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**A Technical Note on Comparative Dynamics**  
**in a Fiscal Competition Model**

by

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# A Technical Note on Comparative Dynamics in a Fiscal Competition Model

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This note discusses the comparative dynamic analysis in “Fiscal competition in space and time” by David Wildasin (Journal of Public Economics, Vol. 87, 2003) from a technical point of view.

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**JEL-Classification:** H21, H77, O40, C61

## 1 Introduction

Fiscal competition often occurs in the form of tax competition for a mobile tax base. It is often assumed that productive capital is the mobile tax base. Capital accumulation is a process where the stock of capital for a given point of time is the result of investment flows in the past. A natural task on the research agenda therefore is: the analysis of the dynamics of capital tax competition. Almost all models of tax competition are static and therefore their results are limited to the “long run” or steady state situations. WILDASIN’S [2003] paper shows that this limit matters when capital accumulation is a time consuming process.

He shows that a government that has to decide about the taxation of mobile capital faces a trade-off. As has been known in the literature for a long time, capital taxation hurts workers. Less capital means that labor is less productive and this in turn depresses wages. But this disadvantage from the perspective of workers matters only in the long run. Capital taxation has also beneficial effects as it generates revenue that can be redistributed in favor of workers. Wildasin shows that the speed of capital de-accumulation in response of capital taxation – i.e. how promptly the deterioration of productivity of workers comes into effect – is crucial for the decision of a government that has to weigh immediate benefits against future disadvantages.

To introduce dynamics into the model, Wildasin uses a standard adjustment cost function for investment. In his model, a convex adjustment cost function has the effect that the adjustment of the capital allocation across jurisdictions after a change in capital

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taxation is not immediate. From a normative point of view, this means that the tax rate chosen by a local jurisdiction is positive.

From a technical point of view, WILDASIN [2003] is a paper in comparative dynamics. The point that comparative static models may be misleading for the analysis of tax incidence has been made earlier by BOADWAY [1979, p. 505]: “Since the process of tax shifting through a change in capital accumulation takes time, comparative steady-state analysis may be quite inappropriate for considering the effects of tax changes over more limited time horizons.” Boadway studies “... the effect of tax changes on the growth path of an economy between two arbitrary points of time in a single-sector neoclassical model ...”. Wildasin uses a similar methodology and applies it to the problem of fiscal competition. Fiscal or tax competition here means a situation where the possibility for capital owners to shift capital to another jurisdiction results in lower tax rates than those that would occur in the case of a closed economy. The analysis in WILDASIN [2003] is about the optimal policy of a local jurisdiction, given this constraint. Wildasin does not solve for the Nash equilibrium for a system of jurisdictions. This note doesn’t do that, either.

The main motivation for this note is to state clearly the basics of comparative dynamics in a fiscal competition model in which dynamics are driven by the accumulation of capital like in traditional Ramsey-type models of optimal growth. It hopefully has helped to improve the author’s understanding of the methodology of comparative dynamics in this special context. And maybe it is useful for others that have the intention to work in the field of dynamic models in fiscal competition.

## 2 The model

For a detailed introduction of variables and the model setup, the reader is referred to WILDASIN [2003]. The model is very similar to the model of an open-economy with an exogenous interest rate that is discussed in BARRO & SALA-I-MARTIN [1995, ch. 3.5]. I will keep the description of the model setup and assumptions very brief.

A representative firm chooses its investment rate  $i(t)$  in the local capital stock  $k(t)$ . The initial value of the capital stock in  $t = 0$  is labeled  $k_0$ . The alternative investment opportunity is to invest in assets that bear an interest at rate  $r$ . The interest rate is assumed to be exogenous to the jurisdiction. The firm produces with a neoclassical production function  $f(k(t))$ , where the size of the labor force has been normalized to one. The firm’s investment decision is determined in part by the local tax rate on capital,  $\tau$ , which is assumed to be a constant flat-rate tax. Hence, the tax rate is a parameter in the dynamic optimization problem. The other major determinant of firm’s behavior is a convex adjustment cost function  $c(i(t))$ . It will turn out that it is its curvature that determines the speed of a capitalist’s reaction to a variation in the capital tax rate.

The current-value Hamiltonian for the decision problem of the firm is:

$$\mathcal{H} = f(k(t)) - c(i(t))k(t) - \tau k(t) - i(t)k(t) - w(t) + \lambda(t)(i(t) - \delta)k(t), \quad (1)$$

where  $\lambda(t)$  is the costate variable of capital,  $\delta$  is the depreciation rate and  $w(t)$  is the wage rate of workers.

