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FULL-TRUTHFUL IMPLEMENTATION IN NASH EQUILIBRIA

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Full-Truthful Implementation in Nash Equilibria*

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Abstract

We consider *full-truthful Nash implementation*, which requires that truth telling by each agent should be a Nash equilibrium of a direct revelation mechanism, and that the set of Nash equilibrium outcomes of the mechanism should coincide with the f-optimal outcome. We show that *restricted monotonicity* together with an auxiliary condition called *boundedness* is both necessary and sufficient for full-truthful Nash implementation. We also prove that full-truthful Nash implementation is equivalent to secure implementation (Saijo et al. (2005)). This gives us an alternative characterization of securely implementable social choice functions.

Keywords: Restricted Monotonicity, Direct Revelation Mechanisms, Nash Implementation, Truthful Implementation, Secure Implementation.

1 Introduction

The implementation problem is that a mechanism designer, who cannot observe the true preferences of each agent, devises a mechanism whose equilibrium outcomes always coincide with the social goal given by a social choice correspondence. The Nash equilibrium concept has often been used as an equilibrium concept in the complete information case, where each agent knows not only own true preferences but also the true preferences of every other agent, while the mechanism designer cannot observe agents' true preferences. In the seminal paper on Nash implementation, Maskin (1999) showed that *monotonicity* is necessary for Nash implementation, and that monotonicity plus *no veto power* is sufficient for Nash implementation when there are three or more agents. The gap between necessity and sufficiency for Nash implementation has subsequently been closed by Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991), etc.

However, positive results listed above rely on complicated mechanisms used in the constructive proofs, where agents are often forced to announce an outcome, an integer, etc. in addition to a preference profile. Such mechanisms have been criticized not only for practicability but also for a theoretical reason. Jackson (1992) criticized such mechanisms for employing an integer game, because mechanisms involving the integer game fail to satisfy the *best response property* regardless of the use of the Nash equilibrium concept. So, it is important to investigate what kinds of social choice functions can be Nash implemented by "simple" mechanisms which rule out integer games.

For the simplicity of mechanisms, we consider implementation by *direct revelation mechanisms*, i.e., mechanisms where agents are required to report own preferences only. Much attention has recently focused on the mechanisms from a practical perspective as well as a theoretical viewpoint. For example, Roth (1984) analyzed a direct revelation mechanism used for the National Resident Matching Program, Abdulkadiroğlu and Sönmez (2003) studied direct revelation mechanisms for school choice, and Roth et al. (2004) proposed direct revelation mechanisms for kidney exchange. Besides, direct revelation mechanisms satisfy *self-relevancy* (see Hurwicz (1960)), the requirement that each agent should be asked to reveal information about only herself.[1] So, the mechanisms seem attractive from the viewpoint of informational decentralization. However, few would study Nash implementation by direct revelation mechanisms. This would be partly because the revelation principle for Nash implementation does not hold in general.

Nevertheless, in this paper, we restrict attention to Nash implementation by a direct revelation mechanism where truth telling by each agent is a Nash equilibrium of the mechanism, which we call *full-truthful Nash implementa-*

[1] Tatamitani (2001) considered Nash implementation by self-relevant mechanisms, where each agent is required to announce own preferences, an outcome, and an agent index.

tion. This restricts the class of implementable social choice functions, since the revelation principle for Nash implementation cannot hold. Indeed, the class of fully-truthfully Nash implementable social choice functions is limited to the class smaller than that of truthfully Nash implementable social choice functions, which is equivalent to that of truthfully dominant strategy implementable social choice functions, as shown by Dasgupta et al. (1979). However, the requirement that truthful revelation by each agent should be a Nash equilibrium of the mechanism would be attractive from a practical standpoint. If there are multiple equilibria of a direct revelation mechanism violating the requirement, then agents could be hard to predict each other's actions, which could lead to miscoordination. But, if truthful reporting by each agent is a Nash equilibrium of the mechanism, then it would be a focal point, and so agents would be able to coordinate their actions.

