partial y_G} - (1-\pi_{BB})\frac{\partial \Phi}{\partial y_B}\right]$$

■

B.4 Back to Proposition 2

Using the above results, we prove that:

$$k = B, G \quad \frac{2(y_G-y_B)}{2-\pi_{GG}-\pi_{BB}}\frac{\partial P_k}{\partial V[y]} = \frac{1-\pi_{kk}}{(1-\pi_{BB})(1-\pi_{GG})}\frac{(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}}{1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}} > 0$$

$$\eta^1\,(P+y)\frac{\partial x_k^1}{\partial V[y]} = -\,\eta^2\,(P+y)\frac{\partial x_k^2}{\partial V[y]} = -\,\frac{\kappa}{\beta\widehat{M}}\frac{\partial P_k}{\partial V[y]} < 0$$

These equations show that: $x_G^1 < x_B^1$, $x_G^2 > x_B^2$ and $P_G > P_B$.

We can also very easily prove that $\frac{\partial E[P]}{\partial V[y]} = \frac{\partial E[x^1]}{\partial V[y]} = \frac{\partial E[x^2]}{\partial V[y]} = 0$.

Finally, we prove the following results regarding the impact of the variance:

$$\frac{\partial V[P]}{\partial V[y]} = \frac{(1-\pi_{BB})(1-\pi_{GG})}{(2-\pi_{GG}-\pi_{BB})^2}2(y_G-y_B)\left(\frac{\partial P_G}{\partial V[y]} - \frac{\partial P_B}{\partial V[y]}\right)\frac{P_G-P_B}{y_G-y_B} = 2\left(\frac{(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}}{1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}}\right)^2$$

$$(P+y)^2\frac{\partial V[x^i]}{\partial V[y]} = \left(\frac{\kappa}{\eta^i\,\beta\,\widehat{M}}\right)^2\frac{\partial V[P]}{\partial V[y]} = 2\left(\frac{1}{\eta^i}\frac{(\pi_{GG}+\pi_{BB}-1)\kappa}{1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}}\right)^2$$

C Proof of proposition 3

We study the evolution of prices and quantities around the symmetric equilibrium $\alpha^1 = \alpha^2 = \alpha$, where asset prices P_G and P_B and asset holdings x_G and x_B are defined as ($k = B, G$):

$$P_k = \beta\sum_{j=B,G}\pi_{k,j}\left(1+\alpha\left(u'\left(\delta+(P_j+y_j)\,\frac{V}{\eta^1+\eta^2}\right)-1\right)\right)(P_j+y_j) \quad \text{and} \quad (\eta^1+\eta^2)\,x_k = V$$

C.1 Quantities

We differentiate (10) with respect to α^l ($l = 1, 2$) in the vicinity of the symmetric equilibrium.

$$\eta^1\,\frac{\partial x_k^1}{\partial \alpha^l} = -\,\eta^2\,\frac{\partial x_k^2}{\partial \alpha^l} = (1_{l=2}-1_{l=1})\frac{\eta^1\,\eta^2}{\eta^1+\eta^2}\frac{\sum_{j=B,G}\pi_{k,j}\left(u'\left(\delta+(P_j+y_j)\,\frac{V}{\eta^1+\eta^2}\right)-1\right)(P_j+y_j)}{\alpha\sum_{j=B,G}\pi_{k,j}\,u''\left(\delta+(P_j+y_j)\,\frac{V}{\eta^1+\eta^2}\right)(P_j+y_j)^2}$$

We deduce that $\frac{\partial x_k^1}{\partial \alpha^l} > 0$ ($u'' < 0$) for $k = B, G$. Moreover, we have after some algebra $\frac{\partial x_B^1}{\partial \alpha^l} > \frac{\partial x_G^1}{\partial \alpha^l}$ iff: $-\frac{u''\left(\delta+(P_G+y_G)\,\frac{V}{\eta^1+\eta^2}\right)}{u'\left(\delta+(P_G+y_G)\,\frac{V}{\eta^1+\eta^2}\right)-1}(P_G+y_G) > -\frac{u''\left(\delta+(P_B+y_B)\,\frac{V}{\eta^1+\eta^2}\right)}{u'\left(\delta+(P_B+y_B)\,\frac{V}{\eta^1+\eta^2}\right)-1}(P_B+y_B)$, which is identical to $\kappa > 0$, cf. (43).