Using the Maximum Principle, the process of capital accumulation by an optimizing

firm can be described as follows.

$$\dot{k}(t) = (i(t) - \delta) k(t) \quad (2a)$$

$$\dot{\lambda}(t) = -f'(k(t)) + \Psi(\lambda(t)) + \tau + \lambda(t)(r + \delta) \quad (2b)$$

$$\lambda(t) = 1 + c'(i(t)) \quad (2c)$$

$$k(0) = k_0 > 0 \text{ is given; } \lim_{t \rightarrow \infty} (\lambda(t)e^{-rt}k(t)) = 0, \quad (2d)$$

where (2c) has been used to substitute for  $\lambda$  and  $\Psi = c(i(t)) + i(t)(1 - \lambda(t)) = c(i(t)) + i(t)c'(i(t))$ . The investment rate  $i(t)$  can implicitly be determined by the first-order condition (2c). The boundary conditions in (2d) are standard.<sup>1</sup> (2a) and (2b) together are the canonical equations.

To simplify the notation, assume the following quadratic specification of adjustment costs:

$$c(i(t))k(t) = \frac{b}{2}i(t)^2k(t), \quad (3)$$

where  $c(0) = 0$ ,  $c' = bi(t)$ ,  $c'' = b$ . Parameter  $b$  can be interpreted as a measure of the mobility of capital: The lower  $b$ , the cheaper it is to adjust the capital stock. The total costs of investing one unit of capital are  $i(t)(1 + c(i(t))k(t)) = i(t)\left(1 + \frac{b}{2}i(t)^2k(t)\right)$ . This implies that the investment rate can be expressed as a function of the costate variable  $\lambda$ , using the first-order condition (2c):

$$\begin{aligned} \lambda(t) = 1 + c'(i(t)) &\Leftrightarrow \lambda(t) = 1 + bi(t) \Leftrightarrow \\ i(t) &= \frac{\lambda(t) - 1}{b}, \end{aligned} \quad (4)$$

and this in turn

$$\begin{aligned} c(i) &= \left(\frac{\lambda - 1}{b}\right)^2 \frac{b}{2} \\ c'(i) &= \lambda - 1 \\ \Psi &= \frac{(\lambda - 1)^2}{2b}. \end{aligned}$$

(2) is a system of ordinary differential equations. Its solution  $\{k(t, \tau), \lambda(t, \tau)\}$  depends on the parameter  $\tau$ . Rewrite (2) for the functional form of the adjustment cost function defined in (3) and get

$$\dot{k}(t) = \left(\frac{\lambda(t) - 1}{b} - \delta\right) k(t) \quad (5a)$$

$$\dot{\lambda}(t) = -f'(k(t)) + \tau + \lambda(t)(r + \delta) + \frac{(\lambda - 1)^2}{2b} \quad (5b)$$

$$k(0) = k_0 > 0 \text{ is given; } \lim_{t \rightarrow \infty} (\lambda(t)e^{-rt}k(t)) = 0. \quad (5c)$$

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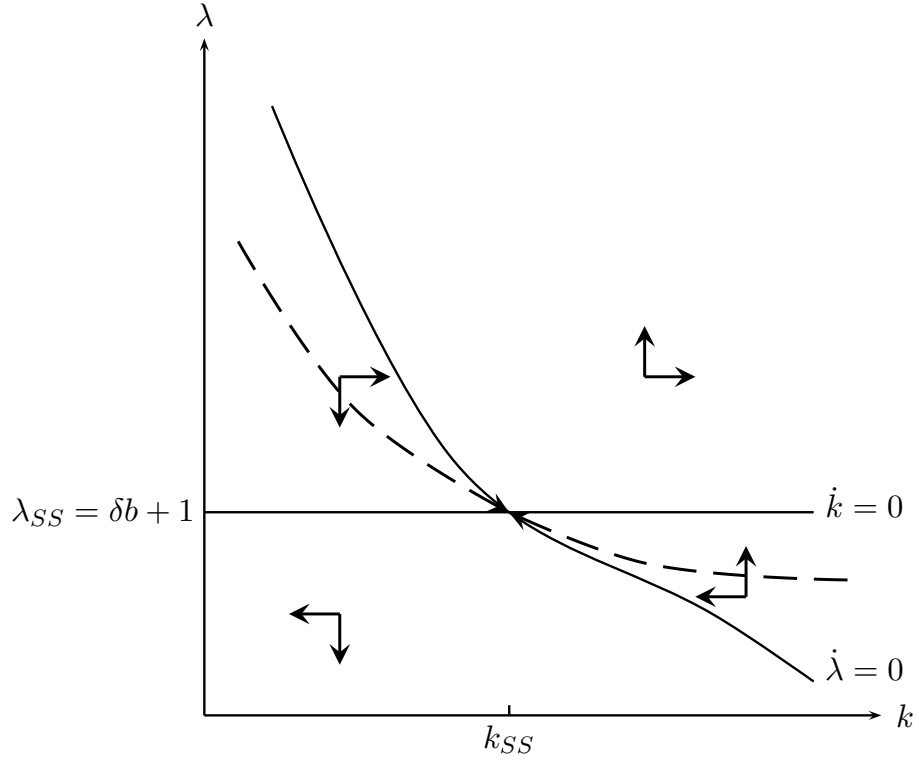
<sup>1</sup>The initial value of the capital stock,  $k_0$ , is given historically. The other condition is not an initial condition but a terminal or transversality condition. From an economic perspective, the following condition is more intuitive:  $\lim_{t \rightarrow \infty} [k(t)e^{-rt}] \geq 0$ . It states that the discounted capital stock must be non-negative at the end of time. But this can be rewritten – given that a few conditions discussed in the literature hold – as in (2d). The intuition is that the value of capital must be asymptotically zero as it would be irrational not to make use of something valuable. This is achieved if the shadow price of capital  $\lambda$  is zero asymptotically (when capital is positive for  $t \rightarrow \infty$ ). See the mathematical appendix in BARRO & SALA-I-MARTIN [1995] or CAPUTO [2005, ch. 14] for references on the topic of transversality conditions in control problems with an infinite time horizon. See also FEICHTINGER & HARTL [1986, ch. 2.6], who discuss the conditions for  $\lim_{t \rightarrow \infty} e^{-rt}\lambda(t)$  being a necessary condition for an optimal solution.

