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# AN IMPOSSIBILITY THEOREM IN MATCHING PROBLEMS

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# An Impossibility Theorem in Matching Problems*

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**Summary.** This paper studies the possibility of strategy-proof rules yielding satisfactory solutions to matching problems. Alcalde and Barberá (1994) show that efficient and individually rational matching rules are manipulable in the one-to-one matching model. We pursue the possibility of strategy-proof matching rules by relaxing efficiency to the weaker condition of *respect for unanimity*. Our first result is positive. We prove that *a strategy-proof rule exists that is individually rational and respects unanimity.* However, this rule is unreasonable in the sense that a pair of agents who are the best for each other are matched on only rare occasions. In order to explore the possibility of better matching rules, we introduce the natural condition of "respect for pairwise unanimity." *Respect for pairwise unanimity* states that a pair of agents who are the best for each other should be matched, and an agent wishing to stay single should stay single. Our second result is negative. We prove that *no strategy-proof rule exists that respects pairwise unanimity.* This result implies Roth (1982) showing that stable rules are manipulable. We then extend this to the many-to-one matching model.

*JEL classification:* C78; D71; D78

*Keywords:* Matching problems; Strategy-proofness

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## 1. Introduction

We study the possibility of designing strategy-proof rules that yield satisfactory solutions to matching problems. By matching problems, we refer to the several important allocation problems in two-sided matching markets where agents, from the start, belong to one of two disjoint sets: for example, workers and firms, students and colleges, and athletes and teams. Allocations in these markets are matchings, assigning each agent in one side of the market the agent(s) in the other side.

A *matching rule* chooses a matching for each preference profile. A matching rule is *efficient* if it always chooses a matching such that no other matching exists that would make all agents better off. A matching rule is *individually rational* if an agent is never assigned to a partner to whom the agent prefers staying single. Individual rationality is necessary for agents to voluntarily participate matchings. A matching is *blocked* by a pair if each agent in the pair prefers the other in it to the assigned partner. A matching rule is *stable* if a matching rule is individually rational, and for any preference profile, the chosen matching is not blocked by any pair. Stability guarantees the rights of all agents in the sense of not compelling them into unacceptable matches.

Because the agents' preferences are not known to others, there may be incentives for agents to misrepresent their preferences in order to manipulate the final outcome in their favor. As a result, the chosen matching may not be socially desirable relative to the agents' true preferences. Therefore, matching rules need to be immune to such strategic misrepresentation to certainly choose desirable matchings based on agents' true preferences. A matching rule is *strategy-proof* if it is a dominant strategy for each agent to announce its true preference.

The possibility of matching rules satisfying desirable properties has been explored by many studies. Gale and Shapley (1962) prove that a stable rule, called the "Gale–Shapley mechanism", exists. Roth (1982) shows that all stable matching rules containing the Gale–Shapley mechanism are not strategy-proof. Alcalde and Barberá (1994) pursue the possibility of a strategy-proof rule by relaxing stability to efficiency and individual rationality, and show the impossibility of designing matching rules satisfying efficiency, individually rationality and strategy-proofness.

In this paper, we pursue the possibility of a strategy-proof matching rule by relaxing efficiency or employing a substitutive concept. A preference profile is *unanimous* if, unless an agent prefers to stay single, the partner the agent most prefers also prefers the agent. A matching rule *respects unanimity* if for any unanimous preference profile, every agent is matched to the partner the agent prefers most. Our first result is positive. We prove that *there exists a strategy-proof rule that is individually rational and respects unanimity.* However, this rule is unreasonable in the sense that a pair of agents who are the best for each other are matched on only rare occasions.

In order to explore the possibility of better matching rules, we introduce a natural condition, which we call "respect for pairwise unanimity". A matching rule *respects pairwise unanimity* if a pair of agents who are the best for each other should be matched, and an agent wishing to stay single should stay single. Compared with stability, respect for pairwise unanimity "weakly"guarantees the rights of all agents. Our second result is negative. We prove that *there exists no strategy-proof rule that respects pairwise unanimity.* Since stability implies respect for pairwise unanimity, this result implies Roth's (1982) negative result.

Section 2 introduces the one-to-one matching model and presents our results. Section 3 extends the negative result of the one-to-one matching model to the many-to-one matching model. Section 4 concludes.

## 2. One-to-One Matchings

### *2.1. One-to-one matching model*

Here, we consider the one-to-one matching model, known as the marriage problem. Let $M = \{m_1, m_2, \ldots, m_n\}$ be the set of men, and $W = \{w_1, w_2, \ldots, w_l\}$ be the set of women. We assume that both $M$ and $W$ are finite and disjoint sets. We also assume that $n \geq 2$ and $l \geq 2$.

Each $m_i \in M$ has a preference relation $P(m_i)$ on $W \cup \{m_i\}$. For each man $m_i \in M$, the alternative $m_i$ implies that $m_i$ stays single. We assume that preferences are strict. For $x$, $x' \in W \cup \{m_i\}$, $xP(m_i)x'$ means that $m_i$ prefers $x$ to $x'$. Each woman $w_j \in W$ has a similar preference $P(w_j)$ on $M \cup \{w_j\}$. Let $\mathcal{P}(m_i)$ denote the set of all possible preferences for $m_i$, and let $\mathcal{P}(w_j)$ denote the set of all possible preferences for $w_j$. We denote preference profiles by $P$. Let $\mathcal{P} = \prod_{i=1}^{n} \mathcal{P}(m_i) \times \prod_{j=1}^{l} \mathcal{P}(w_j)$ be the set of all possible preference profiles. Given a profile $P$, an agent $x \in M \cup W$ and a preference $P'(x)$, we denote by $P/P'(x)$ the profile obtained from $P$ by changing the preferences of $x$ from $P(x)$ to $P'(x)$, and keeping all other preferences unchanged. For all preferences $P(m_i) \in \mathcal{P}(m_i)$, $b(P(m_i))$ denotes the most preferred element in $W \cup \{m_i\}$. Similarly, for all preferences $P(w_j) \in \mathcal{P}(w_j)$, $b(P(w_j))$ denotes the most preferred element in $M \cup \{w_j\}$.

A (one-to-one) *matching* is a function $a : M \cup W \to M \cup W$ such that

(1) $\quad [a(m_i) \notin W \Rightarrow a(m_i) = m_i]$ and $[a(w_j) \notin M \Rightarrow a(w_j) = w_j]$, and

(2) $\quad a(m_i) = w_j \Leftrightarrow a(w_j) = m_i$.

Condition (1) requires that individuals who are not matched with agents of the opposite set must stay single. Condition (2) requires that if a man $m_i$ is matched to a woman $w_j$, then this woman $w_j$ should be matched to that man $m_i$.