C.2 Prices

We derive price expressions relative (12) to α^l $(l = 1, 2)$ in the vicinity of the symmetric equilibrium.

$$\left(1 - \beta\pi_{GG}M_G - \beta\pi_{BB}M_B + \beta^2(\pi_{GG} + \pi_{BB} - 1)M_G M_B\right) \begin{bmatrix} \frac{\partial P_G}{\partial \alpha^l} \\ \frac{\partial P_B}{\partial \alpha^l} \end{bmatrix} =$$

$$\beta \frac{\eta^1 1_{l=1} + \eta^2 1_{l=2}}{\eta^1 + \eta^2} \begin{bmatrix} (\pi_{GG} - \beta M_B(\pi_{GG} + \pi_{BB} - 1))\,\Delta_G + (1 - \pi_{GG})\,\Delta_B \\ (\pi_{BB} - \beta M_G(\pi_{GG} + \pi_{BB} - 1))\,\Delta_B + (1 - \pi_{BB})\,\Delta_G \end{bmatrix} > 0$$

$$\text{with: } M_j = 1 + \alpha \left(u'\left(\delta + (P_j + y_j)\frac{V}{\eta^1 + \eta^2}\right) - 1\right) \left(1 + \frac{(P_j + y_j)V}{\eta^1 + \eta^2} \frac{u''\left(\delta + (P_j + y_j)\frac{V}{\eta^1 + \eta^2}\right)}{u'\left(\delta + \frac{(P_j + y_j)V}{\eta^1 + \eta^2}\right) - 1}\right)$$

$$\Delta_j = \left(u'\left(\delta + (P_j + y_j)\frac{V}{\eta^1 + \eta^2}\right) - 1\right)(P_j + y_j) \qquad j = B, G$$

To prove the last result, note that the sign of $\frac{\partial P_G}{\partial \alpha^l} - \frac{\partial P_B}{\partial \alpha^l}$ is the sign of $(1 - \beta M_B)\,\Delta_G - (1 - \beta M_G)\,\Delta_B$, which is always positive. Indeed for $V \approx 0$, it collapses to $(1 - \beta(1 + \alpha(u'(\delta) - 1)))\,(u'(\delta) - 1)\,(P_G + y_G - P_B - y_B) > 0$.

D Proof of Proposition 4

This proof is similar to the one of Section A proving the equilibrium existence in the economy without options.

E Proof of Proposition 5

Since $\alpha^1 > \alpha^2$, (24) implies that $u'\left(\delta + (P_B + y_B)\,x^1\right) < u'\left(\delta + (P_B + y_B)\,x^2\right)$ and therefore that $x^1 > x^2$, since u' is decreasing. Moreover, $s^1 < 0$ if $\alpha^1(u'\left(\delta + (P_G + y_G)\,x^1\right) - 1) < \alpha^2(u'\left(\delta + (P_G + y_G)\,x^2\right) - 1)$ or if $\frac{u'\left(\delta + (P_G + y_G)\,x^1\right) - 1}{u'\left(\delta + (P_G + y_G)\,x^2\right) - 1} < \frac{u'\left(\delta + (P_B + y_B)\,x^1\right) - 1}{u'\left(\delta + (P_B + y_B)\,x^2\right) - 1}$. It holds if $\pi \mapsto \frac{u'\left(\delta + \pi\,x^1\right) - 1}{u'\left(\delta + \pi\,x^2\right) - 1}$ is decreasing, or $\kappa > 0$ (cf. (43)).

F Proof of Proposition 6

F.1 Quantities

Deriving (23) and (24) relative to y_l, $l = B, G$ in the vicinity of the symmetric equilibrium yields:

$$\eta^1\,(P + y)\frac{\partial x^1}{\partial y_l} = -\,\eta^2\,(P + y)\frac{\partial x^2}{\partial y_l} = -\kappa\left(\frac{\partial P_B}{\partial y_l} + 1_{l=B}\right)$$

$$\eta^1\,(P + y)\frac{\partial x^1}{\partial y_l} + \eta^1(P - K)\frac{\partial s^1}{\partial y_l} = -\,\eta^2\,(P + y)\frac{\partial x^1}{\partial y_l} - \eta^2(P - K)\frac{\partial s^2}{\partial y_l} = -\kappa\left(\frac{\partial P_G}{\partial y_l} + 1_{l=G}\right)$$

F.2 Prices

Differentiating (19) with respect to y_l in the vicinity of the symmetric equilibrium yields exactly the same expression as in the no-option economy (cf. (45)).