Given all parameters and the initial value of  $k_0 = k(0)$ , the economy grows towards a steady state  $\{K_{SS}(\tau), \lambda_{SS}(\tau)\}$ , that is defined by

$$\delta b + 1 = \lambda_{SS} \quad (6a)$$

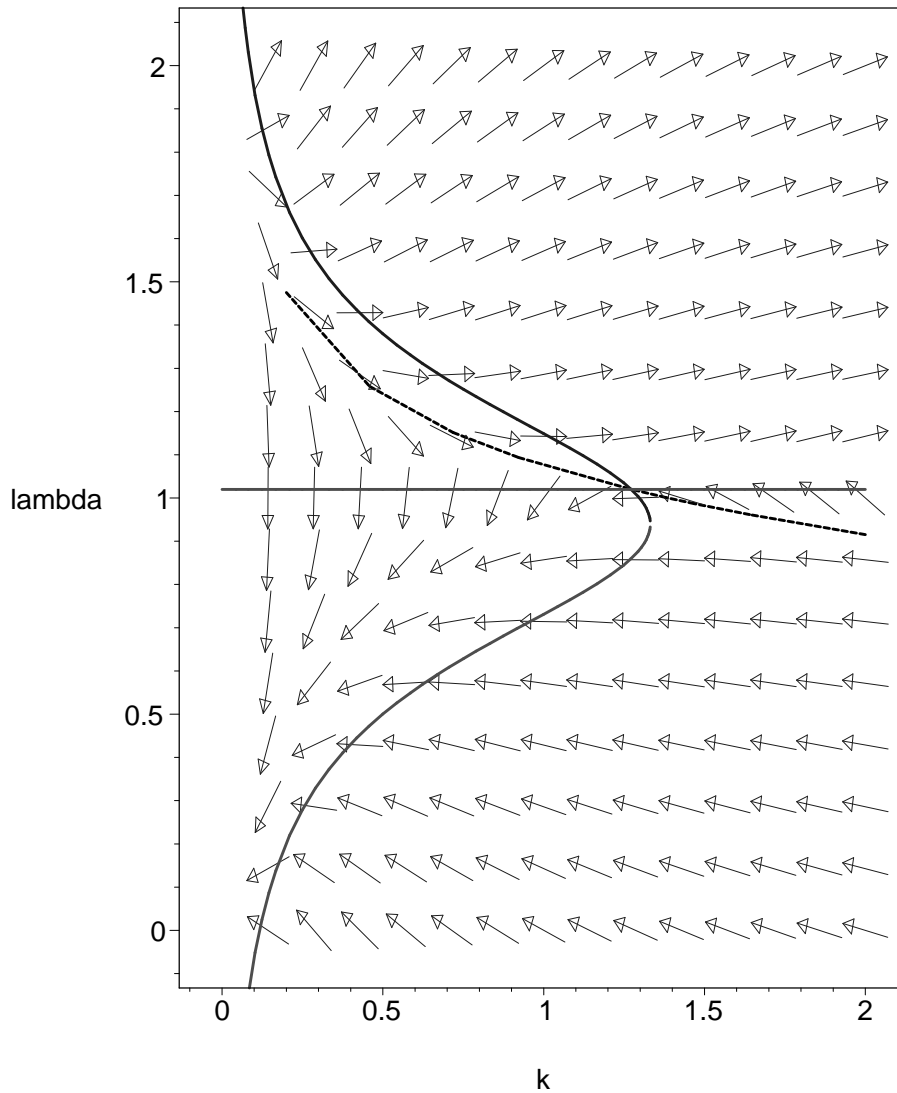
$$\tau + (\delta b + 1)(r + \delta) + \frac{\delta^2 b}{2} = f'(k_{SS}). \quad (6b)$$

The system is saddle-point stable. See figure 1, which displays the phase diagram.<sup>2</sup>



**Figure 1:** Phase Diagram

<sup>2</sup>The  $\dot{k} = 0$  locus is given from (5a) by  $\lambda = \delta b + 1$ . The  $\dot{\lambda} = 0$  locus is given implicitly from (5b) by  $0 = \frac{\lambda^2}{2b} + \lambda(r + \delta - \frac{1}{b}) - f'(k) + \tau + \frac{1}{2b}$ .



