Effects of Patent Length on R&D: A Quantitative DGE Analysis

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Abstract

This paper develops an R&D-growth model and calibrates the model to aggregate data of the US economy to quantify a structural relationship between patent length, R&D and consumption. Under parameter values that match the empirical flow-profit depreciation rate of patents and other key features of the US economy, extending the patent length beyond 20 years leads to a negligible increase in R&D despite equilibrium R&D underinvestment. In contrast, shortening the patent length leads to a significant reduction in R&D and consumption. Finally, this paper also analytically derives and quantifies a dynamic distortionary effect of patent length on capital investment.

Keywords**:** innovation-driven growth, intellectual property rights, patent length, R&D

JEL classification**:** O31, O34

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> [T]here is much to be done to understand the pace of technological progress in frontier economies. Our models of endogenous technological change give us the basic framework for thinking about how profit incentives shape investments in new technologies. … But most of our understanding of these issues is qualitative. For example, in the context of the economics of innovation, we lack a framework – similar to that used for the analysis of the effects of capital and labor income taxes and indirect taxes in public finance – which could be used to analyze the effects of … IPR policies … on innovation and economic growth.
>
> – Daron Acemoglu (2009, p. 873)

1. Introduction

Is the patent length an effective policy instrument to increase R&D? The statutory term of patent in the US was 17 years from 1861 to 1995 and then extended to 20 years as a result of the TRIPS agreement.[1] Given that there is underinvestment in R&D in the market economy,[2] why hasn't the term of patent been lengthened to increase R&D? This paper attempts to provide an answer to these questions by developing an R&D-growth model and calibrating the model to aggregate data of the US economy to quantify a structural relationship between patent length, R&D and consumption.

We find that the quantitative effects of patent length on R&D and consumption depend on the flow-profit depreciation rate of patents. Bessen (2008) estimates a flow-profit depreciation rate of 14% for US patents. At such a high depreciation rate, extending the patent length beyond 20 years leads to a very small percent change in the market value of patents and hence has a negligible effect on R&D. In contrast, shortening the patent length leads to a significant reduction in R&D and consumption. In other words, there is an asymmetric effect between extending and shortening the patent length.

[1] The World Trade Organization's Agreement on Trade-Related Aspects of Intellectual Property Rights (TRIPS), initiated in the 1986-94 Uruguay Round, extends the statutory term of patent in the US from 17 years (counting from the issue date when a patent is granted) to 20 years (counting from the earliest claimed filing date) to conform with the international standard. Due to the difference in the starting date, the effective patent extension was minimal.

[2] See, for example, Jones and Williams (1998, 2000).

This study also identifies and analytically derives a *dynamic* distortionary effect of patent length on capital investment that has been neglected by previous studies, which focus mostly on the *static* distortionary effect of markup pricing. The dynamic distortion arises because when the patent length increases, the fraction of monopolistic industries goes up and the resulting higher aggregate markup increases the wedge between the social marginal product of capital and the rental price. As a result, the equilibrium capital-investment rate decreases and deviates from the social optimum. The numerical analysis also quantifies this distortionary effect on capital investment.

Laitner (1982) is the first study to show that in an overlapping-generation (OLG) model, the presence of monopolistic profits creates the usual static distortion as well as a dynamic distortion on capital accumulation due to the crowding-out effect on portfolio space. Chou and Shy (1993) analyzes this crowding-out effect of patent length in an OLG model. The present study shows that in an R&D-growth model with an infinitely-lived representative household, patent length creates a different dynamic distortionary effect on capital investment by driving a wedge between the social marginal product of capital and the rental price.

This paper also complements the studies in the patent-design literature that are mostly based on a qualitative partial-equilibrium setting by providing a quantitative dynamic general-equilibrium (DGE) analysis on patent length.[3] The seminal study on patent length is Nordhaus (1969), and he concludes that the optimal patent length should balance between the static distortionary effect of markup pricing and the dynamic gain from enhanced innovation. In an exogenous-growth model, Judd (1985) concludes that the optimal patent length should be infinite. In contrast, Futagami and Iwaisako (2003, 2007) show that the optimal patent length can be finite in an endogenous-growth model. While it is interesting to characterize the optimal duration of patent, the present study complements these qualitative analyses by providing a quantitative framework that can be calibrated to data to quantify the extent of R&D underinvestment and the effects of patent length on R&D and consumption.

[3] See Scotchmer (2004) for a comprehensive review of the patent-design literature. O'Donoghue and Zweimuller (2004) provide one of the first studies that merge the patent-design and endogenous-growth literatures.

In terms of quantitative analysis on patent length, an important study is Kwan and Lai (2003), and they find substantial welfare gains from extending the effective lifetime of patent. There is an important reason for the contradicting results between Kwan and Lai (2003) and the present study. By using the same final-goods production function as in Romer (1990), Kwan and Lai (2003) necessarily restrict the size of the markup to the inverse of the capital share. This usually innocuous parameter restriction equates the balanced-growth rate of profit per patent to the population growth rate that is nonnegative. Relaxing this restriction implies that at the empirical flow-profit depreciation rate of patents, the amount of profit captured by a patent declines sharply overtime rendering patent extension ineffective in increasing R&D. In a related study, Chu (2009) considers a different aspect of patent protection by quantifying the effects of blocking patents on R&D in a quality-ladder growth model with overlapping patent rights.

The rest of this study is organized as follows. Section 2 illustrates the intuition of the main results. Section 3 describes the model, and Section 4 defines the equilibrium. Section 5 calibrates the model, and the final section concludes. Proofs are relegated to Appendix A.

2. Intuition of the main results

The first result of this paper is that at the empirical flow-profit depreciation rate of patents, extending the patent length beyond 20 years leads to a negligible increase in R&D. Denote $M_0(T)$ as the market value of an invention patented at time 0 with a patent length of T years. Suppose that the flow profit captured by the invention at time t is given by $\pi_t = \pi_0 \exp(g_\pi t)$, where g_π is the flow-profit growth rate that may be negative. Then, $M_0(T)$ can be determined by the following condition

(1) $$M_0(T) = \int_0^T e^{-(r-g_\pi)t}\pi_0 dt = \left(\frac{1-e^{-(r-g_\pi)T}}{r-g_\pi}\right)\pi_0 .$$

The percent change in the market value of a patent from extending the patent length by τ years is

(2) $$\frac{M_0(T+\tau)-M_0(T)}{M_0(T)} = \frac{e^{-(r-g_\pi)T}-e^{-(r-g_\pi)(T+\tau)}}{1-e^{-(r-g_\pi)T}} .$$

We consider flow-profit growth rates of {-10%, -15%, -20%} that cover the empirical estimate of a 14% flow-profit depreciation rate in Bessen (2008).[4] Given these estimates and an asset return of 7%, the percent changes in the market value of an invention by extending the patent length from 20 years to 25 years are {2.0%, 0.8%, 0.3%}, which are very small. On the other hand, the percent changes from shortening the patent length by 5 years are {-4.6%, -2.5%, -1.3%}, which are more significant. In other words, there is an asymmetric effect between extending and shortening the patent length on the market value of patents. The model in Section 3 provides a growth-theoretic framework that can be calibrated to quantify the resulting effects on R&D and consumption.

The second result of the paper is the dynamic distortionary effect (i.e. an increase in the patent length decreases the equilibrium capital-investment rate). To illustrate the intuition behind this result, the no-arbitrage condition equates the capital's rental price R to the sum of the real interest rate r and the capital-depreciation rate δ such that $R = r + \delta$, which is constant and exogenously determined along the balanced-growth path. The aggregate markup creates a wedge (denoted by $d \leq 1$) between the rental price and the marginal product of capital such that

$$R = d(T)MPK(K/Y). \tag{3}$$

$d(T)$ is decreasing in the patent length T while the marginal product of capital *MPK* is decreasing in the capital-output ratio *K/Y*. Intuitively, an increase in the patent length raises the fraction of monopolistic industries and hence the aggregate markup. A higher aggregate markup increases the wedge (i.e. a smaller d) between the rental price and the social marginal product of capital. As a result, the steady-state capital-output ratio and capital-investment rate must fall to satisfy (3). The R&D-growth model serves the useful purpose in providing a framework to quantify the magnitude of this wedge.

