E C O N O M I C S B U L L E T I N

Multi−Unit Auctions with Synergy

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Abstract

In this note we examine four standard multi−unit sealed−bid auctions in the presence of synergy. The structure of the equilibrium bidding strategy under each rule is quite intuitive. Whether the equilibrium involves "bundle−bidding" or "separating−bidding" strategy depends on the presence of the "exposure problem" and the pressure of "demand reduction" in each case. When the bidders can implicitly coordinate to avoid the "exposure problem" and the pressure of "demand reduction," the equilibrium strategy can be calculated using parallels with unit−demand auctions. However, in the presence of the "exposure problem" well−behaved symmetric equilibria that can be characterized by the first−order condition of bidders' maximization problem may not exist in at least some situations.

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1. Introduction

More than two decades of research in single-object auctions notwithstanding, multi-object auctions have been left relatively less explored. However, as observed recently, even standard results from single-object auctions do not generally extend to the multi-object case. Furthermore, there are certain issues that come up only in the multi-object context making it necessary to develop a separate theory of multi-object auctions. In this note we consider an auction for multiple units of an object and explore the bidding behavior under different sealed-bid rules in the presence of synergies.

In spite of the difficulties in obtaining closed form descriptions, some interesting patterns have already emerged in the equilibrium bidding behavior of multi-unit auctions. Noussair (1995), and Engelbrecht-Wiggans and Kahn (1998a) considered multi-unit auctions with each bidder having diminishing marginal values for two successive units. They showed that under a uniform-price rule (with price equal to the highest-losing bid) bidders tend to bid less aggressively on the second unit. This is the *demand reduction* effect of uniform-price auction. Tenorio (1997), and Ausubel and Cramton (2002) observed the same phenomenon in more tractable models of multi-unit uniform-price auctions. The intuition behind demand reduction is quite simple. Since the price is set by the highest losing bid in the auction a bidder's second unit can determine the price for the first unit. Therefore, rather than being competitive and drive up the price in the auction, in equilibrium the bidders can implicitly coordinate to make low bids on the second unit, thereby keeping the price for the first unit low whenever it is won.

Katzman (1995), and Engelbrecht-Wiggans and Kahn (1998b) looked into multi-unit auctions as above under the pay-your-bid or discriminatory-price rule in which a bidder pays the value of his bid for each unit won. In particular, they considered symmetric continuous equilibrium strategies that can be characterized using the first-order conditions from the bidder's payoff maximization problem. They showed that whenever there exists such an equilibrium it involves a bidder bidding more aggressively on the second unit than on the first. Thus in equilibrium bidders bid the same amount on both units when the marginal values for the two units are close enough.

In all these cases, a bidder's value for the second unit is assumed to be less than that for the first unit. More recently, there has been research interest in auctions of objects with synergies. Two 20 MHz nationwide radio spectrum licenses are often worth more to the bidders than the sum of their individual values because of the richer uses to which they can be put. Similar phenomenon is observed in many auctions for procurement and construction contracts, as well. The difficulty of studying auctions with synergies arises because a bidder who attempts to bid on the synergy may not win the bundle that provides the synergy, and may settle for a collection of object that is worth less than the price paid. This is called the *exposure problem* where in his eagerness to bid on the synergy between objects, a bidder runs the risk of exposing himself to a loss that will result if the bidder cannot win the desired objects at the end.

Krishna and Rosenthal (1996) described equilibrium bidding behavior and obtained some comparative static results in multi-object auctions with single-object demand "local" and multi-object demand "global" bidders. A global bidder's values for the two objects being superadditive. Menezes and Monteiro (2003), and Jeitschko and Wolfstetter (1998) considered different models of sequential auctions with positive and negative synergies, and examined the equilibrium bidding strategies as well as the expected prices.

Kagel and Levin (2000) compared theoretical predictions and the bidding behavior of human global bidder against computer bidders with single-unit demands in experiments with sealed-bid and ascending-bid versions of the uniform-price auction. Katok and Roth (2001) investigated the potential implication of strategic behavior by the bidders with single-unit demands, as well, in a similar model by allowing all bidders to be human subjects.

In this note we add to this line of work and study the bidding behavior in multi-unit sealed-bid auctions where all bidders have multi-unit demands with variable synergies. Standard sealed-bid auction rules differ in whether there are pressures of demand reduction, and/or whether bidders can implicitly coordinate to get around the exposure problem. We find that the structure of equilibrium bidding strategies in these auctions, if they exist, is quite intuitive.