**Figure 2:** *Phase Diagram - numerical example.*

This figure shows the phase diagram for a numerical example corresponding to (5). For the production function, I use a Cobb-Douglas specification:  $f(k) = k^\alpha$  with  $\alpha = 0.3$ . The other parameters have been set to:  $\delta = 0.05$ ,  $r = 0.1$ ,  $\tau := 0.1$ ,  $b = 0.4$ . The saddle-path (indicated by the dashed line) is drawn for an initial capital stock of  $k(0) = 0.2$  (to the left from the equilibrium point) and for  $k(0) = 2$  (right). In both cases, the second condition used to define the boundary-value problem is that  $\lambda$  reaches its equilibrium value  $\lambda_{SS} = 1.475$  after 100 Periods. This defines a boundary-value problem that has been solved with numerical methods. The corresponding Maple 9.5 workfile is available from the author on request.

### 3 Comparative Dynamics

The next step is to characterize the comparative dynamic behavior of the system described in the preceding section.

In a comparative *static* analysis, one is interested in the comparison of an initial equilibrium with another equilibrium that results from the change of a parameter of the model. In a comparative *dynamic* analysis of an optimal control problem, the focus is on the effect of a parameter variation not only on the difference between an “old” steady state equilibrium and a “new” one, which corresponds to the new value of a parameter. Comparative dynamic analysis aims also to show how the optimal solution of the control problem is altered due to the parameter change.

There are several ways to perform a comparative dynamic analysis. CAPUTO [1990b] distinguishes three different approaches:

The first approach is to investigate (or assume) stability of a dynamic system and linearize it around its steady state.<sup>3</sup> For the linearized system, a closed-form solution can then be calculated and the effect of a parameter change investigated. The result is an approximation of the comparative dynamics in a small neighborhood of the steady state. This procedure is only possible for control problems with an infinite time horizon and when they are autonomous in either current-value or present-value terms.<sup>4</sup>

The second approach is to do a comparative dynamic analysis has been introduced into the economics literature by ONIKI [1973]. The idea is to make use of the Peano Theorem in the derivation of a variational differential equation system that needs to be solved in order to get the comparative dynamics of the system. WILDASIN’s [2003] analysis is an example of this approach. His model is a fortunate example as one can derive a closed-form solution for the variational equations. In cases where this is not possible, a graphical solution is an alternative. Of course graphical solutions are feasible for models with only one state variable. An advantage in comparison to the linearization approach is that the results are not limited to the effects of a (infinitesimal small) change of a parameter on the optimal solutions in the neighborhood of the steady state and is not only applicable to control problems that are of the infinite-horizon type and autonomous.

A third approach, which resembles the application of duality theory in comparative statics, has been put forward by CAPUTO [1990b]. This approach shares the advantages of the second approach of applying the Peano Theorem but has the advantage to be useful even when there is more than one state variable. CAPUTO [2003], moreover, claims that only the third approach is able to analyze closed-loop solutions of optimal control problems.<sup>5</sup>

The next subsection will present an application of the second approach. The question that will be discussed is how a small variation of a parameter (here: the tax rate  $\tau$ ) affects the optimal solution of the control problem from the last section, namely the optimal investment plan of a firm that is subject to taxation. The approach chosen allows to calculate the effects on the entire time paths of the state and costate variables.<sup>6</sup>

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<sup>3</sup>Note that (5) is not linear.

<sup>4</sup>Another approach is to rely for comparative dynamic analysis on the Laplace transforms of the endogenous variables, see JUDD [1982]. Central for this approach is the discussion of border conditions. A generalization is found in BARELLI & DE ABREU PESSÓA [2005].

<sup>5</sup>The same author has also published a textbook in optimal control problems and their analysis that highlights the application of duality-theory to comparative dynamics, see CAPUTO [2005].

<sup>6</sup>It will turn out that the effect of an increase in the tax rate on the time path of the capital stock will be negative for all future periods. See HUGGETT [2003] for conditions that ensure the monotonicity of

### 3.1 Step 1: differentiation

In the framework of the model addressed in this note, comparative dynamic analysis means to discuss the effect of a parameter-variation (here:  $d\tau$ ) on the system of differential equations (5). The solution  $\{k(t, \tau), \lambda(t, \tau)\}$  of this system – the time path from  $k = k_0$  towards the steady state – depends on time  $t$ , on the two boundary conditions (5c), and on the parameter  $\tau$ . Denote this solution by

$$\{k(t, \tau), \lambda(t, \tau)\}. \quad (7)$$

Note that there is no initial condition for  $\lambda$ . Instead, a transversality condition is used to ensure that the optimal solution of the firm's maximisation problem is on the saddle-path.