[4] Bessen (2008) estimates that the flow-profit depreciation rate of patents is about 14% in the US. This estimate is in line with the earlier literature on estimating the market value of patents based on European patent renewal data. See, for example, Pakes (1986), Schankerman and Pakes (1986) and Lanjouw (1998).

3. The model

The variety-expanding model is a modified version of Romer (1990). The basic framework is modified to introduce a finite patent length denoted by T for each invented variety of intermediate goods. The final goods are produced with labor and a composite of intermediate goods. An intermediate-goods industry is monopolistic if the producer owns an unexpired patent, and the industry becomes competitive once the patent expires. The relative price between the monopolistic and competitive goods leads to the usual *static* distortionary effect that reduces the output of final goods. The markup in the monopolistic industries drives a wedge between the social marginal product of capital and the rental price. Consequently, it leads to an additional *dynamic* distortionary effect that causes the equilibrium capital-investment rate to deviate from the social optimum. To prevent the model from overstating the social benefits of R&D and hence the extent of R&D underinvestment, we follow Comin (2004) to assume that total factor productivity (TFP) growth is driven by R&D as well as an exogenous process of productivity improvement. In addition, the first-generation R&D-growth models exhibit scale effects that are contradicted by the empirical evidence in Jones (1995a).[5] In the present study, scale effects are eliminated as in Jones (1995b). Upon eliminating scale effects, the model becomes a semi-endogenous growth model, in which the long-run growth rate is determined by exogenous parameters and R&D has a level effect in the long run. The components of the model are presented in Sections 3.1–3.6, and the analysis focuses on the balanced-growth path.

3.1. Household

There is a representative household whose lifetime utility is given by

(4)
$$U=\int_0^\infty e^{-(\rho-n)t}\left(\frac{c_t^{1-\sigma}-1}{1-\sigma}\right)dt\,,$$

where $\sigma>0$ is the inverse of the elasticity of intertemporal substitution and ρ is the exogenous subjective discount rate. The household has $L_t=e^{nt}$ members at time t, and $n>0$ is the exogenous

[5] See Jones (1999) for an excellent discussion on scale effects in R&D-growth models.

population growth rate. To ensure that utility is bounded, $\rho > n + (1-\sigma)g_c$ where g_c is the balanced-growth rate of c_t, which is the per capita consumption of final goods (the numeraire). The household maximizes utility subject to a sequence of budget constraints $\dot{a}_t = (r_t - n)a_t + w_t - c_t$. Each member of the household inelastically supplies one unit of homogenous labor in each period to earn a wage income w_t. a_t is the per capita holding of assets that consist of physical capital and patents, and r_t is the real rate of return. From the household's intertemporal optimization, the familiar Euler equation is

(5) $$\dot{c}_t / c_t = (r_t - \rho)/\sigma .$$

The steady-state real interest rate is $r = \rho + g_c\sigma$.

3.2. Final goods

The sector producing the final goods is characterized by perfect competition, and the producers take both the output and input prices as given. In particular, the final-goods production function is

(6) $$Y_t = Z_t^{1-\alpha} L_{y,t}^{1-\alpha} \left(\int_0^{V_t} X_t^{\alpha\eta}(j)dj \right)^{1/\eta}$$

for $\eta \in (0, 1/\alpha)$. Y_t is the output of final goods. $Z_t = Z_0 \exp(g_Z t)$ represents an exogenous process of productivity improvement that is freely available to final-goods firms. $L_{y,t}$ is the number of production workers. $X_t(j)$ is the amount of intermediate goods of variety $j \in [0, V_t]$, in which V_t is the number of varieties that have been invented as of time *t*.

The production function in (6) nests Romer (1990) as a special case with $\eta = 1$ and $Z_t = 1$ for all *t*. For $\eta = 1$, the monopolistic markup μ is equal to $1/\alpha$ (i.e. roughly the inverse of the capital share). Therefore, Jones and Williams (2000) propose a more flexible specification that allows η to differ from one. The markup becomes $\mu = 1/(\alpha\eta) > 1$ that relaxes the parameter restriction between the markup and capital share. As for the exogenous process Z_t, this study includes it to avoid the assumption that TFP growth is solely driven by R&D. Later, we use the empirical flow-profit depreciation rate of patents to help calibrating the model. Standard profit maximization yields the following first-order conditions.

(7) $$w_t = (1-\alpha)Y_t / L_{y,t},$$

(8) $$P_t(j) = \alpha Z_t^{1-\alpha} L_{y,t}^{1-\alpha} \left(\int_0^{V_t} X_t^{\alpha\eta}(j)dj \right)^{(1-\eta)/\eta} X_t^{\alpha\eta-1}(j),$$

where $P_t(j)$ is the price of intermediate-goods $j \in [0, V_t]$.

3.3. Intermediate goods

There is a continuum of industries indexed by $j \in [0, V_t]$ producing the differentiated intermediate goods $X_t(j)$. Once a variety has been invented, the production function is $X_t(j) = K_{y,t}(j)$, where $K_{y,t}(j)$ is the amount of capital employed by industry j. The profit function facing the producer(s) of variety j is

(9) $$\pi_t(j) = P_t(j)X_t(j) - R_t K_{y,t}(j).$$

R_t is the rental price of capital. Denote the fraction of monopolistic industries at time *t* by $f_t \in [0,1]$, which is endogenously determined by the patent length *T*. Without loss of generality, the industries are ordered such that industries $j \in [0, V_t f_t]$ are protected by patents and industries $j' \in (V_t f_t, V_t]$ are not protected by patents. Then, the equilibrium prices for $j \in [0, V_t f_t]$ and $j' \in (V_t f_t, V_t]$ are respectively

(10) $$P_t(j) = \mu R_t,$$

(11) $$P_t(j') = R_t.$$

3.4. Aggregate production function and the static distortionary effect

The total amount of capital employed by the intermediate-goods sector at time t is

(12) $$K_{y,t} = \int_0^{V_t} X_t(j)dj = V_t[f_t X_t(j) + (1-f_t)X_t(j')].$$

Lemma 1: *The aggregate production function for the final goods is*

(13) $$Y_t = s_t A_t^{1-\alpha} Z_t^{1-\alpha} L_{y,t}^{1-\alpha} K_{y,t}^{\alpha},$$

where A_t *is the level of R&D-driven TFP and is defined as*

(14) $$A_t^{1-\alpha} \equiv V_t^{(1-\alpha\eta)/\eta},$$

and s_t *is defined as*

(15) $$s_t \equiv \frac{[(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} f_t + 1 - f_t]^{1/\eta}}{[(\alpha\eta)^{1/(1-\alpha\eta)} f_t + 1 - f_t]^{\alpha}} \in (0,1].$$

s_t *is strictly less than one for* $f_t \in (0,1)$ *and equal to one for* $f_t \in \{0,1\}$. *In addition,* $\partial s_t / \partial f_t < 0$ *when*

$$f_t < \hat{f} \equiv \frac{1}{1-\alpha\eta}\left(\frac{1}{1-(\alpha\eta)^{1/(1-\alpha\eta)}} - \frac{\alpha\eta}{1-(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)}}\right) < 1.$$

Proof: See Appendix A.■

The variable s_t captures the static distortionary effect of patent length in creating a monopolistic markup in patent-protected industries. Intuitively, the markup in patent-protected industries distorts production towards competitive industries and reduces the total output of final goods. Increasing the fraction of monopolistic industries worsens this static distortionary effect when $f_t < \hat{f}$. The static distortionary effect is non-monotonic in f_t because when all industries are monopolistic, the relative-price distortion disappears.