Let $A$ be the set of all possible matchings.

**Definition.** A matching $a$ is *(Pareto) efficient* at preference profile $P$ if there does not exist another matching $a' \neq a$ such that for all $x \in M \cup W$,

$$a'(x) \neq a(x) \Rightarrow a'(x)P(x)a(x).$$

**Definition.** A matching $a$ is *individually rational* at profile $P$ if each individual who is matched prefers her or his partner to staying single; i.e.,

$$[a(m_i) \in W \Rightarrow a(m_i)P(m_i)m_i] \text{ for all } m_i \in M, \text{ and}$$

$$[a(w_j) \in M \Rightarrow a(w_j)P(w_j)w_j] \text{ for all } w_j \in W.$$

**Definition.** A matching $a$ is *blocked by a pair* $(m_i, w_j) \in M \times W$ at profile $P$ if $w_j P(m_i) a(m_i)$ and $m_i P(w_j) a(w_j)$. A matching $a$ is *stable* at profile $P$ if it is individually rational and it is not blocked by any pair in $M \times W$.

A *matching rule* on $\mathcal{P}$ is a function $f$ from $\mathcal{P}$ to $A$.

In this paper, we title the result of the following algorithm[1], the *M-optimal matching rule*:

*Step 1.* (a) Each man proposes to his most preferred woman.

(b) Each woman rejects the proposal of any man to whom she prefers staying single. Each woman who receives more than one proposal rejects all but her most preferred. Any man whose proposal is not rejected at this point is kept engaged.

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*Step k.* (a) Any man who was rejected in the previous step proposes to his most preferred woman among those who have not yet rejected him, so long as a woman remains to whom he prefers to staying single and has not yet proposed.

(b) Each woman receiving proposals rejects any from men to whom she prefers staying single, and also rejects all but her most preferred among the group consisting of the new proposers, together with any man she has kept engaged from the previous step.

The algorithm stops after any step where no man is rejected. At this point, every man is either engaged to a woman or has been rejected by every woman on his list of women he prefers to staying single. Now each man who is engaged with a woman is matched with her. Each woman who did not receive any proposals from men she prefers to staying single, and each man who is rejected by all women he prefers to staying single, will remain single. This completes the description of the algorithm.

We call the similar algorithm with women proposing, the *W-optimal matching rule*.

**Remark.** (Theorem 2.8 in Roth and Sotomayor, 1990): The M-optimal and W-optimal matching rules produce stable matchings for any preference profile.

We consider an incentive compatibility requirement, *strategy-proofness*. Strategy-proofness says that for every agent, stating the true preferences should be a dominant strategy.

**Definition.** A matching rule $f$ on $\mathcal{P}$ is *manipulable* by an agent $x \in M \cup W$ at $P \in \mathcal{P}$ via $P'(x) \in \mathcal{P}(x)$ if $f(P/P'(x))(x) P(x) f(P)(x)$. A matching rule $f$ is *strategy-proof* on $\mathcal{P}$ if it is not manipulable at any profile in $\mathcal{P}$ by any agent $x \in M \cup W$ via any preference in $\mathcal{P}(x)$.

We introduce the minimum condition of efficiency, *respect for unanimity*.[2] Respect for unanimity states that for any preference profile where unless an agent prefers staying single, the partner the agent most prefers also prefers the agent, every agent should be matched to their most preferred agent.

[1]We borrow the description of the algorithm from Roth and Sotomayor (1990).

[2]Respect for unanimity is the "minimum" condition of efficiency in the sense that it is a necessary condition for almost all reasonable conditions of efficiency.

**Definition.** A matching rule $f$ *respects unanimity* on $\mathcal{P}$ if for all $P \in \mathcal{P}$,

$$[b(P(b(P(x)))) = x \text{ for all } x \in M \cup W] \Rightarrow [f(P)(x) = b(P(x)) \text{ for all } x \in M \cup W]$$

We also introduce a natural axiom, *respect for pairwise unanimity*. Respect for pairwise unanimity states that a pair of agents who are the best for each other should be matched, and an agent for whom staying single is the best should stay single.

**Definition.** A matching rule $f$ *respects pairwise unanimity* on $\mathcal{P}$ if for all $P \in \mathcal{P}$ and all $x \in M \cup W$,

$$b(P(b(P(x)))) = x \Rightarrow f(P)(x) = b(P(x))$$

**Remark.** Respect for pairwise unanimity implies respect for unanimity.

A matching rule $f$ is *stable* if for all profiles $P \in \mathcal{P}$, $f(P)$ is stable at $P$. A matching rule $f$ is *individually rational* if for all profiles $P \in \mathcal{P}$, $f(P)$ is individually rational at $P$. Finally, a matching rule $f$ is *efficient* if for all profiles $P \in \mathcal{P}$, $f(P)$ is efficient at $P$.

**Remark.** Stability implies respect for pairwise unanimity. However, as Example 1 illustrates, respect for pairwise unanimity does not imply stability.

**Example 1.** Let $n = l = 2$. Consider a preference profile $P' \in \mathcal{P}$ defined below[3]:

$$P' : \begin{Bmatrix} P'(m_1) = w_1 w_2 m_1 & P'(w_1) = m_2 m_1 w_1 \\ P'(m_2) = w_1 w_2 m_2 & P'(w_2) = m_1 m_2 w_2 \end{Bmatrix}.$$

Let $f$ be a matching rule that assigns the same matching as the M-optimal rule except for the preference profile $P'$ and assigns to $P'$ the following matching $a$:[4]

$$f(P') = a : \begin{Bmatrix} m_1 & m_2 & \_ \\ \_ & w_1 & w_2 \end{Bmatrix}.$$

It is easy to see that $f$ respects pairwise unanimity, yet it is not stable.

**Remark.** Both efficiency and respect for pairwise unanimity are sufficient conditions for respect for unanimity. However, as Example 2 illustrates, efficiency and respect for pairwise unanimity are mathematically independent on $\mathcal{P}$.

**Example 2.** Let $n = l = 2$. Consider a preference profile $P' \in \mathcal{P}$ defined below:

$$P' : \begin{Bmatrix} P'(m_1) = w_2 w_1 m_1 & P'(w_1) = m_2 m_1 w_1 \\ P'(m_2) = w_2 w_1 m_2 & P'(w_2) = m_2 m_1 w_2 \end{Bmatrix}.$$

Let $f_1$ be such that for all $P \in \mathcal{P}$ and all $y \in M \cup W$,

(1) if there exists an agent $y \in M \cup W$ such that $b(P(b(P(y)))) = y$, then $f_1(P)(y) = b(P(y))$, and

[3]An ordered list of mates indicates the agent's preference from better to worse among the possible mates.