This paper relates to one by Saijo et al. (2005), who identified conditions necessary and sufficient for *secure implementation*,[2] i.e., double implementation in dominant strategy equilibria and Nash equilibria. We show in Section 4 that full-truthful Nash implementation is equivalent to secure implementation, which gives us an alternative characterization of securely implementable social choice functions (see Corollary 3). The equivalence derives mostly from the result of Dasgupta et al. (1979) stating that truth telling by each agent is a Nash equilibrium of a direct revelation mechanism if and only if it is a dominant strategy equilibrium of the mechanism. So, the requirement of truthful revelation by each agent being a Nash equilibrium could be agreeable, because direct revelation mechanisms violating the requirement cannot securely implement any social choice function.

This paper is organized as follows. Section 2 provides notation and definitions. We identify necessary and sufficient conditions for full-truthful Nash implementation in Section 3. In Section 4, we examine the relationship of full-truthful Nash implementation to secure implementation. Section 5 contains some concluding remarks.

2 Notation and Definitions

Let $N := \{1, 2, \ldots, n\}$ be the set of *agents*, where $2 \leq n < +\infty$. Let A be the set of feasible *outcomes*.

Each agent $i \in N$ has *preferences* over A, which are represented by a complete and transitive binary relation R_i. The strict preference relation and indifference relation associated with R_i are denoted by P_i and I_i, respectively. Let $\mathscr{R}_i$ denote the set of possible preferences for agent $i \in N$. A *domain* is denoted by $\mathscr{R} := \mathscr{R}_1 \times \mathscr{R}_2 \times \cdots \times \mathscr{R}_n$. A *preference profile* is a list $R = (R_1, R_2, \ldots, R_n) \in \mathscr{R}$. It is

[2] Cason et al. (2006) ran experiments to compare the performance of secure and non-secure mechanisms. They reported experimental results which suggest that the notion of secure implementation is important in terms of practical applications.

assumed that each agent can observe not only her own preferences but also all other agents' preferences.

An *environment* is a collection $(N, A, \mathscr{R})$.

Let $LC_i(a; R_i) := \{b \in A \mid aR_i b\}$ be agent i's *lower contour set* of $a \in A$ at $R_i \in \mathscr{R}_i$. For each agent $i \in N$, let $\arg\max_{\bar{A}} R_i := \{a \in \bar{A} \mid aR_i b \text{ for all } b \in \bar{A}\}$ be the set of *maximal elements* in $\bar{A} \subseteq A$ at $R_i \in \mathscr{R}_i$.[3]

A *social choice function* is a single-valued function $f\colon \mathscr{R} \to A$ that assigns a feasible outcome $a \in A$ to each preference profile $R \in \mathscr{R}$. Given a social choice function f, let $O_i(R) := \{a \in A \mid a = f(R'_i, R_{-i}) \text{ for some } R'_i \in \mathscr{R}_i\}$ be agent i's *option set* at $R \in \mathscr{R}$. Note that $O_i(R) = O_i(R'_i, R_{-i})$ for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$.

Let M_i denote a *message space* of agent $i \in N$. We call $m_i \in M_i$ a *message* of agent $i \in N$. A *mechanism* is a pair $\Gamma = (M, g)$, where $M := M_1 \times M_2 \times \cdots \times M_n$ and $g\colon M \to A$ is an *outcome function.* A mechanism (M, g) is called a *direct revelation mechanism* if $M_i = \mathscr{R}_i$ for all $i \in N$. Given a social choice function f, a mechanism (M, g) is called the *associated direct revelation mechanism* if $M_i = \mathscr{R}_i$ for all $i \in N$ and $g = f$. A *message profile* is denoted by $m = (m_1, m_2, \ldots, m_n) \in M$.

A message profile $m^* = (m^*_1, m^*_2, \ldots, m^*_n) \in M$ is a *Nash equilibrium* of (M, g) at $R \in \mathscr{R}$ if, for any $i \in N$, $g(m^*_i, m^*_{-i}) R_i g(m'_i, m^*_{-i})$ for any $m'_i \in M_i$. Let $NE^\Gamma(R) \subseteq M$ denote the set of Nash equilibria of $\Gamma = (M, g)$ at $R \in \mathscr{R}$. Let $g(NE^\Gamma(R)) := \{a \in A \mid a = g(m) \text{ for some } m \in NE^\Gamma(R)\}$ be the set of *Nash equilibrium outcomes* of $\Gamma = (M, g)$ at $R \in \mathscr{R}$.