F.3 Back to Proposition 6

We derive (23) and (24) with respect to $V[y]$ and obtain after some manipulations:

$$\eta^1\,(P + y)\frac{\partial x^1}{\partial V[y]} = -\,\eta^2\,(P + y)\frac{\partial x^2}{\partial V[y]} = \kappa\,\frac{2 - \pi_{GG} - \pi_{BB}}{2(y_G - y_B)(1 - \pi_{GG})}\,\frac{1}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}} > 0$$

$$\eta^1\,(P - K)\frac{\partial s^1}{\partial V[y]} = -\,\kappa\,\frac{2 - \pi_{GG} - \pi_{BB}}{2(y_G - y_B)}\,\frac{1}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}}\,\frac{2 - \pi_{BB} - \pi_{GG}}{(1 - \pi_{BB})(1 - \pi_{GG})} < 0$$

The derivatives of the asset price in (19) relative to $V[y]$ express as ($l = B, G$):

$$\frac{2(y_G - y_B)}{2 - \pi_{GG} - \pi_{BB}} \frac{\partial P_l}{\partial V[y]} = (1_{l=G} - 1_{l=B}) \frac{1}{1 - \pi_{ll}} \frac{(\pi_{GG} + \pi_{BB} - 1)\beta \widehat{M}}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta \widehat{M}}$$

Finally we obtain $\frac{\partial Q_G}{\partial V[y]} > \frac{\partial Q_B}{\partial V[y]}$ from the derivatives of (20):

$$\frac{\partial Q_k}{\partial V[y]} = \pi_{k,G} \frac{2 - \pi_{GG} - \pi_{BB}}{2(y_G - y_B)} \frac{1}{1 - \pi_{BB}} \frac{\beta \widehat{M}}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta \widehat{M}} > 0$$

G Proof of Proposition 7

As in section C, we consider the evolution of prices and quantities around the symmetric equilibrium $\alpha^1 = \alpha^2 = \alpha$, where the asset prices are P_G and P_B and those of options are resp. Q_G and Q_B.

G.1 Quantities

Differentiating market participation conditions (23) and (24) yields:

$$\eta^1 (P_B + y_B) \frac{\partial x^1}{\partial \alpha^l} = (1_{l=2} - 1_{l=1}) \frac{\eta^1 \eta^2}{\eta^1 + \eta^2} \frac{u'\left(\delta + (P_B + y_B) \frac{V}{\eta^1 + \eta^2}\right) - 1}{\alpha\, u''\left(\delta + (P_B + y_B) \frac{V}{\eta^1 + \eta^2}\right)}$$

$$\eta^1 (P_G + y_G) \frac{\partial x^1}{\partial \alpha^l} + \eta^1 (P_G - K) \frac{\partial s^1}{\partial \alpha^l} = (1_{l=2} - 1_{l=1}) \frac{\eta^1 \eta^2}{\eta^1 + \eta^2} \frac{u'\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right) - 1}{\alpha\, u''\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right)}$$

We deduce that $\frac{\partial x^1}{\partial \alpha^1} > 0$ and $\frac{\partial s^1}{\partial \alpha^l} < 0$ iif $-\frac{u''\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right)}{u'\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right) - 1} > -\frac{u''\left(\delta + (P_B + y_B) \frac{V}{\eta^1 + \eta^2}\right)}{u'\left(\delta + (P_B + y_B) \frac{V}{\eta^1 + \eta^2}\right) - 1}$, which is the same condition as $\kappa > 0$ (cf. (43)).

G.2 Prices

We differentiate the expressions of both the asset and the option prices with respect to α^l, $l = B, G$.