[4]We use the same notation as Roth and Sotomayor (1990): a pair $(m_i, w_j)$ on the same vertical are matched to each other and an agent with no mate on its vertical stays single.

(2) otherwise, $f_1(P)(y) = y$.

Then, $f_1$ assigns to $P'$ the following matching $a_1$:

$$f_1(P') = a_1 : \begin{Bmatrix} m_1 & m_2 & \_ \\ \_ & w_2 & w_1 \end{Bmatrix}.$$

It is easy to see that $f_1$ respects pairwise unanimity, yet it is not efficient.

Meanwhile, let $f_2$ be a matching rule that assigns to each profile $P \in \mathcal{P}$ a matching that matches $m_1$ to $b(P(m_1))$ and $m_2$ to his most preferred agent in $(W \cup \{m_2\}) \setminus \{b(P(m_1))\}$. Then, $f_2$ assigns to $P'$ the following matching $a_2$:

$$f_2(P') = a_2 : \begin{Bmatrix} m_1 & m_2 \\ w_2 & w_1 \end{Bmatrix}.$$

It is easy to see that $f_2$ is efficient, yet it does not respect pairwise unanimity.

**Remark.** Both individual rationality and respect for pairwise unanimity are necessary conditions for stability. However, as Example 3 illustrates, individual rationality and respect for pairwise unanimity are mathematically independent on $\mathcal{P}$.

**Example 3.** Let $n = l = 2$. Consider a preference profile $P' \in \mathcal{P}$ defined below:

$$P' : \begin{Bmatrix} P'(m_1) = w_2 m_1 w_1 & P'(w_1) = m_2 m_1 w_1 \\ P'(m_2) = w_2 w_1 m_2 & P'(w_2) = m_2 m_1 w_2 \end{Bmatrix}.$$

Let $f_1$ be a matching rule that assigns the same matching as the M-optimal matching rule except for $P'$ and assigns to $P'$ the following matching $a_1$:

$$f_1(P') = a_1 : \begin{Bmatrix} m_1 & m_2 \\ w_1 & w_2 \end{Bmatrix}.$$

It is easy to see that $f_1$ respects pairwise unanimity, yet it is not individually rational.

Meanwhile, let $f_2$ be a matching rule that assigns the same matching as the M-optimal matching rule except for $P'$ and assigns to $P'$ the following matching $a_2$:

$$f_2(P') = a_2 : \begin{Bmatrix} m_1 & m_2 \\ w_2 & w_1 \end{Bmatrix}.$$

It is easy to see that $f_2$ is individually rational, yet it does not respect pairwise unanimity.

### *2.2. Results in one-to-one matchings*

Alcalde and Barberá (1994) show that efficient and individually rational matching rules must be manipulable. Thus we pursue the possibility of strategy-proof rules by relaxing efficiency to the weaker condition of respect for unanimity.

We call the following rule $f$, the *minimum unanimous rule*: First we divide $\mathcal{P}$ into the three subsets $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ defined below:

$\mathcal{P}_1$ : The set of preference profiles $P$ such that for all $x \in M \cup W$, $b(P(b(P(x)))) = x$.

$\mathcal{P}_2$ : The set of preference profiles $P$ such that there exists $(m_i, w_j) \in M \times W$ such that

(i) $b(P(b(P(x)))) = x$ for all $x \in M \cup W \setminus \{m_i, w_j\}$,

(ii) $w_j P(m_i) m_i$ and $m_i P(w_j) w_j$, and

(iii) $b(P(m_i)) \neq w_j$ or $b(P(w_j)) \neq m_i$.

$\mathcal{P}_3$ : $= \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$.

Then the minimum unanimous rule $f$ assigns a matching to each profile $P \in \mathcal{P}$ by following Directions 1, 2, and 3:

Direction 1 : For all preference profiles $P \in \mathcal{P}_1$ and all $x \in M \cup W$, $f(P)(x) = b(P(x))$.

Direction 2 : For all preference profiles $P \in \mathcal{P}_2$ and all $x \in (M \cup W) \setminus \{m_i, w_j\}$, $f(P)(x) = x$, $f(P)(m_i) = w_j$ and $f(P)(w_j) = m_i$.

Direction 3 : For all preference profiles $P \in \mathcal{P}_3$ and all $x \in M \cup W$, $f(P)(x) = x$.

Our first result shows that the minimum unanimous rule $f$ is strategy-proof and individually rational, and respects unanimity.

**Proposition 1.** *The minimum unanimous rule is strategy-proof and individually rational, and respects unanimity.*

**Proof.** By Direction 1, $f$ respects unanimity on $\mathcal{P}$. By Directions 1, 2 and 3, $f$ is individually rational on $\mathcal{P}$. It suffices to show that $f$ is strategy-proof on $\mathcal{P}$.

First, consider a profile $P \in \mathcal{P}_1$. Since everyone is matched to her or his best by Direction 1, anyone cannot manipulate at such a profile.

Second, consider a profile $P \in \mathcal{P}_2$. If $m_{i'} \in M \setminus \{m_i\}$ tries to manipulate at $P$, he would be single by Directions 2 and 3, and cannot be better off. Meanwhile, if the man $m_i$ tries to manipulate at $P$, he would be matched to $w_j$ or be single by Directions 1, 2 and 3, and cannot be better off. Similarly, any woman cannot manipulate at $P$.

Third, consider a profile $P \in \mathcal{P}_3$. Note that everyone stays single by Direction 3. Pick $m_i$ from $M$ arbitrarily.

Assume that there exists $w_j \in W$ such that (i$'$) $b(P(b(P(x)))) = x$ for all $x \in (M \cup W) \setminus \{m_i, w_j\}$, and (ii$'$) $m_i P(m_i) w_j$ and $m_i P(w_j) w_j$. Then, if the man $m_i$ tries to manipulate at $P$ via any $P'(m_i) \in \mathcal{P}(m_i)$ such that $m_i P'(m_i) w_j$, $P/P'(m_i)$ is still in $\mathcal{P}_3$, and he would stay single by Direction 3. On the other hand, if the man $m_i$ tries to manipulate at $P$ via any $P''(m_i) \in \mathcal{P}(m_i)$ such that $w_j P''(m_i) m_i$, then $P/P''(m_i)$ is in $\mathcal{P}_1$ or $\mathcal{P}_2$, and he would be matched with the woman $w_j$ to whom he prefers staying single by Directions 1 and 2.

Next assume that there does not exist $w_j \in W$ for whom (i$'$) and (ii$'$) hold. Then, even if the man $m_i$ manipulates via any $P'(m_i) \in \mathcal{P}(m_i)$ at $P$, $P/P'(m_i)$ is still in $\mathcal{P}_3$, and he would stay single.