3.5. National income account identities and capital accumulation

The market-clearing condition for the final goods is

(16) $$Y_t = C_t + I_t.$$

$C_t = L_t c_t$ is aggregate consumption, and I_t is investment in physical capital. The correct value of gross domestic product (GDP) should include the value of investment in R&D such that

(17) $$GDP_t = Y_t + w_t L_{r,t} + R_t K_{r,t}.$$ [6]

$L_{r,t}$ and $K_{r,t}$ are respectively the number of workers and the amount of capital in the R&D sector. The amount of monopolistic profit and the factor payments for workers and capital in the intermediate-goods sector are

(18) $$w_t L_{y,t} = (1-\alpha)Y_t,$$

(19) $$R_t K_{y,t} = \alpha Y_t d_t,$$

(20) $$\pi_t V_t f_t = \alpha Y_t (1-d_t),$$

where d_t is determined by f_t (the fraction of monopolistic industries) and is defined as

(21) $$d_t \equiv \frac{(\alpha\eta)^{1/(1-\alpha\eta)} f_t + 1 - f_t}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} f_t + 1 - f_t} \in [\alpha\eta, 1].$$

A larger f_t reduces d_t and increases the wedge between the social marginal product of capital and the rental price. As will be shown below, this decrease in d_t also leads to a lower capital-investment rate. Therefore, d_t captures the dynamic distortionary effect of patent length on capital accumulation.

The market-clearing condition for physical capital is

(22) $$K_t = K_{y,t} + K_{r,t}.$$

K_t is the total amount of capital in the economy at time *t*. The law of motion for capital is

(23) $$\dot{K}_t = I_t - \delta K_t.$$

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[6] In the national income account, private spending in R&D is treated as an expenditure on intermediate goods. Therefore, the values of capital investment and GDP in the data are I_t and Y_t respectively. The Bureau of Economic Analysis and the National Science Foundation's R&D satellite account provides preliminary estimates on the effects of including R&D as an intangible asset in the national income accounts.

δ is the capital-depreciation rate. Denote the balanced-growth rate of capital by g_K. Then, the steady-state capital-investment rate can be expressed as

(24) $$i \equiv I_t / Y_t = (g_K + \delta) K_t / Y_t \ .$$

The no-arbitrage condition $r_t = R_t - \delta$ implies that the steady-state capital-output ratio is

(25) $$\frac{K_t}{Y_t} = \frac{\alpha d}{(1-k_r)(r+\delta)},$$

where $k_r \equiv K_{r,t} / K_t$ is the steady-state R&D share of capital. Substituting (25) into (24) yields

(26) $$i = \left(\frac{\alpha d}{1-k_r} \right) \frac{g_K + \delta}{r + \delta}.$$

In the Romer model, (skilled) labor is the only input for R&D (i.e. $k_r = 0$). Therefore, the distortionary effect of patent length on the capital-investment rate is unambiguously negative (i.e. as T increases, d decreases and hence i decreases as well). In the present model with $k_r > 0$, there is an opposing effect through k_r. Intuitively, extending the patent length raises the private return to R&D and increases k_r. Proposition 1 in Section 4.3 shows that the negative distortionary effect dominates this positive effect.

3.6. R&D

The market value of a patent on a variety invented at time t is the present value of the stream of monopolistic profits captured by the patent until it expires at time $t + T$ given by

(27) $$M_t(T) = \int_t^{t+T} e^{-r(\tau - t)} \pi_\tau d\tau = \Omega(T) \pi_t ,$$

where $\Omega(T) \equiv (1 - e^{-(r-g_\pi)T}) / (r - g_\pi)$ is the present-value discount factor. The marginal effect of the patent length on $\Omega(T)$ is given by $\Omega'(T) = e^{-(r-g_\pi)T}$, and this marginal effect depends positively on the profit growth rate g_π. Therefore, a highly negative profit growth rate (i.e. a high flow-profit depreciation rate) renders patent extension ineffective in raising the market value of a patent.

The number of inventions obtained by R&D entrepreneur $k \in [0,1]$ is

$$\lambda_t(k) = \overline{\varphi}_t L_{r,t}^{1-\beta}(k) K_{r,t}^{\beta}(k), \tag{28}$$

where $\overline{\varphi}_t$ is R&D productivity that entrepreneurs take as given. This specification nests the "knowledge-driven" specification in Romer (1990) as a special case with $\beta = 0$ and the "lab equipment" specification in Rivera-Batiz and Romer (1991) and Jones and Williams (2000) with $\beta = \alpha$. The R&D sector is characterized by perfect competition, and the amount of profit earned by R&D entrepreneur k is

$$\pi_{r,t}(k) = M_t \lambda_t(k) - w_t L_{r,t}(k) - R_t K_{r,t}(k). \tag{29}$$

The first-order conditions for the R&D sector are

$$(1-\beta) M_t \overline{\varphi}_t (K_{r,t} / L_{r,t})^{\beta} = w_t, \tag{30}$$

$$\beta M_t \overline{\varphi}_t (K_{r,t} / L_{r,t})^{\beta-1} = R_t. \tag{31}$$

(30) and (31) together with (18) and (19) determine the resource allocation between production and R&D.

To eliminate scale effects and introduce R&D externalities, we follow Jones and Williams (2000) to assume that R&D productivity $\overline{\varphi}_t$ is given by

$$\overline{\varphi}_t = \varphi V_t^{\phi} (K_{r,t}^{\beta} L_{r,t}^{1-\beta})^{\gamma-1}. \tag{32}$$

$\phi \in (-\infty,1)$ captures the positive externality $\phi \in (0,1)$ or the negative externality $\phi \in (-\infty,0)$ in intertemporal knowledge spillovers. $\gamma \in (0,1]$ captures the negative intratemporal-duplication externality or the so-called "stepping-on-toes" effects. The law of motion for the number of varieties is

$$\dot{V}_t = \int_0^1 \lambda_t(k) dk = \overline{\varphi}_t K_{r,t}^{\beta} L_{r,t}^{1-\beta} = \varphi V_t^{\phi} (K_{r,t}^{\beta} L_{r,t}^{1-\beta})^{\gamma}. \tag{33}$$

On the balanced-growth path, $K_{r,t} = \int_0^1 K_{r,t}(k) dk$ increases at g_K and $L_{r,t} = \int_0^1 L_{r,t}(k) dk$ increases at the exogenous population growth rate n. Therefore, the balanced-growth rate of V_t denoted by g_V is

(34) $$g_V \equiv \frac{\dot{V}_t}{V_t} = \varphi \frac{(K_{r,t}^{\beta} L_{r,t}^{1-\beta})^{\gamma}}{V_t^{1-\phi}} = \left(\frac{\gamma\beta}{1-\phi}\right) g_K + \left(\frac{\gamma(1-\beta)}{1-\phi}\right) n \,.$$

Finally, the steady-state fraction f of varieties that are patent-protected is given by

(35) $$f(T) \equiv \frac{\widetilde{V}_t}{V_t} = \frac{\dot{V}_t - \dot{V}_{t-T}}{\dot{V}_t} = 1 - e^{-g_V T} \,,$$

where $\widetilde{V}_t$ is the number of patent-protected varieties and $\dot{V}_t - \dot{V}_{t-T}$ is the net increase in the number of patent-protected varieties at time *t*.