$$\frac{\partial P_k}{\partial \alpha^l} = \beta \frac{\eta^1 1_{l=1} + \eta^2 1_{l=2}}{\eta^1 + \eta^2} \sum_{j=B,G} \pi_{k,j} \Delta_j + \beta \sum_{j=B,G} \pi_{k,j} M_j \frac{\partial P_j}{\partial \alpha^l}$$

with: $$M_j = 1 + \alpha \left(u'\left(\delta + (P_j + y_j) \frac{V}{\eta^1 + \eta^2}\right) - 1 \right) \left(1 + \frac{(P_j + y_j) V}{\eta^1 + \eta^2} \frac{u''\left(\delta + (P_j + y_j) \frac{V}{\eta^1 + \eta^2}\right)}{u'\left(\delta + \frac{(P_j + y_j) V}{\eta^1 + \eta^2}\right) - 1} \right) \qquad j = B, G$$

$$\Delta_j = \left(u'\left(\delta + (P_j + y_j) \frac{V}{\eta^1 + \eta^2}\right) - 1 \right) (P_j + y_j)$$
$$+ 1_{l=i} \beta \sum_{j=B,G} \pi_{k,j} \left(u'\left(\delta + (P_j + y_j) \frac{V}{\eta^1 + \eta^2}\right) - 1 \right) (P_j + y_j)$$

The calculation is very similar to that in the no-option economy, and we find analogously that:

$$(1 - \beta \pi_{GG} M_G - \beta \pi_{BB} M_B + \beta^2 (\pi_{GG} + \pi_{BB} - 1) M_G M_B) \begin{bmatrix} \frac{\partial P_G}{\partial \alpha^l} \\ \frac{\partial P_B}{\partial \alpha^l} \end{bmatrix} =$$
$$\beta \frac{\eta^1 1_{l=1} + \eta^2 1_{l=2}}{\eta^1 + \eta^2} \begin{bmatrix} (\pi_{GG} - \beta M_B (\pi_{GG} + \pi_{BB} - 1)) \Delta_G + (1 - \pi_{GG}) \Delta_B \\ (\pi_{BB} - \beta M_G (\pi_{GG} + \pi_{BB} - 1)) \Delta_B + (1 - \pi_{BB}) \Delta_G \end{bmatrix} > 0$$

We easily deduce that $\frac{\partial Q_G}{\partial \alpha^l} > \frac{\partial Q_B}{\partial \alpha^l} > 0$ from the following derivatives:

$$\frac{\partial Q_k}{\partial \alpha^l} = \beta \pi_{k,G} \left(1 + \alpha \left(u'\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right) - 1 \right) \left(1 + \frac{(P_G - K) \frac{V}{\eta^1 + \eta^2} u''\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right)}{u'\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right) - 1} \right) \right) \frac{\partial P_G}{\partial \alpha^l}$$
$$+ \beta \frac{\eta^1 1_{l=1} + \eta^2 1_{l=2}}{\eta^1 + \eta^2} \pi_{k,G} \left(u'\left(\delta + (P_G + y_G) \frac{V}{\eta^1 + \eta^2}\right) - 1 \right) (P_G - K)$$

H Proof of Proposition 8

H.1 General case

We deduce the instantaneous utilities which we denote by $u^{\star,i}_{k,j}$, where $\star$ is the agent's class, i his type, and $k,l = B,G$ the aggregate states, using notably budget constraints:

$$\begin{aligned}
u^{ee,i}_{k,j} &= u\left(u'^{-1}(1)\right) - u'^{-1}(1) - Q_j s^i_j + (P_j - K)^+ s^i_k + (P_j + y_j)\, x^i_k - P_j\, x^i_j \\
u^{ue,i}_{k,j} &= u\left(u'^{-1}(1)\right) - u'^{-1}(1) - Q_j\, s^i_j - P_j\, x^i_j \\
u^{eu,i}_{k,j} &= u\left(\delta + (P_j + y_j)\, x^i_k + (P_j - K)^+\, s^i_k\right) \\
u^{uu,i}_{k,j} &= u\,(\delta)
\end{aligned}$$

Denoting $u = u\left(u'^{-1}(1)\right) - u'^{-1}(1)$, the instantaneous utility vector $\mathcal{U}^i$ of an agent i is written as $\mathcal{U}^i = \left[u^{ee,i}_{G,G}, u^{ee,i}_{B,G}, u^{ee,i}_{G,B}, u^{ee,i}_{B,B}, u^{ue,i}_{G,G}, \ldots, u^{eu,i}_{G,G}, \ldots, u^{uu,i}_{G,G}, \ldots, u^{uu,i}_{B,B}\right]^\top$ or:

$$\mathcal{U}^i = \begin{bmatrix}
u - (Q_G - (P_G - K))\, s^i_G + y_G\, x^i_G \\
u - \left(Q_G\, s^i_G - (P_G - K)\, s^i_B\right) + (P_G + y_G)\, x^i_B - P_G\, x^i_G \\
u - Q_B\, s^i_B + (P_B + y_B)\, x^i_G - P_B\, x^i_B \\
u - Q_B\, s^i_B + y_B\, x^i_B \\
u - Q_G\, s^i_G - P_G\, x^i_G \\
u - Q_G\, s^i_G - P_G\, x^i_G \\
u - Q_B\, s^i_B - P_B\, x^i_B \\
u - Q_B\, s^i_B - P_B\, x^i_B \\
u\left(\delta + (P_G + y_G)\, x^i_G + (P_G - K)\, s^i_G\right) \\
u\left(\delta + (P_G + y_G)\, x^i_B + (P_G - K)\, s^i_B\right) \\
u\left(\delta + (P_B + y_B)\, x^i_G\right) \\
u\left(\delta + (P_B + y_B)\, x^i_B\right) \\
u\,(\delta)\; 1_4
\end{bmatrix} \tag{46}$$

In order to calculate intertemporal utility, we need an expression for the transition matrix M^i, depending on the agent's type. The matrix can be written as $M^i = \Omega^i \otimes \Lambda$, with:

$$\Omega^i = \begin{bmatrix}
\alpha^i & 0 & 1-\alpha^i & 0 \\
\alpha^i & 0 & 1-\alpha^i & 0 \\
0 & 1-\rho^i & 0 & \rho^i \\
0 & 1-\rho^i & 0 & \rho^i
\end{bmatrix} = P^i\, D^i\, (P^i)^{-1}, \text{ with } P^i = \begin{bmatrix}
1 & 1-\alpha^i & 0 & 1-\alpha^i \\
1 & 0 & \rho^i & 1-\alpha^i \\
1 & -\alpha^i & 0 & -(1-\rho^i) \\
1 & 0 & -(1-\rho^i) & -(1-\rho^i)
\end{bmatrix}$$

and $D^i = \text{Diag}(1\ 0\ 0\ \alpha^i + \rho^i - 1)$

$$\Lambda = \begin{bmatrix}
\pi_{GG} & 0 & 1-\pi_{GG} & 0 \\
\pi_{GG} & 0 & 1-\pi_{GG} & 0 \\
0 & 1-\pi_{BB} & 0 & \pi_{BB} \\
0 & 1-\pi_{BB} & 0 & \pi_{BB}
\end{bmatrix} = Q\,\Delta\, Q^{-1}, \text{ with } Q = \begin{bmatrix}
1 & 1-\pi_{GG} & 0 & 1-\pi_{GG} \\
1 & 0 & \pi_{BB} & 1-\pi_{GG} \\
1 & -\pi_{GG} & 0 & -(1-\pi_{BB}) \\
1 & 0 & -(1-\pi_{BB}) & -(1-\pi_{BB})
\end{bmatrix}$$

and $\Delta = \text{Diag}(1\ 0\ 0\ \pi_{GG} + \pi_{BB} - 1)$

The ex-ante welfare U^i of a type i-agent is equal to the intertemporal sum of instantaneous utilities:

$$\begin{aligned}
U^i &= \sum_{k=0}^{\infty} \beta^k\, (M^i)^k\, \mathcal{U}^i = \left[\sum_{k=0}^{\infty} \beta^k\, (\Omega^i \otimes T)^k\right] \mathcal{U}^i = \left[\sum_{k=0}^{\infty} \beta^k\, (\Omega^i)^k \otimes T^k\right] \mathcal{U}^i = \left[\sum_{k=0}^{\infty} \beta^k\, (P^i\, (D^i)^k\, (P^i)^{-1}) \otimes (Q\,\Delta^k\, Q^{-1})\right] \mathcal{U}^i \\
&= \left[\sum_{k=0}^{\infty} \beta^k\ (P^i \otimes Q)\left((D^i)^k \otimes \Delta^k\right)\left((P^i)^{-1} \otimes Q^{-1}\right)\right] \mathcal{U}^i = (P^i \otimes Q)\ \left[\sum_{k=0}^{\infty} \beta^k\ \left((D^i)^k \otimes \Delta^k\right)\right]\ \left((P^i)^{-1} \otimes Q^{-1}\right) \mathcal{U}^i
\end{aligned}$$

We denote by U_a^i and U_o^i the ex-ante welfare levels in the economy respectively without and with options. Analogously $\mathcal{U}_a^i$ and $\mathcal{U}_o^i$ are the instantaneous utility expressions. We compute the derivatives $\frac{\partial \mathcal{U}_o^i}{\partial y_1}$ and $\frac{\partial \mathcal{U}_a^i}{\partial y_1}$ in the vicinity of the equilibrium without uncertainty, where the option is not traded and both levels of welfare are equal to each other.