Therefore, the man $m_i$ cannot manipulate at any profiles in $\mathcal{P}_3$. Since $m_i$ is picked up arbitrarily from $M$, any man cannot manipulate at any profile in $\mathcal{P}_3$. Similarly, any woman cannot manipulate at any profile in $\mathcal{P}_3$. □

Proposition 1 appears a positive result. However, it has one negative aspect in that the minimum unanimous rule is unreasonable in the sense that it does not respect pairwise unanimity and leaves all agents single for most preference profiles. Therefore, we explore the possibility of better strategy-proof rules that are individual rational and respect pairwise unanimity on $\mathcal{P}$. However, we prove that there exists no strategy-proof matching rule that respects pairwise unanimity on $\mathcal{P}$ as below.

**Proposition 2.** *There exists no strategy-proof matching rule that respects pairwise unanimity on $\mathcal{P}$.*

**Proof.** First we prove the statement for the case with $n = l = 2$ . Later we will explain how to extend the proof to the cases where $n \geq 3$ or $l \geq 3$.

We assume that the rule $f$ respects pairwise unanimity, and prove that it must be manipulable. Note that for the case with $n = l = 2$, the set of all possible matchings $A$ is the following:

$$A = \left\{ a_1 : \begin{Bmatrix} m_1 & m_2 \\ w_1 & w_2 \end{Bmatrix}, a_2 : \begin{Bmatrix} m_1 & m_2 \\ w_2 & w_1 \end{Bmatrix}, a_3 : \begin{Bmatrix} m_1 & m_2 & \_ \\ \_ & w_1 & w_2 \end{Bmatrix}, a_4 : \begin{Bmatrix} m_1 & m_2 & \_ \\ \_ & w_2 & w_1 \end{Bmatrix}, \right.$$
$$\left. a_5 : \begin{Bmatrix} m_1 & m_2 & \_ \\ w_1 & \_ & w_2 \end{Bmatrix}, a_6 : \begin{Bmatrix} m_1 & m_2 & \_ \\ w_2 & \_ & w_1 \end{Bmatrix}, a_7 : \begin{Bmatrix} m_1 & m_2 & \_ & \_ \\ \_ & \_ & w_1 & w_2 \end{Bmatrix} \right\}.$$

(1) Let $P^1 \in \mathcal{P}$ be such that

$$P^1 : \begin{Bmatrix} P^1(m_1) = w_1 w_2 m_1 & P^1(w_1) = m_2 m_1 w_1 \\ P^1(m_2) = w_2 w_1 m_2 & P^1(w_2) = m_1 m_2 w_2 \end{Bmatrix}.$$

The set of all matchings satisfying respect for pairwise unanimity for $P^1$ is equal to $A$.

(2) Let $P^2 = P^1/P^{1'}(w_1)$ where $P^{1'}(w_1) = m_2 w_1 m_1$. That is,

$$P^2 : \begin{Bmatrix} P^2(m_1) = w_1 w_2 m_1 & P^2(w_1) = m_2 w_1 m_1 \\ P^2(m_2) = w_2 w_1 m_2 & P^2(w_2) = m_1 m_2 w_2 \end{Bmatrix}.$$

The set of all matchings satisfying respect for pairwise unanimity for $P^2$ is equal to $A$.

(3) Let $P^{2'}(w_1) = w_1 m_2 m_1$. Then, $f(P^2/P^{2'}(w_1))(w_1) = w_1$ by respect for pairwise unanimity.
(4) Let $P^{2'}(m_1) = w_2 w_1 m_1$. Then, $f(P^2/P^{2'}(m_1))(m_1) = w_2$ by respect for pairwise unanimity.
(5) Let $P^{2'}(m_2) = w_1 w_2 m_2$. Then, $f(P^2/P^{2'}(m_2))(m_2) = w_1$ by respect for pairwise unanimity.
(6) Let $P^6 = P^1/P^{1'}(m_1)$ where $P^{1'}(m_1) = w_1 m_1 w_2$. That is,

$$P^6 : \begin{Bmatrix} P^6(m_1) = w_1 m_1 w_2 & P^6(w_1) = m_2 m_1 w_1 \\ P^6(m_2) = w_2 w_1 m_2 & P^6(w_2) = m_1 m_2 w_2 \end{Bmatrix}.$$

The set of all matchings satisfying respect for pairwise unanimity for $P^6$ is equal to $A$.

(7) Let $P^{6'}(m_1) = m_1 w_1 w_2$. Then, $f(P^6/P^{6'}(m_1))(m_1) = m_1$ by respect for pairwise unanimity.
(8) Let $P^{6'}(w_2) = m_2 m_1 w_2$. Then, $f(P^6/P^{6'}(w_2))(w_2) = m_2$ by respect for pairwise unanimity.
(9) Let $P^{6'}(w_1) = m_1 m_2 w_1$. Then, $f(P^6/P^{6'}(w_1))(w_1) = m_1$ by respect for pairwise unanimity.

(10) Let $P^{1''}(m_1) = w_2w_1m_1$. Then, $f(P^1/P^{1''}(m_1))(m_1) = w_2$ by respect for pairwise unanimity.
(11) Let $P^{1'}(m_2) = w_1w_2m_2$. Then, $f(P^1/P^{1'}(m_2))(m_2) = w_1$ by respect for pairwise unanimity.

Now we prove that $f$ is manipulable using the above preferences and matchings.

Note that Cases 1, 2, 3 and 4 below cover all the possible matchings of $f$ for $P^1$. We show that $f$ is manipulable for each case.

Case 1 : $f(P^1) = a_1$.

If $f(P^2) = a_2$ or $a_3$, that is, if $f(P^2)(w_1) = m_2$, $w_1$ can manipulate at $P^1$ via $P^{1'}(w_1)$ by $m_2P^1(w_1)m_1$. If $f(P^2) = a_1$ or $a_5$, that is, if $f(P^2)(w_1) = m_1$, $w_1$ can manipulate at $P^2$ via $P^{2'}(w_1)$ by (3) and $w_1P^2(w_1)m_1$. If $f(P^2) = a_4$, $m_1$ can manipulate at $P^2$ via $P^{2'}(m_1)$ by (4) and $w_2P^2(m_1)m_1$. If $f(P^2) = a_6$ or $a_7$, $m_2$ can manipulate at $P^2$ via $P^{2'}(m_2)$ by (5) and $w_1P^2(m_2)m_2$.

Case 2 : $f(P^1) = a_2$.