4. Decentralized equilibrium

This section firstly defines the equilibrium and characterizes the balanced-growth path. Then, we derive the socially optimal allocations and discuss the dynamic distortionary effect on capital investment. The decentralized equilibrium is a sequence of allocations $\{c_t, a_t, X_t(j), Y_t, I_t, K_{y,t}, L_{y,t}, K_{r,t}, L_{r,t}, K_t, L_t\}_{t=0}^{\infty}$ and a sequence of prices $\{w_t, r_t, R_t, P_t(j), M_t\}_{t=0}^{\infty}$. Also, in each period,

(a) the representative household chooses $\{c_t, a_t\}$ to maximize utility taking $\{w_t, r_t\}$ as given;

(b) final-goods firms choose $\{X_t(j), L_{y,t}\}$ to maximize profits taking $\{P_t(j), w_t\}$ as given;

(c) monopolistic firms $j \in [0, V_t f_t]$ in the intermediate-goods sector choose $\{P_t(j), K_{y,t}(j)\}$ to maximize profits taking $\{R_t\}$ as given;

(d) competitive firms $j' \in (V_t f_t, V_t]$ in the intermediate-goods sector choose $\{K_{y,t}(j')\}$ to maximize profits taking $\{P_t(j'), R_t\}$ as given;

(e) entrepreneurs $k \in [0,1]$ in the R&D sector choose $\{L_{r,t}(k), K_{r,t}(k)\}$ to maximize profits taking $\{w_t, R_t, M_t\}$ as given;

(f) the market for final goods clears such that $Y_t = C_t + I_t$;

(g) there is full employment of capital such that $K_t = K_{y,t} + K_{r,t}$; and

(h) there is full employment of labor such that $L_t = L_{y,t} + L_{r,t}$.

4.1. Balanced-growth path

Equating (18) and (30) and then imposing the following balanced-growth condition

(36) $$V_t = \overline{\varphi}_t K_{r,t}^{\beta} L_{r,t}^{1-\beta} / g_V$$

on the resulting expression yield the steady-state R&D share of labor given by

(37) $$\frac{l_r}{1-l_r} = \frac{1-\beta}{1-\alpha}\left(\frac{(1-e^{-(r-g_\pi)T})g_V}{r-g_\pi}\right)\frac{\alpha(1-d)}{f},$$

where $l_r \equiv L_{r,t} / L_t$. Similarly, equating (19) and (31) and then imposing (36) yield

(38) $$\frac{k_r}{1-k_r} = \frac{\beta}{\alpha}\left(\frac{(1-e^{-(r-g_\pi)T})g_V}{r-g_\pi}\right)\frac{\alpha(1-d)}{f}.$$

The balanced-growth rates of various variables are given as follows. From the aggregate production function in (13) and the steady-state capital-investment rate in (26), we see that

(39) $$g_Y = g_K = g_I = g_C = g_A + g_Z + n.$$

Therefore, $g_c = g_C - n = g_A + g_Z$. From the definition of R&D-driven TFP in (14),

(40) $$g_A = \frac{1}{1-\alpha}\left(\frac{1-\alpha\eta}{\eta}\right)g_V.$$

Finally, from (20), the balanced-growth rate of monopolistic profit per patent is

(41) $$g_\pi = g_Y - g_V = g_A + g_Z + n - g_V.$$

Note that g_π must equal $g_Z + n > 0$ when $\eta = 1$ because $g_A = g_V$. However, as η increases (i.e. a smaller markup $\mu = 1/\alpha\eta$), g_V increases relative to g_A. Therefore, holding g_A constant, an increase in η leads to a decrease in g_π. Eventually, g_π becomes negative for a small enough markup.

Denote ξ as the fraction of long-run TFP growth driven by R&D such that $\xi g_{TFP} = (1-\alpha)g_A$ and $(1-\xi)g_{TFP} = (1-\alpha)g_Z$. If $\xi = 1$, then the value of g_π is pinned down by g_{TFP}, n, α and η according to (40) and (41). However, this implied value of g_π may be biased due to the misspecification of the model. Therefore, we allow ξ to differ from one and calibrate this parameter from the data. Firstly, we use the empirical estimate of g_π in (41), which pins down a unique value of g_V for given values of g_{TFP}, α and n from the data. Then, given g_V, (40) determines a unique value of g_A and hence ξ for given values of g_{TFP}, α and η. Finally, combining (34), (39) and (40) yields the following balanced-growth condition that can be used to calibrate the externality parameters γ and ϕ.

(42) $$g_V = \left(\frac{1-\phi}{\gamma} - \frac{\beta}{1-\alpha}\left(\frac{1-\alpha\eta}{\eta}\right)\right)^{-1}(\beta g_Z + n).$$

We impose the following parameter restriction $\frac{1-\phi}{\gamma} > \frac{\beta}{1-\alpha}\left(\frac{1-\alpha\eta}{\eta}\right)$ to ensure that $g_V > 0$.

4.2. Socially optimal allocations

To derive the socially optimal capital-investment rate i^* and R&D shares of labor l_r^* and capital k_r^*, the social planner maximizes

(43) $$U = \int_0^\infty e^{-(\rho-n)t}\left(\frac{[(1-i_t)Y_t/L_t]^{1-\sigma}-1}{1-\sigma}\right)dt$$

subject to (i) the aggregate production function expressed in terms of $l_{r,t}$ and $k_{r,t}$ given by

(44) $$Y_t = V_t^{(1-\alpha\eta)/\eta} Z_t^{1-\alpha}(1-k_{r,t})^\alpha (1-l_{r,t})^{1-\alpha} K_t^\alpha L_t^{1-\alpha},$$

(ii) the law of motion for capital expressed in terms of i_t given by

(45) $$\dot{K}_t = i_t Y_t - \delta K_t,$$

and (iii) the law of motion for the number of varieties expressed in terms of $l_{r,t}$ and $k_{r,t}$ given by

(46) $$\dot{V}_t = V_t^{\phi}(k_{r,t})^{\beta\gamma}(l_{r,t})^{(1-\beta)\gamma} K_t^{\beta\gamma} L_t^{(1-\beta)\gamma}\varphi .$$

After deriving the first-order conditions, the social planner solves for i^*, l_r^* and k_r^* on the balanced-growth path.[7] The socially optimal capital-investment rate i^* is

(47) $$i^* = \left(\frac{\alpha}{1-k_r^*}\right)\frac{g_K+\delta}{r+\delta},$$

and the socially optimal R&D shares of labor l_r^* and capital k_r^* are respectively

(48) $$\frac{l_r^*}{1-l_r^*} = \frac{1-\beta}{1-\alpha}\left(\frac{\gamma g_V}{r-g_Y+(1-\phi)g_V}\right)\frac{1-\alpha\eta}{\eta},$$

(49) $$\frac{k_r^*}{1-k_r^*} = \frac{\beta}{\alpha}\left(\frac{\gamma g_V}{r-g_Y+(1-\phi)g_V}\right)\frac{1-\alpha\eta}{\eta}.$$

Comparing (37) and (38) with (48) and (49) shows the various sources of R&D externality (i) the negative externality in intratemporal duplication $\gamma \in (0,1]$, (ii) the positive or negative externality in intertemporal knowledge spillovers $\phi \in (-\infty,1)$, (iii) the positive externality from the dynamic surplus-appropriability problem (due to the finite patent length) given by $1-e^{-(r-g_\pi)T} < 1$, and (iv) the positive externality from the static surplus-appropriability problem given by $\alpha(1-d)/f < (1-\alpha\eta)/\eta$ for all *T*.[8] Given the presence of positive and negative externalities, it requires a numerical calibration that will be performed in Section 5 to determine whether the market economy overinvests or underinvests in R&D.

4.3. The dynamic distortionary effect

If the market economy underinvests in R&D, the government may want to increase the patent length to reduce the extent of this market failure. However, an increase in patent length would worsen the dynamic distortionary effect on capital accumulation. Therefore, the government needs to trade off the gains from increasing R&D against the losses from the dynamic distortion as well as the static distortion. Proposition

[7] The derivations are provided in an appendix available from the author upon request.
[8] It can be shown that $(1-d)/f$ is increasing in T. As $T \to \infty$, $\alpha(1-d)/f = \alpha(1-\alpha\eta) < (1-\alpha\eta)/\eta$.

1 provides the condition under which an increase in patent length would move the equilibrium capital-investment rate i away from the social optimum i^*.

Proposition 1: *The decentralized equilibrium capital-investment rate* i *is below the socially optimal rate* i^* *if either (i) there is underinvestment in R&D or (ii) labor is the only factor input for R&D. In addition, an increase in the patent length always reduces the equilibrium capital-investment rate* i.