Once the solution (7) is found, one can substitute back into the system (5):

$$\dot{k}(t, \tau) = \left( \frac{\lambda(t, \tau) - 1}{b} - \delta \right) k(t, \tau) \quad (8a)$$

$$\dot{\lambda}(t, \tau) = -f'(k(t, \tau)) + \tau + \lambda(t, \tau)(r + \delta) + \frac{(\lambda(t, \tau) - 1)^2}{2b} \quad (8b)$$

$$k(0) = k_0 > 0 \text{ is given; } \lim_{t \rightarrow \infty} (\lambda(t, \tau) e^{-rt} k(t)) = 0. \quad (8c)$$

The Peano Theorem states that the solution of (8), i.e.  $\left\{ \frac{dk(t)}{d\tau}, \frac{d\lambda(t)}{d\tau} \right\}$ , satisfies a system of variational equations as it is shown below:

$$\left( \frac{dk(t)}{d\tau} \right) = \left( \frac{\lambda(t, \tau) - 1}{b} - \delta \right) \frac{dk(t)}{d\tau} + \left( \frac{k(t, \tau)}{b} \right) \frac{d\lambda(t)}{d\tau} + 0 \quad (9a)$$

$$\left( \frac{d\lambda(t)}{d\tau} \right) = -f''(k(t, \tau)) \frac{dk(t)}{d\tau} + \left( r + \delta + \frac{(\lambda(t, \tau) - 1)}{b} \right) \frac{d\lambda(t)}{d\tau} + 1 \quad (9b)$$

$$\frac{dk(0)}{d\tau} = 0, \quad \lim_{t \rightarrow \infty} \frac{d\lambda(t)}{d\tau} = 0. \quad (9c)$$

For a detailed statement of the Peano Theorem see ONIKI [1973, p. 273]. Note first that the endogenous variables in (9),  $\frac{dk(t)}{d\tau}$  and  $\frac{d\lambda(t)}{d\tau}$ , depend on time. They are the variables I am interested in: how does the time path of the state variable  $k$  and of the co-state variable  $\lambda$  change when the parameter  $\tau$  changes.

The system of variational equations, (9), is to be evaluated at the initial solution (7), the one that depends on the initial parameter value  $\tau$ . It is now ready to be solved for  $\frac{dk(t)}{d\tau}$  and  $\frac{d\lambda(t)}{d\tau}$ .

*Remark 1 (boundary conditions):* The Peano theorem also states the initial conditions of the system of variational equations. In this application it is assumed that the change of the tax-rate does not change the initial value of the state variable  $k_0 = k(0)$  as it is historically given. Nor does it change the initial time itself – time starts in  $t = 0$  regardless of a possible perpetuation of the parameter.<sup>7</sup> This said, the first of the two conditions in (9c) is a direct application of the Peano Theorem as stated in ONIKI [1973, eq. (26)].

The second condition in (8c) is an end-point condition. Assume that  $\lim_{t \rightarrow \infty} k(t) > 0$ . Given that the Inada Conditions hold, a situation where capital approaches zero cannot be an optimal investment plan. This means that the discounted shadow price of capital

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comparative dynamic statements.

<sup>7</sup>The Peano Theorem also deals with situations in which the initial value or the initial time, or both, depend on the parameter.



must approach zero. The end-point condition in (8c) is equivalent to  $\lim_{t \rightarrow \infty} \lambda(t) = 0$ . Note also that it is possible to transform any endpoint condition into a condition on the initial value of the co-state variable. In a phase diagram, the endpoint condition in (8c) makes sure that for any initial value of the state variable, the corresponding value of the co-state variable is such that the system converges to a steady state. Hence, there always exists a corresponding initial condition for an endpoint condition as in (8c).

The second condition in (9c) is the variational-equations endpoint condition corresponding to  $\lim_{t \rightarrow \infty} \lambda(t) = 0$ . If there was a condition  $\lambda_0 = \lambda(0)$  in (8c), the Peano Theorem would state that  $\frac{d\lambda(0)}{d\tau} = 0$ . The second condition shown in (9c) is the corresponding endpoint-condition.<sup>8</sup>

*Remark 2 (solvability of the variational system):* System (9) cannot be solved analytically as the entries in the Jacobian are not constants. This problem will be tackled in the next step. Note that it is a system with time-varying constants. One way to deal with this situation is to use phase diagrams, as there are only two equations. This is done in CAPUTO [1990a] and in ONIKI [1973]. To allow for an analytical solution, the next step is necessary – and this is the approach taken by BOADWAY [1979] and also in WILDASIN [2003].

### 3.2 Step 2: it is an transition between steady states

The problem with system (9) is that the entries in the Jacobian are not constant as they contain the original solutions (7) which are generally not constant over time. This would be different if I had chosen to substitute the initial system by its linearized version. But the drawback of a linearization is that the investigation of the dynamic system is then limited to a small neighborhood of the steady state. To avoid this limitation, the idea in BOADWAY [1979]; WILDASIN [2003] is to make a strong assumption, discussed in detail in the following, that has the advantage that (9) turns out to be a system with constant coefficients.