If $f(P^6) = a_1$ or $a_5$, that is, if $f(P^6)(m_1) = w_1$, $m_1$ can manipulate at $P^1$ via $P^{1'}(m_1)$ by $w_1P^1(m_1)w_2$. If $f(P^6) = a_2$ or $a_6$, that is, if $f(P^6)(m_1) = w_2$, $m_1$ can manipulate at $P^6$ via $P^{6'}(m_1)$ by (7) and $m_1P^6(m_1)w_2$. If $f(P^6) = a_3$ or $a_7$, that is, if $f(P^6)(w_2) = w_2$, $w_2$ can manipulate at $P^6$ via $P^{6'}(w_2)$ by (8) and $m_2P^6(w_2)w_2$. If $f(P^6) = a_4$, $w_1$ can manipulate at $P^6$ via $P^{6'}(w_1)$ by (9) and $m_1P^6(w_1)w_1$.

Case 3 : $f(P^1) = a_3$, $a_4$ or $a_7$.

Because $f(P^1)(m_1) = m_1$, $m_1$ can manipulate at $P^1$ via $P^{1''}(m_1)$ by (10) and $w_2P^1(m_1)m_1$.

Case 4 : $f(P^1) = a_5$ or $a_6$.

Because $f(P^1)(m_2) = m_2$, $m_2$ can manipulate at $P^1$ via $P^{1'}(m_2)$ by (11) and $w_1P^1(m_2)m_2$.

Next we explain how to prove the statement for the cases where $n \geq 3$ or $l \geq 3$. Let the preferences of agents $y \in (M \cup W) \setminus \{m_1, m_2, w_1, w_2\}$ be such that $b(P(y)) = y$ for all $(M \cup W) \setminus \{m_1, m_2, w_1, w_2\}$. Then, all agents $y \in (M \cup W) \setminus \{m_1, m_2, w_1, w_2\}$ would stay single in matchings satisfying respect for pairwise unanimity. Therefore, the proof for these cases is identical to the above proof. □

**Remark.** Since stability implies respect for pairwise unanimity, our result implies Roth (1982) showing that all stable matching rules must be manipulable.[5]

### 3. Many-to-One Matchings

In Section 2, we considered the matching problems on the one-to-one matching model. However, in terms of economic phenomena, many-to-one matchings in two-sided markets are typical, where one side of the market consists of institutions and the other side of individuals: for example, colleges and students, firms and workers, hospitals and interns. Accordingly, in this

[5] Alcalde and Barberá (1994) also extend Roth's result by relaxing stability to efficiency and individual rationality. However, note that our arguments considered several matchings that satisfy our conditions, but not efficiency and individual rationality.

section we extend the negative result in Section 2 to the many-to-one matching model, commonly known as the college admissions problem.

*3.1. Many-to-one matching model*

Let $C = \{C_1, C_2, \ldots, C_n\}$ be the set of colleges and $S = \{s_1, s_2, \ldots, s_l\}$ be the set of students. We assume that both $C$ and $S$ are finite and disjoint sets. We also assume that $n \geq 2$ and $l \geq 2$.

Each college $C_i$ has a quota $q_{C_i}$, which indicates the maximum number of positions it may fill. We assume that each $q_{C_i}$ is a positive integer. Hence, a matching of this model assigns each student to at most one college and each college to at most its quota of students.

Each student $s_j \in S$ has a preference relation $P(s_j)$ on $M(s_j) \equiv \{C_1, \ldots, C_n, s_j\}$. Let $\mathcal{P}(s_j)$ denote the set of all possible preferences for $s_j \in S$. Each college $C_i \in C$ has a preference relation $P(C_i)$ on $M(C_i) \equiv \{G \subseteq S : |G| \leq q_{C_i}\}$.

**Definition.** (Roth and Sotomayor 1990): A preference $P(C_i)$ is *responsive* if

(1) for all $G \subseteq S$ with $|G| < q_{C_i}$ and all $s_j \in S \setminus G$,

$$(G \cup \{s_j\}) P(C_i) G \Leftrightarrow \{s_j\} P(C_i) \emptyset, \text{ and}$$

(2) for all $G \subseteq S$ with $|G| < q_{C_i}$ and all $s_j, s_k \in S \setminus G$,

$$(G \cup \{s_j\}) P(C_i) (G \cup \{s_k\}) \Leftrightarrow \{s_j\} P(C_i) \{s_k\}.$$

We assume that preferences of all colleges are responsive. Let $\mathcal{P}(C_i)$ denote the set of all possible responsive preferences for $C_i \in C$. We assume that preferences are strict. We denote preference profiles by $P$. Let $\mathcal{P} = \prod_{i=1}^{n} \mathcal{P}(C_i) \times \prod_{j=1}^{l} \mathcal{P}(s_j)$ be the set of all possible preference profiles.

**Definition.** A *matching a* is a function $a : C \cup S \to 2^{C \cup S}$ such that:

(1) for all $s_j \in S$, $a(s_j) \in C \cup \{s_j\}$,

(2) for all $C_i \in C$, $a(C_i) \subseteq S$ and $|a(C_i)| \leq q_{C_i}$, and

(3) for all $(C_i, s_j) \in C \times S$, $a(s_j) = C_i \Leftrightarrow s_j \in a(C_i)$.

Let $A$ be the set of all possible matchings.

**Definition.** The *best* $b(P(C_i), G)$ is the most preferred subset of $G$. That is, $b(P(C_i), G)$ is the subset of $G$ such that $b(P(C_i), G) P(C_i) G'$ for all $G' \subseteq G$ such that $G' \in M(C_i) \setminus \{b(P(C_i), G)\}$.

**Definition.** A matching $a$ is *blocked by a student* $s_j \in S$ at $P \in \mathcal{P}$ if $s_j P(s_j) a(s_j)$. A matching $a$ is *blocked by a college* $C_i \in C$ at $P \in \mathcal{P}$ if $a(C_i) \neq b(P(C_i), a(C_i))$.

Note that since colleges' preferences are responsive, a matching $a$ is blocked by a college $C_i \in C$ at $P \in \mathcal{P}$ if there exists a student $s_j \in a(C_i)$ such that $\emptyset P(C_i) \{s_j\}$.

**Definition.** A matching $a$ is *individually rational* at $P \in \mathcal{P}$ if it is not blocked by any agent $y \in (C \cup S)$ at $P \in \mathcal{P}$.

**Definition.** A matching $a$ is *blocked by a pair* $(C_i, s_j) \in C \times S$ at $P \in \mathcal{P}$ if $C_i P(s_j) a(s_j)$ and $a(C_i) \neq b(P(C_i), a(C_i) \cup \{s_j\})$. A matching $a$ is *stable* at $P \in \mathcal{P}$ if it is not blocked by any student $s_j \in S$, any college $C_i \in C$, or any pair $(C_i, s_j) \in C \times S$.