A message profile $m^* = (m^*_1, m^*_2, \ldots, m^*_n) \in M$ is a *dominant strategy equilibrium* of (M, g) at $R \in \mathscr{R}$ if, for any $i \in N$, $g(m^*_i, m'_{-i}) R_i g(m'_i, m'_{-i})$ for any $m'_i \in M_i$ and any $m'_{-i} \in M_{-i}$. Let $DSE^\Gamma(R) \subseteq M$ be the set of dominant strategy equilibria of $\Gamma = (M, g)$ at $R \in \mathscr{R}$. Let $g(DSE^\Gamma(R)) := \{a \in A \mid a = g(m) \text{ for some } m \in DSE^\Gamma(R)\}$ be the set of *dominant strategy equilibrium outcomes* of $\Gamma = (M, g)$ at $R \in \mathscr{R}$.

A mechanism $\Gamma = (M, g)$ *Nash implements* a social choice function f if $g(NE^\Gamma(R)) = f(R)$ for any $R \in \mathscr{R}$.[4] A social choice function is *Nash implementable* if there exists a mechanism that Nash implements it. A social choice function is *directly implementable in Nash equilibria* if there exists a direct revelation mechanism which Nash implements it.

A direct revelation mechanism $\Gamma = (\mathscr{R}, g)$ *truthfully Nash implements* a social choice function f if $R \in NE^\Gamma(R)$ and $g(R) = f(R)$ for any $R \in \mathscr{R}$.[5] A social choice function is *truthfully implementable in Nash equilibria* if there exists a direct revelation mechanism which truthfully Nash implements it.

A direct revelation mechanism $\Gamma = (\mathscr{R}, g)$ *fully-truthfully Nash implements* a social choice function f if $R \in NE^\Gamma(R)$ and $g(NE^\Gamma(R)) = f(R)$ for any $R \in \mathscr{R}$. A social choice function is *fully-truthfully implementable in Nash equilibria* if

[3]Note that $\arg\max_{\bar{A}} R_i$ may be empty.

[4]To simplify notation, we write $f(R)$ instead of $\{f(R)\}$.

[5]As we focus on social choice functions, truthful Nash implementation can be defined as $R \in NE^\Gamma(R)$ and $g(R) = f(R)$ for any $R \in \mathscr{R}$, instead of as $R \in NE^\Gamma(R)$ and $g(R) \in f(R)$ for any $R \in \mathscr{R}$.

there exists a direct revelation mechanism which fully-truthfully implements it in Nash equilibria.

A mechanism $\Gamma = (M, g)$ *dominant strategy implements* a social choice function f if $g(DSE^{\Gamma}(R)) = f(R)$ for any $R \in \mathscr{R}$. A social choice function is *dominant strategy implementable* if there exists a mechanism that dominant strategy implements it. A direct revelation mechanism $\Gamma = (\mathscr{R}, g)$ *truthfully dominant strategy implements* a social choice function f if $R \in DSE^{\Gamma}(R)$ and $g(R) = f(R)$ for any $R \in \mathscr{R}$.

3 Characterizations

In this section, we seek to characterize social choice functions which are fully-truthfully implementable in Nash equilibria.

We begin by proving the *associated direct revelation principle*, which says that when attention is restricted to truthfully implementable social choice functions, the class of social choice functions that are directly implementable in Nash equilibria is equivalent to that of social choice functions which can be Nash implemented by the associated direct revelation mechanisms.

Theorem 1 (The Associated Direct Revelation Principle)**.** *A social choice function f is fully-truthfully implementable in Nash equilibria if and only if it is fully-truthfully implemented in Nash equilibria by the associated direct revelation mechanism.*

Proof. The if part. This part follows straight from the definition of full-truthful Nash implementation.

The only if part. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism. Since f is fully-truthfully implementable in Nash equilibria, there is a direct revelation mechanism $\bar{\Gamma} = (\mathscr{R}, g)$ such that $R \in NE^{\bar{\Gamma}}(R)$ and $g(NE^{\bar{\Gamma}}(R)) = f(R)$ for all $R \in \mathscr{R}$. Since $\bar{\Gamma}$ truthfully implements f in Nash equilibria, we have $g = f$ by definition, which implies $\bar{\Gamma} = \Gamma$. Hence, $R \in NE^{\Gamma}(R)$ and $g(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$. □

The following example demonstrates that if we give up truthful implementation in Nash equilibria, then there is a social choice function that cannot be Nash implemented by the associated direct revelation mechanism, but which is directly Nash implementable.