There is one special case where the original solution  $\{k(t, \tau), \lambda(t, \tau)\}$  is time independent: if the system initially has been in a steady state. This is the case if the initial value  $k_0$  for  $k$  in (5) happens to be a steady state value, labeled  $k_{SS}(\tau)$  and the corresponding value of the shadow price of capital is  $\lambda_{SS}(\tau)$ . Then the solution (7) – the solution prior to a change in the parameter – can be written as

$$\{k(t, \tau) = k_{SS}(\tau) = \text{constant}, \quad \lambda(t, \tau) = \lambda_{SS}(\tau) = \text{constant}\}.$$

The system of variational equations then is:

$$\left(\frac{dk(t, \tau)}{d\tau}\right) = 0 \frac{dk(t)}{d\tau} + \left(\frac{k_{SS}(\tau)}{b}\right) \frac{d\lambda(t)}{d\tau} \quad (10a)$$

$$\left(\frac{d\lambda_{SS}(\tau)}{d\tau}\right) = -f''(k_{SS}(\tau)) \frac{dk(t)}{d\tau} + (r + 2\delta) \frac{d\lambda(t)}{d\tau} + 1 \quad (10b)$$

$$\frac{dk_{t=0}}{d\tau} = 0, \quad \lim_{t \rightarrow \infty} \frac{d\lambda(t)}{d\tau} = 0. \quad (10c)$$

This system is now ready to be solved *analytically* for  $\frac{dk(t)}{d\tau}$  and  $\frac{d\lambda(t)}{d\tau}$ .

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<sup>8</sup>Here I follow CAPUTO [1990a, footnote 3] who claims that the uniqueness of the solution (7) implies that the terminal boundary condition can always be rewritten as an initial condition and that therefore the Peano Theorem applies. There is no need for an initial condition for  $\lambda$  or  $\frac{d\lambda}{d\tau}$ . The terminal condition in WILDASIN [2003, p. 2586] has the same effect.

### 3.3 Step 3: solving

It is relatively straightforward to verify the solution found in WILDASIN [2003]: The Jacobi-Matrix of coefficients is, given the form of the adjustment cost function,

$$\begin{bmatrix} 0 & k_{SS}/b \\ -f''(k_{SS}) & r + 2\delta \end{bmatrix}.$$

The corresponding eigenvalues are real and distinct, one of them being negative and one positive:

$$\begin{bmatrix} \rho_1 = \frac{br+2b\delta + \sqrt{b(br^2+4br\delta+4b\delta^2-4f''(k_{SS})k_{SS})}}{2b} > 0 \\ \rho_2 = \frac{br+2b\delta - \sqrt{b(br^2+4br\delta+4b\delta^2-4f''(k_{SS})k_{SS})}}{2b} < 0 \end{bmatrix}.$$

The long-term solution (particular solution) is found from (10) and is

$$\lim_{t \rightarrow \infty} \frac{d\lambda_t}{d\tau} = 0 \quad (11a)$$

$$\lim_{t \rightarrow \infty} \frac{dK_t}{d\tau} = \frac{1}{f''(k_{SS}(\tau))}. \quad (11b)$$

$$(11c)$$

The general solution then is

$$\frac{dk(t)}{d\tau} = a_1 e^{\rho_1 t} + a_2 e^{\rho_2 t} + \frac{1}{f''(k_{SS}(\tau))} \quad (12a)$$

$$\frac{d\lambda(t)}{d\tau} = a_1 e^{\rho_1 t} + a_2 e^{\rho_2 t}. \quad (12b)$$

The remaining constants  $a_1, a_2$  can be calculated from the boundary conditions (10c) and the result for the comparative-dynamic impact of a change in  $\tau$  on  $k(t)$  is

$$\frac{dk(t)}{d\tau} = \frac{1}{f''(k_{SS}(\tau))} (1 - e^{\rho_2 t}). \quad (13)$$

Note that this approach to comparative dynamics calculates the impact of a change in the parameter  $\tau$  for the entire path of the state and costate variable. (13) is not limited to the neighborhood of the initial steady state as there is no linear approximation involved. The result holds globally, see [CAPUTO, 1990a, p. 224].

Linearization of 5 and a differentiation of the linearized system would have taken us to the same result.<sup>9</sup> But the drawback would have been that (13) would be valid only in the neighborhood of the steady state. Outside this neighborhood, the marginal effect of the change of the tax rate on the evolution of the capital stock and on the optimal investment policy would be unknown. The assumption of an initial steady state allows to calculate a global solution.

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<sup>9</sup>The algebraic calculations would have been almost the same. The system of differential equations in  $dk(t)/d\tau$  and  $d\lambda(t)/d\tau$  would be linear and with constant coefficients. A linearization means that a non-linear saddle path is approximated by a linear saddle-path. The assumption of an initial steady state means that the initial conditions is a special case where the saddle path is irrelevant as there is no evolution of the state variable and the co-state variable to the equilibrium point.