Proof: See Appendix A.■

The second part of the proposition is quite intuitive. When the patent length increases, the fraction of monopolistic industries rises. The resulting higher aggregate markup drives a larger wedge between the social marginal product of capital and the rental price. Therefore, the equilibrium capital-investment rate decreases. As for the first part of the proposition, the discrepancy between the equilibrium capital-investment rate and the socially optimal rate arises due to (i) the aggregate markup and (ii) the discrepancy between the equilibrium R&D share of capital k_r and the optimal R&D share k_r^*. Given that the equilibrium capital-investment rate i is increasing in k_r, R&D underinvestment (i.e. $k_r < k_r^*$) is sufficient for $i < i^*$. When there is R&D overinvestment, whether i is below or above i^* depends on the relative magnitude of the markup effect and the R&D-overinvestment effect. For the case in which labor is the only factor input for R&D, k_r equals zero and hence, only the markup effect is present.

5. Calibration

This section firstly calibrates the structural and externality parameters using long-run aggregate data of the US economy and then computes the changes in R&D and consumption by varying the patent length. After that, the dynamic distortionary effect on capital investment is also quantified.

5.1. Structural parameters

The statutory patent length T in the US is 20 years, and the average annual labor-force growth rate n is 0.0166.[9] The annual discount rate ρ and the annual rate of depreciation δ for physical capital are set to conventional values of 0.04 and 0.08 respectively. β is set equal to α corresponding to the lab-equipment specification in Rivera-Batiz and Romer (1991) and Jones and Williams (2000).[10] Once the above parameters are determined, the model provides five steady-state conditions (summarized in (50)-(54) below) to determine the remaining structural parameters $\{\sigma, \alpha, \eta, \xi, g_V\}$ by matching five empirical moments.[11] To identify σ, the ratio of private investment to GDP is set to 0.202.[12] To identify α, the labor share of GDP is set to a conventional value of 0.7. To identify η through the markup, the R&D share of GDP is set to 0.0149.[13] To identify g_V, g_π is set to {-20%, -15%, -10%} based on the estimate in Bessen (2008). Finally, to identify ξ, the long-run TFP growth rate $g_{TFP} \equiv (1-\alpha)(g_A + g_Z)$ is set to 0.0102.[14]

(50) $$\frac{I}{Y} = \frac{\alpha d}{1-k_r}\left(\frac{n+g_c+\delta}{\rho + g_c\sigma + \delta}\right),$$

(51) $$\frac{wL}{Y} = \frac{1-\alpha}{1-l_r},$$

(52) $$\frac{wL_r + RK_r}{Y} = (1-\alpha)\frac{l_r}{1-l_r} + \alpha d \frac{k_r}{1-k_r},$$

(53) $$g_\pi = g_{TFP}/(1-\alpha) + n - g_V,$$

---

9 This number is calculated using data from 1956 to 2006, and the data is from the Bureau of Labor Statistics.

10 We have considered different values of $\beta \in \{0, \alpha, 2\alpha, 3\alpha\}$ as a sensitivity analysis, and the results are robust to these parameter changes.

11 Technically, g_v is not a parameter but determined by exogenous parameters according to (42).

12 This number is calculated using data from 1956 to 2006, and the data is from the Bureau of Economic Analysis. GDP is net of government spending.

13 This number is calculated using data from 1956 to 2004. The data is from the Bureau of Economic Analysis and the National Science Foundation. R&D is net of federal spending, and GDP is net of government spending.

14 This number is calculated using data on multifactor productivity (available from 1956 to 2002) for the private non-farm business sector. The data is from the Bureau of Labor Statistics.

(54) $$g_{TFP} = \frac{1}{\xi}\left(\frac{1-\alpha\eta}{\eta}\right)g_V .$$

For $\alpha = \beta$, $\frac{k_r}{1-k_r} = \frac{l_r}{1-l_r} = \left(\frac{(1-e^{-(\rho-n+(\sigma-1)g_c+g_V)T})g_V}{\rho-n+(\sigma-1)g_c+g_V}\right)\frac{\alpha(1-d)}{f}$.

Table 1 lists the calibrated structural parameters along with the implied markup $\mu = 1/(\alpha\eta)$ and the implied real interest rate $r = \rho + g_c\sigma$.

Table 1: Calibrated structural parameters

g_π	σ	α	η	ξ	g_V	μ	r
-0.10	2.87	0.31	2.97	0.34	0.13	1.08	0.08
-0.15	2.91	0.31	3.01	0.39	0.18	1.07	0.08
-0.20	2.93	0.31	3.02	0.46	0.23	1.07	0.08

The implied markup is within the empirically plausible range. For example, Laitner and Stolyarov's (2004) estimated markup is 1.09-1.11, and Basu and Fernald (1997) estimate that the aggregate profit share in the US is about 3%. Also, the implied real interest rate is roughly in line with the historical rate of return in the US stock market. The calibrated values for ξ suggest that about 35% to 45% of long-run TFP growth in the US is driven by R&D.

5.2. Externality parameters

For each set of calibrated parameter values, the balanced-growth condition in (42) determines a unique value for $\gamma/(1-\phi)$, which is sufficient to determine the effect of R&D on the balanced-growth level of consumption. However, holding $\gamma/(1-\phi)$ constant, a larger γ implies a faster rate of convergence to the balanced-growth path. Therefore, it is important to consider different values of γ (i.e. the negative externality in intratemporal duplication). Table 2 presents the calibrated values of ϕ (i.e. the externality in intertemporal knowledge spillovers), and the positive values suggest positive knowledge spillovers (i.e. the standing-on-shoulder effect).

Table 2: Calibrated values of ϕ										
g_π / γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
-0.10	0.98	0.97	0.95	0.94	0.92	0.90	0.89	0.87	0.85	0.84
-0.15	0.99	0.98	0.96	0.95	0.94	0.93	0.92	0.91	0.89	0.88
-0.20	0.99	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.92	0.91

5.3. Socially optimal R&D

This section computes the socially optimal R&D share of GDP $(1-\alpha)l_r^*/(1-l_r^*)+\alpha k_r^*/(1-k_r^*)$.

[insert Figure 1 here]

Figure 1 shows that there is underinvestment in R&D unless γ is very small. To reduce the plausible parameter space for γ, we consider the empirical estimates of the social rate of return to R&D. Following Jones and Williams' (1998) derivation, Appendix B shows that the net social rate of return to R&D can be expressed as

(55) $$\tilde{r}=\frac{1+g_Y}{1+g_V}\left(1+g_V\left(\frac{1-\alpha\eta}{\eta}\frac{\gamma}{k_r}+\phi\right)\right)-1.$$

Holding other things constant, $\tilde{r}$ is increasing in γ. Table 3 shows that for the range of values $\gamma \leq 0.2$ that exhibits R&D overinvestment, the implied social rates of return $\tilde{r}$ are less than 8%, which are far below the empirical estimates summarized in Jones and Williams (1998).

Table 3: Implied social rates of return to R&D										
g_π / γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
-0.10	5.0%	7.0%	8.9%	10.8%	12.7%	14.6%	16.5%	18.4%	20.3%	22.2%
-0.15	5.3%	7.4%	9.6%	11.7%	13.9%	16.0%	18.1%	20.3%	22.4%	24.6%
-0.20	5.6%	8.0%	10.5%	12.9%	15.3%	17.8%	20.2%	22.7%	25.1%	27.5%

Given that the empirical estimates of the social return to R&D vary across studies, we will leave it to the readers to decide on their preferred values. For the relevant range $\gamma > 0.2$, there is R&D underinvestment, and this finding is due to the calibration result that a non-negligible fraction of long-run TFP growth is driven by R&D. If the calibrated values for ξ were smaller, then the socially optimal levels of R&D would be lower because a smaller ξ implies a smaller $\gamma/(1-\phi)$, which in turn implies that R&D would have a smaller effect on consumption.

5.4. Patent extension

Given the above finding of R&D underinvestment, a natural question to ask is whether extending the patent length can effectively mitigate this problem. Figure 2 shows that the magnitude of the increase in R&D depends on the flow-profit depreciation rate of patents.