Example 1. Consider an environment $(N, A, \mathscr{R})$ such that $\#N = 2$, $A = \{a, b, c\}$, $\mathscr{R} = \{R_1, \bar{R}_1\} \times \{R_2, \bar{R}_2\}$, and $aP_i bP_i c$ and $c\bar{P}_i a\bar{P}_i b$ for all $i \in N$. A social choice function f is given as follows.

$$f = \begin{array}{cc|c} R_2 & \bar{R}_2 & \\ \hline a & a & R_1 \\ a & c & \bar{R}_1 \end{array}$$

Then, f cannot be Nash implemented by the associated direct revelation mechanism $\Gamma = (\mathscr{R}, f)$. This is because $f(NE^{\Gamma}(\bar{R}_1, \bar{R}_2)) = \{a, c\} \neq \{c\} = f(\bar{R}_1, \bar{R}_2)$, since $NE^{\Gamma}(\bar{R}_1, \bar{R}_2) = \{(R_1, R_2), (\bar{R}_1, \bar{R}_2)\}$.

However, another direct revelation mechanism $\bar{\Gamma} = (\mathscr{R}, g)$ can Nash implement f, where g is defined below.

$$g = \begin{array}{cc|c} R_2 & \bar{R}_2 & \\ \hline \multicolumn{1}{|c|}{c} & \multicolumn{1}{c|}{a} & R_1 \\ \hline \multicolumn{1}{|c|}{a} & \multicolumn{1}{c|}{b} & \bar{R}_1 \\ \hline \end{array}$$

Since $NE^{\bar{\Gamma}}(R_1, R_2) = \{(R_1, \bar{R}_2), (\bar{R}_1, R_2)\}$, $NE^{\bar{\Gamma}}(R_1, \bar{R}_2) = \{(\bar{R}_1, R_2)\}$, $NE^{\bar{\Gamma}}(\bar{R}_1, R_2) = \{(R_1, \bar{R}_2)\}$, and $NE^{\bar{\Gamma}}(\bar{R}_1, \bar{R}_2) = \{(R_1, R_2)\}$, we have $g(NE^{\bar{\Gamma}}(R_1, R_2)) = \{a\} = f(R_1, R_2)$, $g(NE^{\bar{\Gamma}}(R_1, \bar{R}_2)) = \{a\} = f(R_1, \bar{R}_2)$, $g(NE^{\bar{\Gamma}}(\bar{R}_1, R_2)) = \{a\} = f(\bar{R}_1, R_2)$, and $g(NE^{\bar{\Gamma}}(\bar{R}_1, \bar{R}_2)) = \{c\} = f(\bar{R}_1, \bar{R}_2)$, respectively. Thus, f can be Nash implemented by the direct revelation mechanism $\bar{\Gamma}$, although it cannot be Nash implemented by the associated direct revelation mechanism Γ. Note that f cannot be truthfully implemented in Nash equilibria by $\bar{\Gamma}$, although it can be truthfully implemented in Nash equilibria by Γ. □

Invoking the associated direct revelation principle, we restrict attention to the associated direct revelation mechanisms hereafter. We next identify a condition which is necessary for full-truthful Nash implementation by the associated direct revelation mechanisms.

Restricted monotonicity is a version of *monotonicity*[6] (Maskin (1999)), which requires the following. Suppose a change from R to R'. Then, for each agent $i \in N$, if any outcome that was weakly worse for her than $f(R)$ *in her option set* at R when her preference relation was R_i remains weakly worse for her than $f(R)$ when her preference relation is R'_i, then $f(R)$ must still be f-optimal at R'.

Definition 1 (Restricted Monotonicity)**.** A social choice function f satisfies *restricted monotonicity* if, for all $R, R' \in \mathscr{R}$, if $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$.

Remark 1. Restricted monotonicity is stronger than monotonicity by definition.