**Proposition 1 (local/global)** *Result (13) holds globally if the dynamic system of investment and capital accumulation (5) has been in a steady state at time  $t = 0$  (the point in time when the tax-rate change occurs). Otherwise, it is limited to the neighborhood of the steady state.*

This does of course not mean that the closed-form solution for the time path of the capital stock is found. As in a comparative static exercise, (13) is an approximation of the reaction to a infinitesimal small change of the parameter. The actual change needs not to be small. And even if it was small<sup>10</sup> – or turned out to be small once the optimal tax rate is found – there would remain an approximation error.

It is shown in the next section how the comparative dynamic result derived above is used for the determination of the optimal tax rate by a local tax authority.

## 4 Optimal taxation

A well known result from many static models of tax competition is the following property:

$$\frac{dk}{d\tau} = \frac{1}{f''(k)} .$$

In words: an increase of the tax rate on capital by a small (marginal) amount induces an outflow of capital. This leads to a tax-base externality, see WILSON [1999]. Equation (13) above is the dynamic version of that result: In the long run, the marginal reaction of the capital stock will be the same as in a standard static setup. But it will take some time.

In the model discussed here, the strength of the tax-base externality depends on time. Initially, there is no capital flight at all and the local capital stock is a perfectly inelastic tax base. But as time goes on, firms subject to taxation reallocate capital and the tax-base externality becomes stronger. Local governments need to take that into account when they calculate optimal local tax policy. The time profile for the tax base externality matters when the marginal response is used in the first-order condition for the calculation of the optimal tax rate. Equation (8) in WILDASIN [2003] is

$$\frac{dY}{d\tau} = \int_0^{\infty} \left( \underbrace{-k_{SS} f''(k_{SS}) \frac{dk(t)}{d\tau}}_a + \underbrace{\tau \frac{dk(t)}{d\tau} + k_{SS}}_b \right) e^{-rt} dt . \quad (14)$$

The government seeks to find the tax rate that maximizes the life-time wealth,

$$Y = \int_0^{\infty} (f(k(t)) - k(t)f'(k(t)) + \tau k(t)) e^{-rt} ,$$

of worker households.<sup>11</sup> The overall effect of a marginal change in the tax rate on life-time income consists of two effects. Labeled with a) in (14), there is the effect of a marginal

<sup>10</sup>As will be seen in the next section, in the model discussed here, the tax rate a local government chooses depends on the initial capital stock. That in turn – as I have assumed that the initial situation is a steady state – depends on the initial, historically given tax rate in a way that is described in (6). One could of course imagine a situation where initial taxation is very high and hence the tax rate change on top of the historically given one is small. But it is not small in general.

<sup>11</sup>I assumed here for simplicity that the capital stock is owned by foreigners only, hence  $\theta = 0$  in Wildasin's notation. A worker households income then consists of wages and redistribution.

change in the tax rate on wage-income and, labeled with b), the effect it has on revenue from capital taxation as this is redistributed to worker households.<sup>12</sup>

The optimal tax rate chosen by a tax authority, given the initial steady state, is found by equating (14) to zero and is <sup>13</sup>

$$\tilde{\tau} = \frac{rk_{SS}f''(k_{SS})}{\rho_2} > 0. \quad (15)$$

Note that the tax rate corresponding with the initial steady state is labeled  $\tau$  and the optimal tax rate chosen by the local government, given the initial circumstances, is labeled  $\tilde{\tau}$ . If  $\tau \neq \tilde{\tau}$ , there will be a transition away from the initial steady state.

The essential point in WILDASIN [2003] is that the optimal tax rate is positive. Note that the effect b) is an approximation. A change in the tax rate means that there will be a tax rate effect (given the current tax base, revenue increases when the tax rate increases) and the tax base effect (for a given rate, revenues become smaller as the tax base shrinks). Wildasin's result can be stated also like this:

**Proposition 2 (Wildasin)** *The optimal tax rate described in (15) is always positive. It can be greater or smaller than the initial tax rate. Accordingly, there can be an inflow or an outflow of capital from time  $t = 0$  on.*

Proposition 2 follows directly from (15) and from the fact that the initial value for the local capital stock,  $k_{SS}$ , that depends on the initial tax rate  $\tau$ , can be arbitrarily chosen.

Another important point for the interpretation of (15) is to note that it depends on the initial situation (in particular on the initial capital stock and therefore also on the initial tax rate  $\tau$ ), both directly and because of the dependence of the negative Eigenvalue  $\rho_2$  on the initial capital stock. The tax rate that is chosen is not independent from history. The decision characterized in the present model is a one-shot decision, with full commitment, to set a constant capital tax rate. (15) is an open-loop strategy. The fact that it is not history independent is important if one considers the closed-loop strategy where the tax authority can decide again about the optimal tax rate in all subsequent periods. In general, a tax authority that has the possibility to revise its decision in the future will find that the capital stock is lower than initially and hence will choose another tax rate.

**Proposition 3 (time consistency)** *The optimal tax policy described in (15) is not time consistent.*

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<sup>12</sup>Here I use the same approximation as Wildasin does, namely that  $\frac{d(\tau K(t))}{d\tau} = \tau \frac{dk(t)}{d\tau} + k_{SS}$ . The approximation error stems from the difference between  $K_{ss}$  and  $k(t)$ . As will be shown in a moment, the local government strictly sets positive tax rates if capital is less than perfectly mobile. Hence  $K_{ss} > k(t)$ . The effect of a change in the tax rate on redistribution, labeled with b) in (14), is hence overstated.