2. The Model

A seller has $K(\geq 2)$ identical units of an object on sale. There are $n(\geq K/2)$ bidders with bidder i having marginal values V_{1i} for the first unit and V_{2i} for the second unit where $V_{1i} \leq V_{2i}$. We assume that the marginal values are continuous random variables with a joint distribution function $F(v_1, v_2)$ and density $f(v_1, v_2)$ on the support $0 \leq v_1 \leq v_2 \leq 1$, where (v_1, v_2) is a realization of (V_{1i}, V_{2i}).

Under all auction rules each bidder submits two nonnegative bids b_1 and b_2, and the highest K bids are awarded the K units. Thus a bidder receives one or two units depending on whether one of his bids or both his bids are among the K highest bids in the auction. Assume, without loss of generality, that the *bidding constraint* $b_1 \geq b_2$ holds for acceptable bids.

In a *Vickrey auction* a bidder who wins k units pays an amount equal to the k highest losing bids of his rivals.[1] In a *discriminatory-price* or *pay-your-bid auction* a bidder who wins a unit pays precisely the amount of his bid for that unit. A single price for each unit is established in the auction under each *uniform-price rule.* Under the *low-bid uniform-price rule* the price is equal to the lowest-winning bid in the auction whereas under the *high-bid uniform-price rule* the price is equal to the highest-losing bid.

A bidder's strategy under an auction rule is a function $b = (b_1, b_2)$ that takes each value pair $v = (v_1, v_2)$ to a pair of acceptable bids $(b_1(v), b_2(v))$. Throughout this note we consider symmetric Bayes-Nash equilibria in pure strategies. We say that strategy $b(v)$ is in equilibrium if $b(v) \in \arg\max_{\hat{b}:\hat{b}_1 \geq \hat{b}_2} \pi(v, \hat{b})$ for all v where $\pi(v, \hat{b})$ is a bidder's expected payoff from bidding $\hat{b} = (\hat{b}_1, \hat{b}_2)$ when the other bidders follow the bidding strategy b. For reasons of tractability we follow the aforementioned papers in multi-unit auctions, and restrict ourselves to well-behaved equilibria where π is differentiable to allow a first-order condition characterization of an interior solution.

[1] This rule is the appropriate notion of a Vickrey (1961) auction in the case of multi-unit demands with diminishing marginal values. However, when the marginal values are increasing then seeking (nonincreasing) bids for the successive units, as in our case, does not allow the bidders to express their true worths for the different units. The only way that a Vickrey auction can be implemented in its true spirit in this case is by seeking combinatorial bids for every possible package that a bidder may want. Nonetheless, since we do not intend to discus combinatiorial auctions here, we continue to call it by the same name in this paper.

3. Bundle Bidding Equilibrium

An equilibrium in an auction can involve one of two types of bidding strategies: With a *bundle-bidding strategy* a bidder always bids the same amount on both units. With a *separating strategy* while for some values a bidder could bid the same amount on the two units, for other values the bidder submits a larger bid for the first unit. The possibility of equilibrium with separating-bidding strategies has already been observed in other multi-unit auction contexts. Typically, such equilibrium strategies do not have closed form expressions, and are generally characterized by first-order conditions without guarantees of existence (see Engelbrecht-Wiggans and Kahn, 1998b).

We start by discussing the case of two-unit supply, i.e. $K = 2$. In this case if bidders use bundle-bidding strategies a bidder either wins both units or none, thereby avoiding the exposure problem altogether. On the other hand, pressures of demand reduction, whenever present, prevent bidders from using bundle-bidding strategies in equilibrium thus opening them up to the exposure problem, and making bidding on the synergies difficult.[2] Proposition 1 identifies the auctions where bidders can coordinate to bundle-bid and those where they cannot. (All proofs are gathered in the Appendix.)

Proposition 1. Suppose that there are two units on sale. Vickrey auction, discriminatory-price auction, and high-bid uniform-price auction have equilibrium in bundle-bidding strategies while low-bid uniform-price auction does not have such an equilibrium.

In Vickrey and discriminatory-price auctions the second bid does not determine the price for the first unit. Therefore, the pressure of demand reduction is not present in these auctions, and the presence of the synergies makes the bidders bundle-bid as a result. Interestingly, unlike in the low-bid uniform-price auction, in the high-bid uniform-price auction whether a bidder's second bid determines the price for his first unit depends on whether the other bidders bundle-bid or not.