Remark 2. The definition of restricted monotonicity can be rewritten as follows. A social choice function f satisfies *restricted monotonicity* if, for all $R \in \mathscr{R}$ and all $i \in N$, there is a set $C_i(f(R); R) \subseteq O_i(R)$ with $f(R) \in \arg\max_{C_i(f(R);R)} R_i$ such that for all $R' \in \mathscr{R}$, if $f(R) \in \arg\max_{C_i(f(R);R)} R'_i$ for all $i \in N$, then $f(R') = f(R)$. This way of defining restricted monotonicity is analogous to that of defining Condition μ (Moore and Repullo (1990)).

[6]A social choice function f satisfies *monotonicity* if, for all $R, R' \in \mathscr{R}$, if $LC_i(f(R); R_i) \subseteq LC_i(f(R); R'_i)$ for all $i \in N$, then $f(R') = f(R)$. Note that monotonicity is a necessary condition for Nash implementation and is part of the sufficient condition for Nash implementation when there are three or more agents.

Since restricted monotonicity is stronger than monotonicity by Remark 1, it is not clear whether restricted monotonicity is a necessary condition for Nash implementation. The following lemma states that restricted monotonicity is necessary for full-truthful implementation in Nash equilibria by the associated direct revelation mechanisms.

Lemma 1. *If a social choice function f is fully-truthfully implemented in Nash equilibria by the associated direct revelation mechanism, then it satisfies restricted monotonicity.*

Proof. Pick any $R, \bar{R} \in \mathscr{R}$ such that $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism. Since f is truthfully implemented in Nash equilibria by Γ, we have $R \in NE^{\Gamma}(R)$.

Since $R \in NE^{\Gamma}(R)$, it holds that for all $i \in N$, $f(R) R_i f(R'_i, R_{-i})$ for all $R'_i \in \mathscr{R}_i$. This implies $f(R) \in \arg\max_{O_i(R)} R_i$ for all $i \in N$. So, $LC_i(f(R); R_i) \cap O_i(R) = O_i(R)$ for all $i \in N$.

Thus, since $LC_i(f(R); R_i) \cap O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$, we have $O_i(R) \subseteq LC_i(f(R); \bar{R}_i)$ for all $i \in N$. So, $R \in NE^{\Gamma}(\bar{R})$. Since f is Nash implemented by Γ, $f(R) \in f(NE^{\Gamma}(\bar{R})) = f(\bar{R})$. This implies $f(\bar{R}) = f(R)$, because f is a single-valued function. □

We are now ready to characterize fully-truthfully implementable social choice functions in Nash equilibria. Theorem 2 below says that restricted monotonicity together with an auxiliary condition called *boundedness* is both necessary and sufficient for full-truthful Nash implementation. It should be noted that Theorem 2 holds even when $n = 2$.

Definition 2 (Boundedness)**.** A social choice function f satisfies *boundedness* if $\arg\max_{O_i(R)} R_i \neq \emptyset$ for all $R \in \mathscr{R}$ and all $i \in N$.

Remark 3. As we focus on implementation by the associated direct revelation mechanisms, imposing boundedness on a social choice function is equivalent to requiring the associated direct revelation mechanism to satisfy the *best response property*[7] (Jackson et al. (1994)). As mentioned by Jackson et al. (1994), the best response property would be an appropriate restriction in order for the Nash equilibrium concept to make sense. Theorem 2 shows that the restriction is not only sufficient but also necessary for a social choice function to be fully-truthfully implementable in Nash equilibria.

Theorem 2. *A social choice function f is fully-truthfully implementable in Nash equilibria if and only if it satisfies restricted monotonicity and boundedness.*

Proof. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism.

The if part. Step 1: $f(R) \in \arg\max_{O_i(R)} R_i$ for all $R \in \mathscr{R}$ and all $i \in N$.

[7] A mechanism (M, g) satisfies the *best response property* if, for all $R \in \mathscr{R}$, all $i \in N$, and all $m_{-i} \in M_{-i}$, there exists $m_i \in M_i$ such that $g(m_i, m_{-i}) R_i g(m'_i, m_{-i})$ for all $m'_i \in M_i$.

Suppose to the contrary that $f(R) \notin \arg\max_{O_i(R)} R_i$ for some $R \in \mathscr{R}$ and some $i \in N$. Let $b \in A$ be such that $b \in \arg\max_{O_i(R)} R_i$.[8] Then, $b \neq f(R)$. Since $b \in O_i(R)$, $b = f(\bar{R}_i, R_{-i})$ for some $\bar{R}_i \in \mathscr{R}_i$.