[insert Figure 2 here]

At a high flow-profit depreciation rate of 20%, the effect of extending the patent length on R&D is almost negligible. At a lower patent-value depreciation rate of 10%, extending the patent length from 20 to 50 years increases the R&D share of GDP from 0.015 to 0.017, but this level of R&D is still far below the social optimum. This exercise suggests that patent extension is not an effective method to increase R&D. However, shortening the patent length can reduce R&D significantly. Therefore, we find that there is an important asymmetric effect between extending and shortening the patent length on R&D.

The next exercise computes the percent changes in long-run consumption, which also determines steady-state welfare. After dropping some exogenous terms, the balanced-growth level of consumption can be expressed as

$$c(T)=\begin{pmatrix} s(T)^{\frac{\eta(1-\phi)}{\eta(1-\phi)(1-\alpha)-\alpha\gamma(1-\alpha\eta)}}\, i(T)^{\frac{\alpha\eta(1-\phi)+\alpha\gamma(1-\alpha\eta)}{\eta(1-\phi)(1-\alpha)-\alpha\gamma(1-\alpha\eta)}}[1-i(T)] \\ k_r(T)^{\frac{\gamma(1-\alpha\eta)}{\eta(1-\phi)(1-\alpha)-\alpha\gamma(1-\alpha\eta)}}[1-k_r(T)]^{\frac{\eta(1-\phi)}{\eta(1-\phi)(1-\alpha)-\alpha\gamma(1-\alpha\eta)}} \end{pmatrix} \tag{56}$$

for $\alpha=\beta$.[15] Figure 3 plots the percent changes in long-run consumption.

[insert Figure 3 here]

Given the small increases in R&D from patent extension, the maximum effect on consumption is no more than 3% (percent change) at a 10% flow-profit depreciation rate and is as small as 0.6% at a 20% flow-

[15] The derivations are provided in an appendix available upon request. Also, note that $l_r = k_r$ under $\alpha=\beta$.

profit depreciation rate. On the other hand, shortening the patent length can lead to a substantial decrease in consumption. Furthermore, the changes in consumption mostly come from the direct technology effect of R&D on consumption, i.e. $\frac{\gamma(1-\alpha\eta)}{\eta(1-\phi)(1-\alpha)-\alpha\gamma(1-\alpha\eta)}\Delta \ln k_r(T)$.

5.5. Dynamic distortion on capital investment

Proposition 1 derives the sufficient condition under which the equilibrium capital-investment rate is below the socially optimal rate in (47). The next numerical exercise quantifies this discrepancy. Figure 4 plots the socially optimal capital-investment rates along with the US long-run capital-investment rate, and the difference is about 0.017 on average.

[insert Figure 4 here]

The equilibrium capital-investment rate is decreasing in the aggregate markup; therefore, extending the patent length decreases the capital-investment rate and causes it to deviate from the social optimum. Figure 5 plots the equilibrium capital-investment rates at different patent length and shows that extending the patent length would slightly worsen the dynamic distortionary effect on capital investment.

[insert Figure 5 here]

6. Conclusion

This paper provides a growth-theoretic framework that can be calibrated to aggregate data to quantify a structural relationship between patent length, R&D and consumption. At the empirical flow-profit depreciation rate of patents, extending the patent length beyond 20 years has a negligible effect on R&D. Therefore, patent length is not an effective instrument in solving the R&D-underinvestment problem. Although the analysis focuses on the balanced-growth path, taking into consideration the transition dynamics would not alter this policy implication. This is because if the long-run effect of patent extension on consumption is so small, accounting for the potential short-run consumption losses would make the overall welfare gains even more negligible. This finding of a small effect of patent extension on R&D

rationalizes the fact that the patent policy changes in the US during the 80's were related to other aspects of patent protection, such as patentability requirement, patent breadth and blocking patents.[16]

Finally, the readers are advised to interpret the numerical results with some important caveats in mind. Although the variety-expanding model has been generalized to capture more realistic features of the US economy, it is still an oversimplification of the real world. Furthermore, the finding of R&D underinvestment is based on the assumption that the empirical estimates of the social return to R&D and the data on R&D expenditure are not incorrectly measured by an order of magnitude. If it is indeed the opposite case that there is R&D overinvestment in the US economy, then the quantitative analysis suggests that shortening the patent length would be an effective method to reduce R&D.

[16] See Jaffe (2000), Gallini (2002) and Jaffe and Lerner (2004) for a discussion on these policy changes.

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Appendix A: Proofs

Proof of Lemma 1: Combining (8), (10) and (11) yields $X_t(j) = (\alpha\eta)^{1/(1-\alpha\eta)} X_t(j')$. Substituting this condition into (12) and then (6) yields (13). For $f_t \in \{0,1\}$, s_t equals one. Simple differentiation yields

(A1) $$\frac{\partial s_t}{\partial f_t} = s_t \left[\frac{1}{\eta} \left(\frac{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} - 1}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} f_t + 1 - f_t} \right) - \alpha \left(\frac{(\alpha\eta)^{1/(1-\alpha\eta)} - 1}{(\alpha\eta)^{1/(1-\alpha\eta)} f_t + 1 - f_t} \right) \right].$$

At $f_t = \hat{f}$ (defined in Lemma 1), $\partial s_t / \partial f_t = 0$ and

(A2) $$\left. \frac{\partial^2 s_t}{\partial f_t^2} \right|_{f_t = \hat{f}} = -s_t \left[\frac{1}{\eta} \left(\frac{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} - 1}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} \hat{f} + 1 - \hat{f}} \right)^2 - \alpha \left(\frac{(\alpha\eta)^{1/(1-\alpha\eta)} - 1}{(\alpha\eta)^{1/(1-\alpha\eta)} \hat{f} + 1 - \hat{f}} \right)^2 \right] > 0.$$

Note that $(\alpha\eta)^{1/(1-\alpha\eta)} < (\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} < 1$ and $\dfrac{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} - 1}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} f_t + 1 - f_t} > \dfrac{(\alpha\eta)^{1/(1-\alpha\eta)} - 1}{(\alpha\eta)^{1/(1-\alpha\eta)} f_t + 1 - f_t}$.■

Proof of Proposition 1a ($i < i^*$)**:** From (47), the socially optimal capital-investment rate i^* is

(A3) $$i^* = \left(\frac{\alpha}{1 - k_r^*} \right) \frac{g_K + \delta}{r + \delta}.$$

From (26), the market equilibrium capital-investment rate i is

(A4) $$i = \left(\frac{\alpha\, d}{1 - k_r} \right) \frac{g_K + \delta}{r + \delta}.$$

Therefore, either (i) labor being the only input for R&D (i.e. $k_r = k_r^* = 0$) or (ii) R&D underinvestment (i.e. $k_r < k_r^*$) is *sufficient* for $i < i^*$. Recall that $d < 1$ for $T > 0$.■

Proof of Proposition 1b ($\partial i / \partial T < 0$)**:** Recall that $d(T)$ is a function of *T* and is given by

(A5) $$d(T) = \frac{(\alpha\eta)^{1/(1-\alpha\eta)} f(T) + 1 - f(T)}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)} f(T) + 1 - f(T)},$$

where $f(T) = 1 - e^{-g_V T}$. Differentiating $d(T)$ with respect to T yields

(A6) $$\frac{\partial d(T)}{\partial T} = -d(T)\underbrace{\left(\frac{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)}-1}{(\alpha\eta)^{\alpha\eta/(1-\alpha\eta)}f(T)+1-f(T)} - \frac{(\alpha\eta)^{1/(1-\alpha\eta)}-1}{(\alpha\eta)^{1/(1-\alpha\eta)}f(T)+1-f(T)}\right)}_{>0}\underbrace{\frac{\partial f(T)}{\partial T}}_{>0} < 0$$

for $T \in (0,\infty)$. Recall that the equilibrium capital-investment rate is $i(T) = \frac{\alpha\, d(T)}{1-k_r(T)}\left(\frac{g_K+\delta}{r+\delta}\right)$, and differentiating $i(T)$ with respect to T yields