**Remark.** (Sönmez 1996): The set of stable matchings is a singleton for each profile $P \in \mathcal{P}$ on the many-to-one matching model with $q_{C_i} \geq |S|$ for all $C_i \in C$. Here after, we call the matching rule $f$ assigning the associated stable matching to each preference profile, the *stable rule*. The stable rule $f$ can be described as below: for all $P \in \mathcal{P}$ and all $s_j \in S$,

(1) if there exists a college $C_i \in C$ such that $\{s_j\} P(C_i) \emptyset, C_i P(s_j) s_j$ and $C_i P(s_j) C_i^{'}$ for all $C_i^{'} \in C \setminus \{C_i\}$ such that $\{s_j\} P(C_i^{'}) \emptyset$, then $f(P)(s_j) = C_i$, and

(2) otherwise, $f(P)(s_j) = s_j$.

**Definition.** A matching $a \in A$ is *blocked by a coalition* $I \subseteq (C \cup S)$, if there exists another matching $a' \neq a$ such that for all students $s_j \in I$ and all colleges $C_i \in I$,

(1) $a'(s_j) \in I$ and $a'(s_j) P(s_j) a(s_j)$, and

(2) $[s_j \in a'(C_i) \Rightarrow s_j \in (I \cup a(C_i))]$ and $[a'(C_i) P(C_i) a(C_i)]$.

**Definition.** A matching is *group stable* at $P \in \mathcal{P}$ if it is not blocked by any coalition $I \subseteq (C \cup S)$ at $P \in \mathcal{P}$.

**Remark.** (Lemma 5.5. in Roth and Sotomayor, 1990): A matching is group stable if and only if it is stable on $\mathcal{P}$.

**Definition.** A matching $a$ is *(Pareto) efficient* at $P \in \mathcal{P}$ if there is no other matching $a' \neq a$ such that for all $y \in C \cup S$,

$$a'(y) \neq a(y) \Rightarrow a'(y) P(y) a(y).$$

**Definition.** A *matching rule* is a function $f$ from $\mathcal{P}$ to $A$. A matching rule $f$ is *individually rational* if for all profiles $P \in \mathcal{P}$, $f(P)$ is individually rational at $P \in \mathcal{P}$. A matching rule $f$ is *stable* if for all profiles $P \in \mathcal{P}$, $f(P)$ is stable at $P \in \mathcal{P}$. A matching rule $f$ is *efficient* if for all profiles $P \in \mathcal{P}$, $f(P)$ is efficient at $P \in \mathcal{P}$.

**Definition.** A matching rule $f$ is *manipulable* by an agent $y \in C \cup S$ at $P \in \mathcal{P}$ via $P'(y) \in \mathcal{P}(y)$ if $f(P/P'(y))(y) P(y) f(P)(y)$. A matching rule $f$ is *strategy-proof* on $\mathcal{P}$ if it is not manipulable at any $P \in \mathcal{P}$ by any $y \in C \cup S$ via any $P'(y) \in \mathcal{P}(y)$.

For all $y \in C \cup S$ and all $P(y) \in \mathcal{P}(y)$, let $b(P(y))$ be the best element, that is, $b(P(y)) P(y) G$ for all $G \in M(y) \setminus \{b(P(y))\}$.

**Remark.** Notice that $b(P(C_i), S) = b(P(C_i))$ for all $C_i \in C$.

**Definition.** A matching rule $f$ *respects unanimity* on $\mathcal{P}$ if for all $P \in \mathcal{P}$ such that

(1) for all $C_i \in C$, $[b(P(C_i)) = \emptyset]$ or [ for all $s_j \in b(P(C_i)), b(P(s_j)) = C_i$], and

(2) for all $s_j \in S$, $b(P(s_j)) = s_j$ or $s_j \in b(P(b(P(s_j))))$,

for all $y \in C \cup S$, $f(P)(y) = b(P(y))$.

**Remark.** Efficiency implies respect for unanimity.

**Definition.** A matching rule $f$ *respects pairwise unanimity* on $\mathcal{P}$ if for all $P \in \mathcal{P}$,

(1) for all $C_i \in C$ such that $b(P(C_i)) \neq \emptyset$ and all $s_j \in S$ such that $b(P(s_j)) = C_i$ and $s_j \in b(P(C_i))$, $f(P)(s_j) = C_i$,

(2) for all $s_j \in S$ such that $b(P(s_j)) = s_j$, $f(P)(s_j) = s_j$, and

(3) for all $C_i \in C$ such that $b(P(C_i)) = \emptyset$, $f(P)(C_i) = \emptyset$.

**Remark.** Respect for pairwise unanimity implies respect for unanimity.

**Remark.** Stability implies respect for pairwise unanimity.

**Proof.** Suppose that there exists a matching rule $f$ that is stable, yet it does not respect pairwise unanimity on $\mathcal{P}$. Then,

(1) there exists some profile $P \in \mathcal{P}$ and a pair $(C_i, s_j) \in C \times S$ such that $b(P(s_j)) = C_i$ and $s_j \in b(P(C_i))$, and $f(P)(s_j) \neq C_i$,

(2) there exists some profile $P \in \mathcal{P}$ and a student $s_j \in S$ such that $b(P(s_j)) = s_j$, and $f(P)(s_j) \neq s_j$, or

(3) there exists some profile $P \in \mathcal{P}$ and a college $C_i \in C$ such that $b(P(C_i)) = \emptyset$, and $f(P)(C_i) \neq \emptyset$.

Suppose that $f$ satisfies (1). Since $f$ satisfies (1) and $P(C_i)$ is responsive, $f(P)(C_i) \neq b(P(C_i), f(P)(C_i) \cup \{s_j\})$ and $C_i P(s_j) f(P)(s_j)$. Then, $f(P)$ is blocked by the pair $(C_i, s_j) \in C \times S$ at the profile $P \in \mathcal{P}$. Next, suppose that $f$ satisfies (2). Then, $f(P)$ is blocked by the student $s_j$ at the profile $P \in \mathcal{P}$. Similarly, suppose that $f$ satisfies (3). Then, $f(P)$ is blocked by the college $C_i$ at the profile $P \in \mathcal{P}$. This is contradicting stability of the matching rule $f$. □

**Remark.** As Example 4 illustrates, respect for pairwise unanimity does not imply stability.

**Example 4.** Let $n = l = 2$ and $q_{C_1} = q_{C_2} = 2$. Consider a preference profile $P' \in \mathcal{P}$ defined below:

$$P' : \begin{Bmatrix} P'(C_1) = \{s_1, s_2\}\, \{s_1\}\, \{s_2\}\, \emptyset & P'(s_1) = C_2\, C_1\, s_1 \\ P'(C_2) = \{s_2\}\, \emptyset\, \{s_1, s_2\}\, \{s_1\} & P'(s_2) = C_2\, C_1\, s_2 \end{Bmatrix}.$$

Let $f$ be a matching rule that assigns a matching to each profile $P \in \mathcal{P} \setminus \{P'\}$ subject to the stable rule, and assigns to $P'$ the following matching $a$:[6]

$$f(P') = a : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_2\} \end{Bmatrix}.$$

[6]We use the notation used by Sönmez (1996): a pair $(C_i, S') \in C \times 2^S$ on the same vertical are matched to each other and each student who is matched to herself or himself is omitted for ease of notation .