Since $f(\bar{R}_i, R_{-i}) = b \in \arg\max_{O_i(R)} R_i$, it holds that $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(R) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(R)$. So, it follows from $O_i(R) = O_i(\bar{R}_i, R_{-i})$ that $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(\bar{R}_i, R_{-i})$. This implies $LC_i(f(\bar{R}_i, R_{-i}); \bar{R}_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i)$, because $LC_i(f(\bar{R}_i, R_{-i}); R_i) \cap O_i(\bar{R}_i, R_{-i}) \subseteq LC_i(f(\bar{R}_i, R_{-i}); R_i)$. Thus, since $LC_j(f(\bar{R}_i, R_{-i}); R_j) \cap O_j(\bar{R}_i, R_{-i}) \subseteq LC_j(f(\bar{R}_i, R_{-i}); R_j)$ for all $j \neq i$, restricted monotonicity implies $f(R) = f(\bar{R}_i, R_{-i})$, which contradicts $f(R) \neq b = f(\bar{R}_i, R_{-i})$.

Step 2: f satisfies *strategy-proofness*.[9]

Since $f(R) \in \arg\max_{O_i(R)} R_i$ for all $R \in \mathscr{R}$ and all $i \in N$ by Step 1, it follows from the definition of the option set that $f(R) R_i f(R'_i, R_{-i})$ for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$. Thus, f satisfies strategy-proofness.

Step 3: f is fully-truthfully Nash implementable.

Pick any $R \in \mathscr{R}$. Since f satisfies strategy-proofness by Step 2, it is truthfully implemented in dominant strategy equilibria by Γ, i.e., $R \in DSE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. So, $R \in NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$.

Suppose $\bar{R} \in NE^{\Gamma}(R)$. Then, for any $i \in N$, $f(\bar{R}) R_i f(R'_i, \bar{R}_{-i})$ for any $R'_i \in \mathscr{R}_i$. This implies $f(\bar{R}) \in \arg\max_{O_i(\bar{R})} R_i$ for all $i \in N$, implying $LC_i(f(\bar{R}); R_i) \cap O_i(\bar{R}) = O_i(\bar{R})$ for all $i \in N$. So, since $LC_i(f(\bar{R}); \bar{R}_i) \cap O_i(\bar{R}) \subseteq O_i(\bar{R})$ for all $i \in N$, we have $LC_i(f(\bar{R}); \bar{R}_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R}); R_i) \cap O_i(\bar{R})$ for all $i \in N$. This implies $LC_i(f(\bar{R}); \bar{R}_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R}); R_i)$ for all $i \in N$, since $LC_i(f(\bar{R}); R_i) \cap O_i(\bar{R}) \subseteq LC_i(f(\bar{R}); R_i)$ for all $i \in N$. Therefore, restricted monotonicity implies $f(R) = f(\bar{R})$. So, $f(R) = f(\bar{R})$ for any $\bar{R} \in NE^{\Gamma}(R)$. This implies $f(NE^{\Gamma}(R)) = f(R)$. Thus, $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$.

The only if part. By Theorem 1, if f is fully-truthfully Nash implementable, then it is fully-truthfully Nash implemented by Γ. So, Lemma 1 implies that f satisfies restricted monotonicity.

Since f is fully-truthfylly Nash implemented by Γ, it holds that $R \in NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$, i.e., $f(R) R_i f(R'_i, R_{-i})$ for all $R \in \mathscr{R}$, all $i \in N$, and all $R'_i \in \mathscr{R}_i$. This implies $f(R) \in \arg\max_{O_i(R)} R_i$ for all $R \in \mathscr{R}$ and all $i \in N$. Thus, f satisfies boundedness. □

We end this section by briefly discussing the redundancy of boundedness in characterizing fully-truthfully implementable social choice functions in Nash equilibria. The following is due to Dasgupta et al. (1979).

[8] It should be noted that $\arg\max_{O_i(R)} R_i \neq \emptyset$ by boundedness.

[9] A social choice function f satisfies *strategy-proofness* if, for all $R \in \mathscr{R}$ and all $i \in N$, there is no $R'_i \in \mathscr{R}_i$ such that $f(R'_i, R_{-i}) P_i f(R)$.