(A7) $$\frac{\partial i(T)}{\partial T} = i(T)\left(\frac{1}{d(T)}\frac{\partial d(T)}{\partial T} + \frac{1}{1-k_r(T)}\frac{\partial k_r(T)}{\partial T}\right),$$

where

(A8) $$\frac{\partial k_r(T)}{\partial T} = k_r(T)[1-k_r(T)]\left(\frac{(r-g_\pi)e^{-(r-g_\pi)T}}{1-e^{-(r-g_\pi)T}} - \frac{g_V e^{-g_V T}}{1-e^{-g_V T}} - \frac{1}{1-d(T)}\frac{\partial d(T)}{\partial T}\right).$$

Substituting (A8) into (A7) yields

(A9) $$\frac{\partial i(T)}{\partial T} = i(T)\left[\left(\frac{1-d(T)[1+k_r(T)]}{d(T)[1-d(T)]}\right)\underbrace{\frac{\partial d(T)}{\partial T}}_{<0} + k_r(T)\underbrace{\left(\frac{(r-g_\pi)e^{-(r-g_\pi)T}}{1-e^{-(r-g_\pi)T}} - \frac{g_V e^{-g_V T}}{1-e^{-g_V T}}\right)}_{<0}\right],$$

where $r - g_\pi = \rho - n - (1-\sigma)g_c + g_V > g_V$. Also, note that $xe^{-xT}/(1-e^{-xT})$ is decreasing in x for $T \in (0,\infty)$. To complete the proof, we need to show that

(A10) $$1 - d(T)[1+k_r(T)] > 0.$$

Using (38), it can be shown that (A10) is equivalent to

(A11) $$\beta(2d-1)\left(\frac{1-e^{-(r-g_\pi)T}}{r-g_\pi}\right)\bigg/\left(\frac{1-e^{-g_V T}}{g_V}\right) < 1.$$

This inequality holds because (i) $\beta < 1$, (ii) $(2d-1) < 1$, and (iii) $(1-e^{-xT})/x$ is decreasing in x for $T \in (0,\infty)$. ■

Appendix B: The social rate of return to R&D

Jones and Williams (1998) define the social rate of return to R&D as the sum of the additional output produced and the reduction in R&D that is made possible by reallocating one unit of output from consumption to R&D in the current period and then reducing R&D in the next period to leave the subsequent path of technology unchanged. We rewrite the law of motion for R&D technology as

$$\text{(B1)} \qquad \dot{V}_t = G(V_t, \Psi_t) \equiv \varphi V_t^{\phi} \Psi_t^{\gamma},$$

where $\Psi_t \equiv L_{r,t}^{1-\alpha} K_{r,t}^{\alpha}$. The aggregate production function is rewritten as

$$\text{(B2)} \qquad Y_t = F(s_t, V_t, Z_t, L_{y,t}, K_{y,t}) \equiv s_t V_t^{(1-\alpha\eta)/\eta} Z_t^{1-\alpha} L_{y,t}^{1-\alpha} K_{y,t}^{\alpha}.$$

Using the above definition, Jones and Williams (1998) show that the gross social return to R&D is

$$\text{(B3)} \qquad 1+\tilde{r} = \left(\frac{\partial G}{\partial \Psi}\right)_t \left(\frac{\partial F}{\partial V}\right)_{t+1} + \frac{(\partial G/\partial \Psi)_t}{(\partial G/\partial \Psi)_{t+1}} \left(1 + \left(\frac{\partial G}{\partial V}\right)_{t+1}\right).$$

After imposing the balanced-growth conditions, the net social return to R&D becomes

$$\text{(B4)} \qquad \tilde{r} = \frac{1+g_Y}{1+g_V}\left(1 + g_V\left(\frac{1-\alpha\eta}{\eta}\frac{\gamma}{k_r} + \phi\right)\right) - 1.$$