It is easy to see that $f$ respects pairwise unanimity, yet it is not stable.

**Remark.** Both efficiency and respect for pairwise unanimity are necessary conditions for stability. However, as Example 5 illustrates, efficiency and respect for pairwise unanimity are mathematically independent on $\mathcal{P}$.

**Example 5.** Let $n = l = 2$ and $q_{C_1} = q_{C_2} = 2$. Consider the preference profile $P' \in \mathcal{P}$ presented in Example 4. Let $f_1$ be such that, for all $P \in \mathcal{P}$, for all $C_i \in C$, for all $s_j \in S$,

(1) if there exists a pair $(C_i, s_j) \in C \times S$ such that $b(P(s_j)) = C_i$ and $s_j \in b(P(C_i))$, then $f(P)(s_j) = C_i$, and

(2) otherwise, $f_1(P)(C_i) = \emptyset$ or $f_1(P)(s_j) = s_j$.

Then, $f_1$ assigns to $P'$ the following matching $a_1$:

$$f_1(P') = a_1 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_2\} \end{Bmatrix}.$$

It is easy to see that $f_1$ respects pairwise unanimity, yet it is not efficient.

Meanwhile, consider a preference profile $P'' \in \mathcal{P}$ defined below:

$$P'' : \begin{Bmatrix} P''(C_1) = \{s_1\}\,\emptyset\,\{s_1, s_2\}\,\{s_2\} & P''(s_1) = C_1\,C_2\,s_1 \\ P''(C_2) = \{s_1\}\,\emptyset\,\{s_1, s_2\}\,\{s_2\} & P''(s_2) = C_1\,C_2\,s_2 \end{Bmatrix}.$$

Let $f_2$ be a matching rule that assigns a matching to each profile $P \in \mathcal{P} \setminus \{P''\}$ subject to the stable rule and assigns to $P''$ the following matching $a_2$:

$$f_2(P'') = a_2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_1\} \end{Bmatrix}.$$

It is easy to see that $f_2$ is efficient, yet it does not respect pairwise unanimity.

**Remark.** Both individual rationality and respect for pairwise unanimity are necessary conditions for stability. However, as Example 6 illustrates, individual rationality and respect for pairwise unanimity are mathematically independent on $\mathcal{P}$.

**Example 6.** Let $n = l = 2$ and $q_{C_1} = q_{C_2} = 2$. Consider a preference profile $P' \in \mathcal{P}$ defined below:

$$P' : \begin{Bmatrix} P'(C_1) = \{s_2\}\,\emptyset\,\{s_1, s_2\}\,\{s_1\} & P'(s_1) = C_1\,C_2\,s_1 \\ P'(C_2) = \{s_1, s_2\}\,\{s_1\}\,\{s_2\}\,\emptyset & P'(s_2) = C_2\,C_1\,s_2 \end{Bmatrix}.$$

Let $f_1$ be a matching rule that assigns a matching to each profile $P \in \mathcal{P} \setminus \{P'\}$ subject to the stable rule and assigns to $P'$ the following matching $a_1$:

$$f_1(P') = a_1 : \begin{Bmatrix} C_1 & C_2 \\ \{s_1\} & \{s_2\} \end{Bmatrix}.$$

It is easy to see that $f_1$ respects pairwise unanimity, yet it is not individually rational.

Meanwhile, let $f_2$ be a matching rule that assigns a matching to each profile $P \in \mathcal{P} \setminus \{P'\}$ subject to the stable rule and assigns to $P'$ the following matching $a_2$:

$$f_2(P') = a_2 : \begin{Bmatrix} C_1 & C_2 \\ \{s_2\} & \{s_1\} \end{Bmatrix}.$$

It is easy to see that $f_2$ is individually rational, yet it does not respect pairwise unanimity.

*3.2. Results in many-to-one matchings*

We show that in the context of one-to-one matching problems, there is no strategy-proof rule that respects pairwise unanimity. However, there is a significant change in this result when colleges can admit as many students as they wish. Since respect for pairwise unanimity is a necessary condition for stability, the stable rule respects pairwise unanimity. On the other hand, strategy-proofness of the stable rule follows from Sönmez's (1996) finding that the stable rule is strategy-proof on the many-to-one matching model with $q_{C_i} \geq |S|$ for all $C_i \in C$. Therefore, we obtain a positive result as below.

**Proposition 3.** *If $q_{C_i} \geq |S|$ for all $C_i \in C$, then the stable rule is strategy-proof and respects pairwise unanimity on $\mathcal{P}$.*

In the next proposition, we show that our negative result in the one-to-one matching model extends to the many-to-one matching model where a college exists that cannot admit as many students as it would like.

**Proposition 4.** *If $q_{C_i} < |S|$ for some $C_i \in C$, then there is no strategy-proof rule that respects pairwise unanimity on $\mathcal{P}$.*

**Proof.** First, we prove the result for the case with $n = l = 2$. Later, we will explain how to extend the proof to the cases where $n \geq 3$ or $l \geq 3$. Let $q_{C_1} = 1 < |S| = 2$ without loss of generality. Assume that the rule $f$ respects pairwise unanimity, and prove that it must be manipulable. Since the case with $q_{C_2} = 1$ is covered by Proposition 2, it suffices to prove it for cases with $q_{C_2} \geq 2$.