Proposition 1 (Dasgupta et al. (1979))**.** *Suppose that $\mathscr{R}$ is rich.*[10] *Then, if a social choice function satisfies monotonicity, then it satisfies strategy-proofness.*

Proposition 1 together with Remark 1 implies that restricted monotonicity implies strategy-proofness if $\mathscr{R}$ is rich. So, restricted monotonicity implies boundedness if $\mathscr{R}$ is rich. Moreover, if A is finite, then boundednes is automatically satisfied by the completeness and transitivity of preferences regardless of whether or not $\mathscr{R}$ is rich. Thus, we have the following corollary.

Corollary 1. *Suppose that either (i) A is finite or (ii) $\mathscr{R}$ is rich. Then, a social choice function is fully-truthfully implementable in Nash equilibria if and only if it satisfies restricted monotonicity.*

4 The Relationship to Secure Implementation

In this section, we explore the relationship of full-truthful Nash implementation to *secure implementation*[11] (Saijo et al. (2005)), which is identical with double implementation in dominant strategy equilibria and Nash equilibria.

We begin by showing a limitation of full-truthful Nash implementation, which stems mainly from Proposition 2 (Dasgupta et al. (1979)).

Proposition 2 (Dasgupta et al. (1979))**.** *A social choice function is truthfully implemented in Nash equilibria by a direct revelation mechanism if and only if it is truthfully implemented in dominant strategy equilibria by the same direct revelation mechanism.*

Theorem 3. *If a social choice function f is fully-truthfully Nash implemented by the associated direct revelation mechanism, then it is dominant strategy implemented by the associated direct revelation mechanism.*

Proof. Let $\Gamma = (\mathscr{R}, f)$ denote the associated direct revelation mechanism. Since f is truthfully implemented in Nash equilibria by Γ, Proposition 2 implies that it is truthfully implemented in dominant strategy equilibria by Γ. So, $R \in DSE^{\Gamma}(R)$ for all $R \in \mathscr{R}$, implying $R \in DSE^{\Gamma}(R) \subseteq NE^{\Gamma}(R)$ for all $R \in \mathscr{R}$. This implies $f(DSE^{\Gamma}(R)) \subseteq f(NE^{\Gamma}(R))$ for all $R \in \mathscr{R}$. Thus, $f(DSE^{\Gamma}(R)) = f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$, because $f(NE^{\Gamma}(R)) = f(R)$ for all $R \in \mathscr{R}$, and because f is a single-valued function. $\square$

The following example demonstrates that the converse to Theorem 3 does not hold in general, which is in contrast to Proposition 2.

[10] A domain $\mathscr{R}$ is *rich* (Dasgupta et al. (1979)) if, for any $R, R' \in \mathscr{R}$, any $a, a' \in A$, and any $i \in N$, if $aR_i a'$ implies $aR'_i a'$ and $aP_i a'$ implies $aP'_i a'$, then there exists $R'' \in \mathscr{R}$ such that $LC_i(a; R_i) \subseteq LC_i(a; R''_i)$ and $LC_i(a'; R'_i) \subseteq LC_i(a'; R''_i)$. Examples of rich domains are found in Dasgupta et al. (1979).

[11] A mechanism $\Gamma = (M, g)$ *securely implements* a social choice function f if $g(DSE^{\Gamma}(R)) = g(NE^{\Gamma}(R)) = f(R)$ for any $R \in \mathscr{R}$. A social choice function f is *securely implementable* if there exists a mechanism which securely implements it.

Example 2. Consider an environment $(N, A, \mathscr{R})$ for which $\#N = 2$, $A = \{a, b, c\}$, and $\mathscr{R} = \{R_1, \bar{R}_1\} \times \{R_2, \bar{R}_2\}$, where aP_1cP_1b, $b\bar{P}_1c\bar{P}a$, aI_2bP_2c, and $a\bar{P}_2b\bar{I}_2c$. A social choice function f is defined below.

$$f = \begin{array}{cc|c} R_2 & \bar{R}_2 & \\ \hline b & a & R_1 \\ b & c & \bar{R}_1 \end{array}$$

Then, it is easy to check that f can be dominant strategy implemented by the associated direct revelation mechanism, but cannot be fully-truthfully Nash implemented by the associated direct revelation mechanism. □

Theorem 3 together with the revelation principle leads to the following corollary, which indicates the relationship of full-truthful Nash implementation to secure implementation.