Appendix C: Figures

Figure 1: Socially optimal R&D shares of GDP

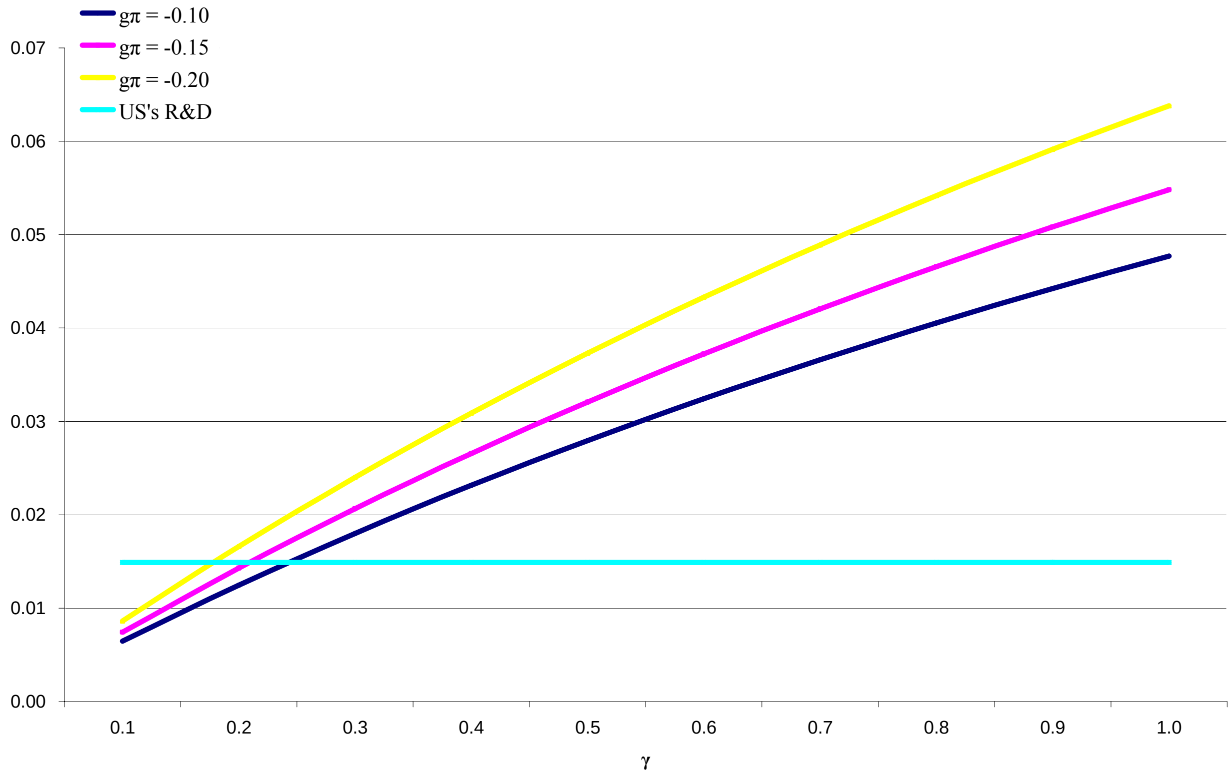

Figure 2: R&D shares of GDP at different patent length

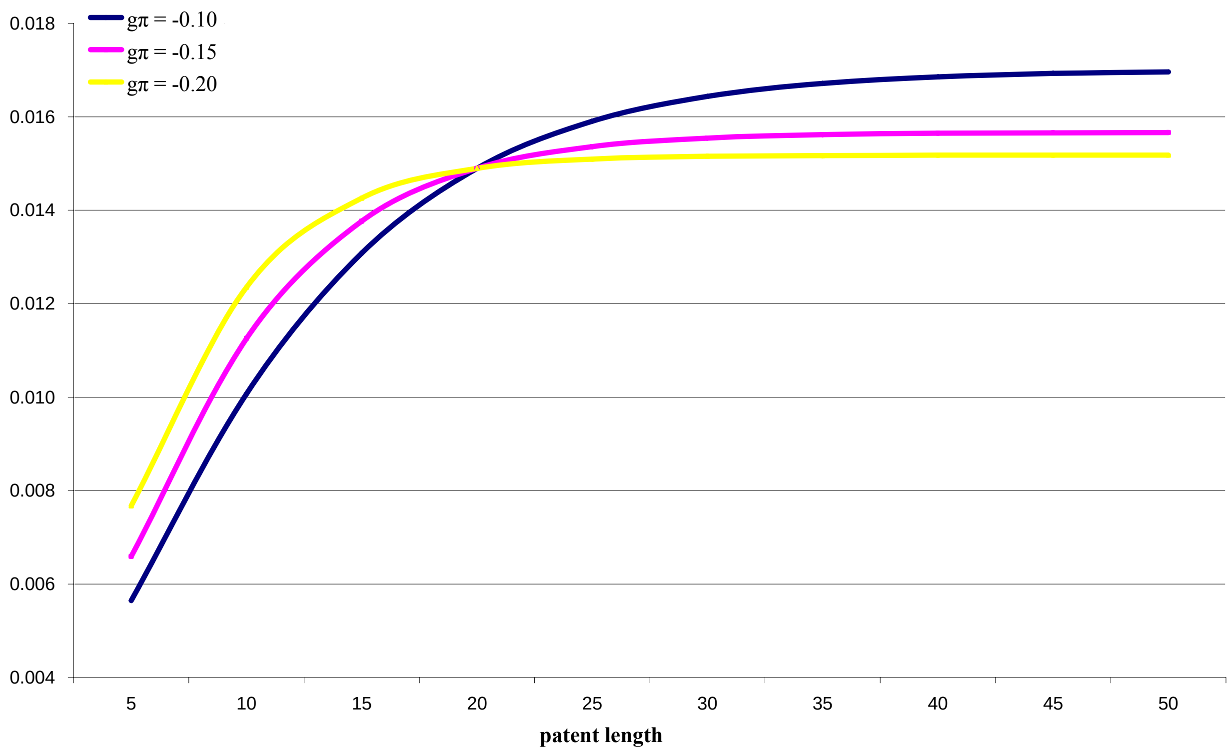

Figure 3: Percent changes in consumption

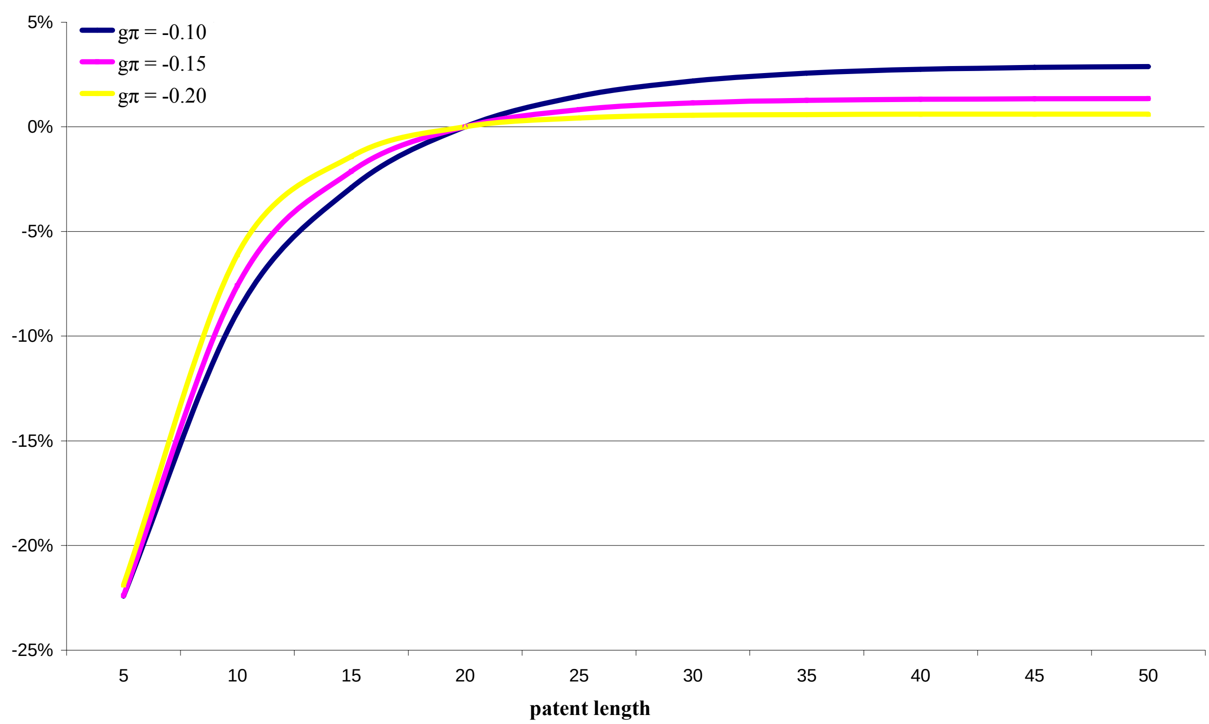

Figure 4: Socially optimal capital-investment rates

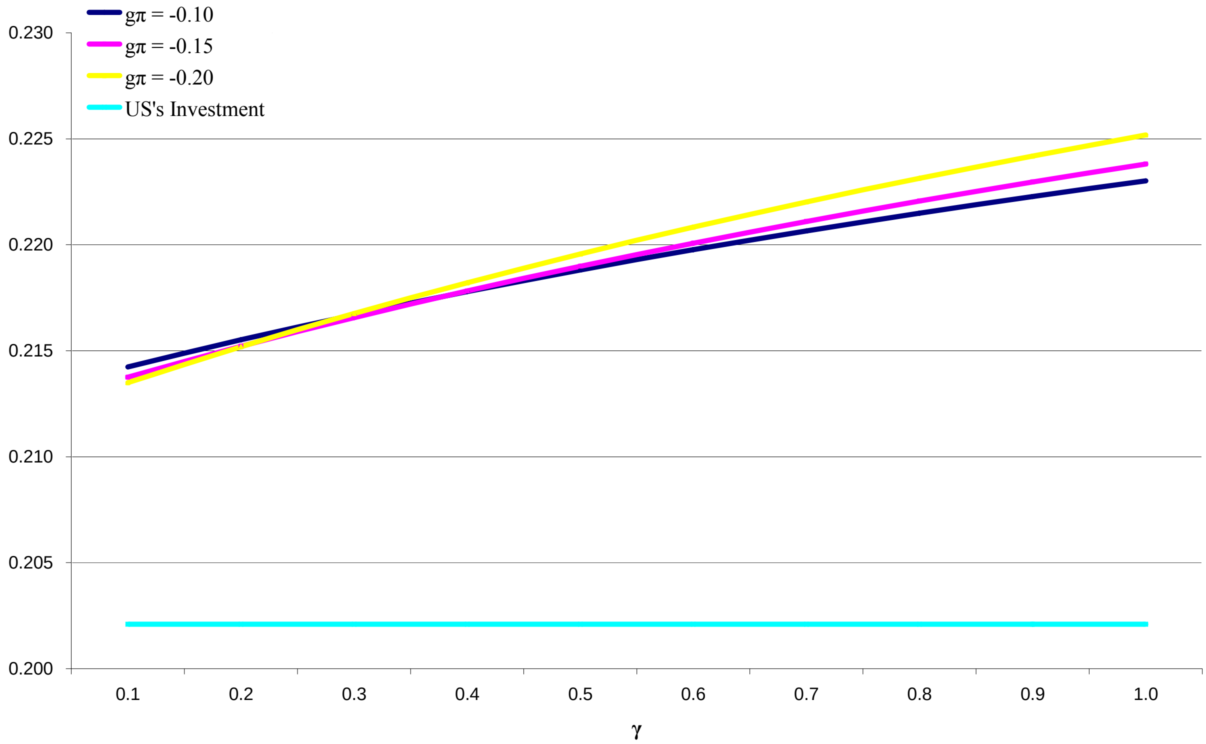

Figure 5: Equilibrium capital-investment rates at different patent length