H.2 In the economy without option

The utility vector (46) and the derivative of $\mathcal{U}_a^1$ relative to y_1 are written as follows without options:

$$\mathcal{U}_a^i = \begin{bmatrix} u + y_G\, x_G^i \\ u - P_G(x_G^i - x_B^i) + y_G\, x_B^i \\ u - P_B(x_B^i - x_G^i) + y_B\, x_G^i \\ u + y_B\, x_B^i \\ u - P_G\, x_G^i \\ u - P_G\, x_G^i \\ u - P_B\, x_B^i \\ u - P_B\, x_B^i \\ u\left(\delta + (P_G + y_G)\, x_G^i\right) \\ u\left(\delta + (P_G + y_G)\, x_B^i\right) \\ u\left(\delta + (P_B + y_B)\, x_G^i\right) \\ u\left(\delta + (P_B + y_B)\, x_B^i\right) \\ u\,(\delta)\ 1_4 \end{bmatrix} \qquad \frac{\partial \mathcal{U}_a^1}{\partial y_G} = \begin{bmatrix} y\,\frac{\partial x_G^1}{\partial y_G} + x \\ -\,P\,\frac{\partial (x_G^1 - x_B^1)}{\partial y_G} + y\,\frac{\partial x_B^1}{\partial y_G} + x \\ P\,\frac{\partial (x_G^1 - x_B^1)}{\partial y_G} + y\,\frac{\partial x_G^1}{\partial y_G} \\ y\,\frac{\partial x_B^1}{\partial y_G} \\ -\frac{\partial P_G}{\partial y_G}\,x - P\,\frac{\partial x_G^1}{\partial y_G} \\ -\frac{\partial P_G}{\partial y_G}\,x - P\,\frac{\partial x_G^1}{\partial y_G} \\ -\frac{\partial P_B}{\partial y_G}\,x - P\,\frac{\partial x_B^1}{\partial y_G} \\ -\frac{\partial P_B}{\partial y_G}\,x - P\,\frac{\partial x_B^1}{\partial y_G} \\ u'_{x^1}\left[(1 + \frac{\partial P_G}{\partial y_G})\,x + (P+y)\,\frac{\partial x_G^1}{\partial y_G}\right] \\ u'_{x^1}\left[(1 + \frac{\partial P_G}{\partial y_G})\,x + (P+y)\,\frac{\partial x_B^1}{\partial y_G}\right] \\ u'_{x^1}\left[\frac{\partial P_B}{\partial y_G}\,x + (P+y)\,\frac{\partial x_G^1}{\partial y_G}\right] \\ u'_{x^1}\left[\frac{\partial P_B}{\partial y_G}\,x + (P+y)\,\frac{\partial x_B^1}{\partial y_G}\right] \\ 0_4 \end{bmatrix}$$

H.3 In the economy with an option

Since portfolios are the same in both states of the world, we have $s_G^i = s_B^i = s^i$ and $x_G^i = x_B^i = x^i$:

$$\mathcal{U}_o^i = \begin{bmatrix} u - (Q_G - (P_G - K))\ s^i + y_G\, x^i \\ u - Q_B\, s^i + y_B\, x^i \\ u - Q_G\, s^i - P_G\, x^i \\ u - Q_B\, s^i - P_B\, x^i \\ u\left(\delta + (P_G + y_G)\, x^i + (P_G - K)\, s^i\right) \\ u\left(\delta + (P_B + y_B)\, x^i\right) \\ u\,(\delta)\ 1_2 \end{bmatrix} \otimes 1_2 \quad \frac{\partial \mathcal{U}_o^1}{\partial y_G} = \begin{bmatrix} -\,(Q - (P - K))\,\frac{\partial s^1}{\partial y_G} + x + y\,\frac{\partial x^1}{\partial y_G} \\ -Q\,\frac{\partial s^1}{\partial y_G} + y\,\frac{\partial x^1}{\partial y_G} \\ -Q\,\frac{\partial s^1}{\partial y_G} - P\,\frac{\partial x^1}{\partial y_G} - \frac{\partial P_G}{\partial y_G}\,x \\ -Q\,\frac{\partial s^1}{\partial y_G} - P\,\frac{\partial x^1}{\partial y_G} - \frac{\partial P_B}{\partial y_G}\,x \\ u'_{x^1}\left[(1 + \frac{\partial P_G}{\partial y_G})x + (P+y)\,\frac{\partial x^1}{\partial y_G} + (P-K)\,\frac{\partial s^1}{\partial y_G}\right] \\ u'_{x^1}\left[\frac{\partial P_B}{\partial y_G}\,x + (P+y)\,\frac{\partial x^1}{\partial y_G}\right] \\ 0_2 \end{bmatrix} \otimes 1_2$$