(1) Let $P^1 \in \mathcal{P}$ be such that

$$P^1 : \begin{Bmatrix} P^1(C_1) = \{s_2\}\,\{s_1\}\,\emptyset & P^1(s_1) = C_1\,C_2\,s_1 \\ P^1(C_2) = \{s_1, s_2\}\,\{s_1\}\,\{s_2\}\,\emptyset & P^1(s_2) = C_2\,C_1\,s_2 \end{Bmatrix}.$$

The set of all matchings satisfying respect for pairwise unanimity for $P^1$ is the following:

$$a^1_1 : \begin{Bmatrix} C_1 & C_2 \\ \{s_1\} & \{s_2\} \end{Bmatrix},\ a^1_2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_2\} \end{Bmatrix}\ \&\ a^1_3 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_1, s_2\} \end{Bmatrix}.$$

(2) Let $P^2 = P^1/P^{1'}(C_2)$ where $P^{1'}(C_2) = \{s_1\}\emptyset\{s_1, s_2\}\{s_2\}$. That is,

$$P^2 : \begin{Bmatrix} P^2(C_1) = \{s_2\}\,\{s_1\}\,\emptyset & P^2(s_1) = C_1\,C_2\,s_1 \\ P^2(C_2) = \{s_1\}\,\emptyset\,\{s_1, s_2\}\,\{s_2\} & P^2(s_2) = C_2\,C_1\,s_2 \end{Bmatrix}.$$

The set of all matchings satisfying respect for pairwise unanimity for $P^2$ is the following:

$$a_1^2 : \begin{Bmatrix} C_1 & C_2 \\ \{s_1\} & \{s_2\} \end{Bmatrix},\ a_2^2 : \begin{Bmatrix} C_1 & C_2 \\ \{s_2\} & \{s_1\} \end{Bmatrix},\ a_3^2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_1\} \end{Bmatrix},\ a_4^2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_2\} \end{Bmatrix},$$
$$a_5^2 : \begin{Bmatrix} C_1 & C_2 \\ \{s_1\} & \emptyset \end{Bmatrix},\ a_6^2 : \begin{Bmatrix} C_1 & C_2 \\ \{s_2\} & \emptyset \end{Bmatrix},\ a_7^2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \emptyset \end{Bmatrix} \ \&\ a_8^2 : \begin{Bmatrix} C_1 & C_2 \\ \emptyset & \{s_1, s_2\} \end{Bmatrix}.$$

(3) Let $P^{2'}(C_2) = \emptyset\{s_1\}\{s_2\}\{s_1, s_2\}$. Then, $f(P^2/P^{2'}(C_2))(C_2) = \emptyset$ by respect for pairwise unanimity.
(4) Let $P^{2'}(s_2) = C_1\, C_2\, s_2$. Then, $f(P^2/P^{2'}(s_2))(s_2) = C_1$ by respect for pairwise unanimity.
(5) Let $P^{2'}(s_1) = C_2\, C_1\, s_1$. Then, $f(P^2/P^{2'}(s_1))(s_1) = C_2$ by respect for pairwise unanimity.
(6) Let $P^{1'}(C_1) = \{s_1\}\{s_2\}\emptyset$. Then, $f(P^1/P^{1'}(C_1))(C_1) = \{s_1\}$ by respect for pairwise unanimity.

Now we prove that $f$ is manipulable using the above preferences and matchings. Note that Cases 1 and 2 below cover all the possible matchings of $f$ for $P^1$. We show that $f$ is manipulable for each case.

Case 1 : $f(P^1) = a_1^1$.

If $f(P^2) = a_2^2$, $a_3^2$ or $a_8^2$, $C_2$ can manipulate at $P^1$ via $P^{1'}(C_2)$ by (2) and $\{s_1\}P^1(C_2)\{s_2\}$ and $\{s_1, s_2\}P^1(C_2)\{s_2\}$. If $f(P^2) = a_1^2$ or $a_4^2$, that is, if $f(P^2)(C_2) = \{s_2\}$, $C_2$ can manipulate at $P^2$ via $P^{2'}(C_2)$ by (3) and $\emptyset P^2(C_2)\{s_2\}$. If $f(P^2) = a_5^2$ or $a_7^2$, that is, if $f(P^2)(s_2) = s_2$, $s_2$ can manipulate at $P^2$ via $P^{2'}(s_2)$ by (4) and $C_1P^2(s_2)s_2$. If $f(P^2) = a_6^2$, $s_1$ can manipulate at $P^2$ via $P^{2'}(s_1)$ by (5) and $C_2P^2(s_1)s_1$.

Case 2 : $f(P^1) = a_2^1$ or $a_3^1$.

Because $f(P^1)(C_1) = \emptyset$, $C_1$ can manipulate at $P^1$ via $P^{1'}(C_1)$ by (6) and $\{s_1\}P^1(C_1)\emptyset$.

Next we explain how to prove the result for the cases where $n \geq 3$ or $l \geq 3$. Just like the above proof, let $q_{C_1} < |S|$ without loss of generality. Let $S_1 \subseteq S \setminus \{s_1, s_2\}$ be such that $|S_1| = q_{C_1} - 1$. Let the preferences of all students $s_j \in S_1$ be such that $b(P(s_j)) = C_1$, those of colleges $C_i \in C \setminus \{C_1, C_2\}$ be such that $b(P(C_i)) = \emptyset$ and those of students $s_j \in S \setminus (\{s_1, s_2\} \cup S_1)$ be such that $b(P(s_j)) = s_j$. Let the preferences $P^1(C_1)$ and $P^{1'}(C_1)$ of $C_1 \in C$ be such that

$P^1(C_1) = (S_1 \cup \{s_2\})\ (S_1 \cup \{s_1\})\ G\ \emptyset$ for all $G \in M(C_1) \setminus \{(S_1 \cup \{s_1\}), (S_1 \cup \{s_2\}), \emptyset\}$, and
$P^{1'}(C_1) = (S_1 \cup \{s_1\})\ (S_1 \cup \{s_2\})\ G\ \emptyset$ for all $G \in M(C_1) \setminus \{(S_1 \cup \{s_1\}), (S_1 \cup \{s_2\}), \emptyset\}$.

Then, in matchings satisfying respect for pairwise unanimity, each student $s_j \in S_1$ would be matched to $C_1 \in C$, each college $C_i \in C \setminus \{C_1, C_2\}$ would be matched to $\emptyset$ and each student $s_j \in S \setminus (\{s_1, s_2\} \cup S_1)$ would be matched to the student $s_j$. Therefore, the proof for these cases is identical to the above proof. □

By Propositions 3 and 4, we have the following characterization of the class of matching problems that admit strategy-proof rules that respect pairwise unanimity.

**Theorem.** *Consider the matching problems with responsive preferences. There exists a strategy-proof rule that respects pairwise unanimity if and only if each college's quota is unlimited.*

## 4. Concluding Remarks

In this paper, we explore the possibility of designing satisfactory matching rules. First, in the one-to-one matching model, we establish that i) *there exists a strategy-proof rule that is individually rational and respects unanimity*, and ii) *there exists no strategy-proof rule that respects pairwise unanimity.* Second, we extended the result ii) to the many-to-one matching model. Our results, together with Roth (1982) and Alcalde and Barberá (1994), suggest the difficulty of designing strategy-proof rules satisfying better than respect for unanimity.

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