<sup>13</sup>Without the approximation error mentioned in the previous footnote, the optimal tax rate would be

$$\tilde{\tau} = \frac{rk_{SS}f''(k_{SS})}{\rho_2} + \frac{\epsilon f''(k_{SS})(\rho_2 - r)}{-\rho_2},$$

where  $\epsilon = \int_0^\infty k(t) - k_{SS}e^{-rt} dt$ . Wildasin's approximation assumes implicitly  $\epsilon = 0$ . The additional term as compared with (15) is a negative number. The property that  $\tilde{\tau} > 0$  holds for the tax rate without the approximation if  $k_{SS}r$  is not too small.

This result is mainly due to the very restrictive assumption of the tax rate being constant over time. An alternative would be to let local government choose a time path of tax rates. The restriction that tax rates are equal in all periods allows to apply the Peano Theorem as taxation in this case is similar to the choice of a parameter. But this comes at the cost of time inconsistency.

The result of positive tax rates in WILDASIN [2003] can also be seen as a variant of a result known from the literature on optimal taxation. Without the restriction of a constant tax rate, optimal taxation of capital usually means to tax capital strongly initially, to choose lower tax rates in the subsequent periods and not to tax capital in the steady state. The restriction of a constant tax rate has the effect that it becomes optimal to choose some average of initially high taxes and lower taxes later on. See, for example, CHARI & KEHOE [1998].

## 5 Summary

This note has discussed in detail the comparative dynamic analysis in WILDASIN [2003]. The importance of the assumption that the economy of the jurisdiction in question is in a steady state initially has been stressed. It allows to derive a global solution for the comparative dynamics of capital taxation when capital owners can either invest in the local capital stock or in financial assets abroad. It has been argued that the optimal local policy in this model setting, where the tax rate is assumed to be constant over time, is not time-consistent.

## Appendix: The time path of the local capital stock for a linearized version of the model

An alternative strategy to solve for the comparative-dynamic properties of the model would have been to linearize around the steady state of the dynamic system (5) – the first approach mentioned in 3 on page 6. Employing the Peano Theorem for the comparative dynamics exercise discussed in this chapter has the advantage that the results are exact and there is no approximation involved. The disadvantage is that the time path of the capital stock is unknown.

Let's consider the following scenario: The economy is initially in a steady state  $\{K_{SS}(\tau), \lambda_{SS}(\tau)\}$  corresponding with an initial tax rate  $\tau$ . The government then sets a tax rate  $\tilde{\tau} > \tau$ . What is the time path of the local capital stock, from  $k_{SS}(\tau)$  to the new steady state, labeled  $\tilde{k}_{SS}(\tilde{\tau})$ ? To derive a closed solution for  $k(t)$ , there is then the possibility to use numerical methods given explicit values for all parameters. An alternative is to use a linearization technique. This appendix provides the linear approximation for the time path of  $k(t)$ , given its initial value and the optimal tax rate  $\tau$ . A textbook exposition of the linearization of a growth model of the Ramsey-type can be found, for example, in BARRO & SALA-I-MARTIN [1995, ch. 2] or BLANCHARD & FISCHER [1989, ch. 2]. Other approaches to derive analytically a closed-form solution of the Ramsey model are based on special assumptions about the structure of the model, see SMITH [2006] and references therein.

The negative, stable root of the dynamic system (5) has already been calculated as  $\rho_2$ .

The dynamics of the local capital stock around its new steady state  $\tilde{k}_{SS}$  are then given by

$$k(t) = \tilde{k}_{SS} + (k_{SS} - \tilde{k}_{SS}) e^{\rho_2 t} , \quad (.A1)$$

where  $\tilde{k}_{SS}$  is given implicitly by

$$\begin{aligned} \tilde{\tau} + (\delta b + 1)(r + \delta) + \frac{\delta^2 b}{2} &= f'(\tilde{k}_{SS}) \\ \Leftrightarrow \frac{rk_{SS}f''(k_{SS})}{\rho_2} + (\delta b + 1)(r + \delta) + \frac{\delta^2 b}{2} &= f'(\tilde{k}_{SS}) \end{aligned} \quad (.A2)$$

Once a functional form for the production function  $f(k)$  is given, the time path of the local capital stock could be calculated as a function of the initial capital stock  $r$ ,  $k_{SS}$ ,  $\rho_2(b, r, \delta, k_{SS})$ ,  $\delta$ ,  $b$ .

Note that this would also allow to calculate the time path of revenue,  $\tilde{\tau}k(t)$ . However, this would not necessarily allow a better approximation of  $d(\tau k(t))/d\tau$  than the one used above,

$$\frac{d(\tau K(t))}{d\tau} = \tau \frac{dk(t)}{d\tau} + k_{SS} ,$$

to derive (15).

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