H.3.1 Effects of option introduction

Using pricing equations, we define $0 < \gamma < 1$ (which does not depend on the agent type) as follows:

$$\gamma = \frac{P}{P+y} = \frac{Q}{P-K} = \beta\left(1 + \alpha^1(u'(\delta + (P+y)x^i) - 1)\right) = \beta\left(1 + \alpha^2(u'(\delta + (P+y)x^i) - 1)\right)$$

The difference of welfares between both economies (without and with option) expresses as:

$$\eta^1 \frac{\partial \mathcal{U}_o^1}{\partial y_G} - \eta^1 \frac{\partial \mathcal{U}_a^1}{\partial y_G} = -\left(\eta^2 \frac{\partial \mathcal{U}_o^2}{\partial y_G} - \eta^2 \frac{\partial \mathcal{U}_a^2}{\partial y_G}\right) = \frac{\kappa}{1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M}} \begin{bmatrix} -(1-\gamma)(1-\pi_{GG}) \\ -(1-\gamma)\pi_{BB} - \gamma(\pi_{GG}+\pi_{BB}-1) \\ \gamma\,(\pi_{GG}+\pi_{BB}) + (1-\gamma)\pi_{GG} \\ \gamma + (1-\gamma)(1-\pi_{BB}) \\ \gamma\,(1-\pi_{GG})\,1_2 \\ \gamma\,\pi_{BB}\,1_2 \\ -\,u'_{x^1}\,(1-\pi_{GG}) \\ -\,u'_{x^1}\,\pi_{BB} \\ u'_{x^1}\,\pi_{GG} \\ u'_{x^1}\,(1-\pi_{BB}) \\ 0_4 \end{bmatrix}$$

To compute ex ante utility, we multiply the above by the vector of weights, which equals:

$$\begin{aligned} W^i = & \left[\frac{\alpha^i\,(1-\rho^i)}{2-\alpha^i-\rho^i},\ \frac{(1-\alpha^i)\,(1-\rho^i)}{2-\alpha^i-\rho^i},\ \frac{(1-\alpha^i)\,(1-\rho^i)}{2-\alpha^i-\rho^i},\ \frac{(1-\alpha^i)\,\rho^i}{2-\alpha^i-\rho^i}\right] \\ & \otimes \left[\frac{\pi_{GG}\,(1-\pi_{BB})}{2-\pi_{GG}-\pi_{BB}},\ \frac{(1-\pi_{GG})\,(1-\pi_{BB})}{2-\pi_{GG}-\pi_{BB}},\ \frac{(1-\pi_{GG})\,(1-\pi_{BB})}{2-\pi_{GG}-\pi_{BB}},\ \frac{(1-\pi_{GG})\,\pi_{BB}}{2-\pi_{GG}-\pi_{BB}}\right] \end{aligned}$$

The difference in the change in ex ante utilities $\frac{\partial U_o^i}{\partial y_G} - \frac{\partial U_a^i}{\partial y_G}$ for the agent $i = 1, 2$ equals:

$$\frac{\partial U_o^1}{\partial y_G} - \frac{\partial U_a^1}{\partial y_G} = -\left(\frac{\partial U_o^2}{\partial y_G} - \frac{\partial U_a^2}{\partial y_G}\right) = \frac{\kappa}{(1-(\pi_{GG}+\pi_{BB}-1)\widehat{\beta})}\,\frac{(1-\pi_{GG})\gamma}{(1-\beta)(2-\pi_{GG}-\pi_{BB})} > 0 \tag{47}$$

Aggregate ex ante welfare is unchanged (if being an agent of type 1 or 2 is equiprobable).

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