Corollary 2. *A social choice function is securely implemented by the associated direct revelation mechanism if and only if it is fully-truthfully Nash implemented by the associated direct revelation mechanism.*

Corollary 3 below follows directly from Theorems 1 and 2, Corollary 2, and results of Saijo et al. (2005) for secure implementation.

Corollary 3. *The following statements are equivalent:*

(i) f is fully-truthfully implementable in Nash equilibria,

(ii) f is securely implementable,

(iii) f is robustly implemented in Bayesian Nash equilibria by the associated direct revelation mechanism,

(iv) f satisfies restricted monotonicity and boundedness,

(v) f satisfies the rectangular property[12] *(Saijo et al. (2005)) and strategy-proofness.*

Corollary 3 tells us that full-truthful Nash implementation is equivalent to secure implementation, which sheds light on the structure of secure implementation. Corollary 3 also provides an alternative characterization of securely implementable social choice functions. In contrast to the characterization by Saijo et al. (2005), our characterization has an advantage of using a version of monotonicity, which is familiar to literature on implementation theory.

[12]A social choice function f satisfies the *rectangular property* if, for all $R, R' \in \mathscr{R}$, if $f(R') I_i f(R_i, R'_{-i})$ for all $i \in N$, then $f(R') = f(R)$.

5 Conclusion

In this paper, we have shown that restricted monotonicity together with boundedness is both necessary and sufficient for full-truthful Nash implementation. Moreover, by proving the equivalence of full-truthful Nash implementation and secure implementation, we have provided an alternative characterization of securely implementable social choice functions. Our characterization sheds new light on the structure of securely implementable social choice functions in terms of monotonicity, a well-known property in implementation theory.

It is true that the requirement of truth telling by each agent being a Nash equilibrium is appealing. But, as demonstrated in Example 1, the requirement restricts the class of Nash implementable social choice functions by direct revelation mechanisms. Direct revelation mechanisms have been recently received a great deal of attention from both theoretical and practical viewpoints. So, an interesting topic for further research is to identify necessary and sufficient conditions for Nash implementation by direct revelation mechanisms.

References

ABDULKADIROĞLU, A. AND T. SÖNMEZ (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

CASON, T. N., T. SAIJO, T. SJÖSTRÖM, AND T. YAMATO (2006): "Secure Implementation Experiments: Do Strategy-Proof Mechanisms Really Work?" forthcoming in *Games and Economic Behavior*.

DASGUPTA, P., P. J. HAMMOND, AND E. MASKIN (1979): "The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility," *Review of Economic Studies*, 46, 185–216.

DUTTA, B. AND A. SEN (1991): "A Necessary and Sufficient Condition for Two-Person Nash Implementation," *Review of Economic Studies*, 58, 121–128.

HURWICZ, L. (1960): "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical Methods in the Social Sciences*, ed. by K. J. Arrow, S. Karlin, and P. Suppes, Stanford University Press, 27–46.

JACKSON, M. O. (1992): "Implementation in Undominated Strategies: A Look at Bounded Mechanisms," *Review of Economic Studies*, 59, 757–775.

JACKSON, M. O., T. R. PALFREY, AND S. SRIVASTAVA (1994): "Undominated Nash Implementation in Bounded Mechanisms," *Games and Economic Behavior*, 6, 474–501.

MASKIN, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66, 23–38.

MOORE, J. AND R. REPULLO (1990): "Nash Implementation: A Full Characterization," *Econometrica*, 58, 1083–1099.

ROTH, A. E. (1984): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory," *Journal of Political Economy*, 92, 991–1016.

ROTH, A. E., T. SÖNMEZ, AND M. U. ÜNVER (2004): "Kidney Exchange," *Quarterly Journal of Economics*, 119, 457–488.

SAIJO, T., T. SJÖSTRÖM, AND T. YAMATO (2005): "Secure Implementation," mimeo., Osaka University.

SJÖSTRÖM, T. (1991): "On the Necessary and Sufficient Conditions for Nash Implementation," *Social Choice and Welfare*, 8, 333–340.

TATAMITANI, Y. (2001): "Implementation by Self-Relevant Mechanisms," *Journal of Mathematical Economics*, 35, 427–444.