The auctions effectively reduce to single-object auctions for the bundle under bundle-bidding. Therefore, in equilibrium a bidder just splits the bid for the bundle into two

[2] We say that a bidder is bidding on the synergy at value (v_1, v_2) with $v_1 < v_2$ if a bidder's bid on the first unit is greater than v_1.

equal bids for the two units, thus bidding on the synergy. Hence, we have for $i = 1, 2$,

$$\begin{aligned} b^*_{Vickrey,i}(v) &= \frac{v_1 + v_2}{2},\ b^*_{high\text{-}bid,i}(v) = \frac{v_1 + v_2}{2} \\ b^*_{discriminatory,i}(v) &= E[\frac{V_1 + V_2}{2} | V_1 + V_2 \leq v_1 + v_2]. \end{aligned}$$

It can be checked that in both the uniform-price auctions if the other bidders use separating-strategies then the best response of a bidder is to use a separating-strategy. This, of course, does not mean that the auction necessarily has a symmetric equilibrium in separating strategies. In fact, it is possible to show that when bidder values are uniformly distributed, the high-bid uniform-price auction does not have a separating equilibrium strategy with a first-order condition representation.

It is not difficult to see that the above results carry over to any auction where each bidder has increasing marginal values on M units, and kM units of the object are sold for some positive integer k $(< n)$. Thus whenever bidders can implicitly coordinate to eliminate the exposure problem and the pressures of demand reduction through bundle-bidding, they will do so in an equilibrium.

In other situations, however, bundle-bidding cannot eliminate the exposure problem. For concreteness consider the case where there are three units on sale (i.e. $K = 3$) and each bidder has increasing marginal values for two units only. It is no longer the case that with bundle-bidding bidders will either win both units or none. Thus there is no way that a bidder can bid on the synergy without worrying about the exposure problem. Nonetheless, because pressure of demand reduction is absent in Vickrey and discriminatory-price auctions (and in low-bid uniform-price auction under bundle-bidding), these rules induce bidders to bundle-bid in equilibrium whenever there exists one.

Proposition 2. Suppose three units of an object are on sale. Under the Vickrey, discriminatory-price, and low-bid uniform-price auction rules the best response of a bidder is to bundle-bid whenever the other bidders are bundle-bidding. The high-bid uniform-price auction does not have an equilibrium in bundle-bidding strategies.

When the other bidders use separating bidding strategies then the pressure of demand reduction reappears in the low-bid uniform-price auction since the second bid can now determine the price for the first unit. In other words, a separating bidding strategy is a best response to other bidders using separating bidding strategies, making an equilibrium in separating bidding strategies possible under both uniform-price rules.

Since bundle-bidding does not reduce the auctions to unit-demand auctions anymore, like the previous case, equilibrium bidding strategies cannot be described by drawing parallels with unit-demand auctions. In fact given the exposure problem an equilibrium may not exist even under the relatively simple structure of the bundle-bidding strategy as the following example suggests.

Example

Consider a Vickrey auction with $K = 3$ and $f(x, y) = 2$. It can be checked that when there are two bidders in the auction there is an equilibrium in bundle-bidding strategy

$$b_1(v_1, v_2) = b_2(v_1, v_2) = v_2.$$

Thus bidders bid on the synergies in the sense that the bid on the first unit is greater than the value of the first unit. However, this is a trivial situation because of the following reason: In this case each bidder wins the first unit for sure and because the first bid does not displace any rival bid, the payment on it is zero. The second bid displaces a rival bid and it wins if the rival's bid is less than his bid. This makes it almost like a single-unit Vickrey auction for the second unit. The exposure problem is really not present in this case, and a second bid (and hence also a first bid) of v_2 becomes a weakly dominant strategy. In fact, just as the intuition suggests this fact is true for general distributions.

When $n = 3$ the exposure problem appears, and the derivative of a bidder's expected payoff function when the other bidders use a bundle-bidding strategy is given by

$$2g(b)[(v_1 - b)(1 - G(b)) + (v_2 - b)G(b)]$$

where $G(\cdot)$ denotes the distribution of the joint (bundle-)bid for a unit by the bidders and $g(\cdot)$ the corresponding density. If $v_1 < v_2$, the expression is positive when $b = v_2$, and negative when $b = v_2$. Thus the best response bid is between v_1 and v_2. Now consider the best response bid at $(0, v_2)$ where $v_2 > 0$. Observe that the above expression is linear in the values, hence if a bidder makes the same bid at two pairs of values he makes the exact same bid for all convex combinations of the values.[3] Hence, if b is the best response bid at $(0, v_2)$, and since b is the best response bid at (b, b) we have $G(b) = v_2 b$. Substituting this in the above expression the first-order condition becomes $-b(1-v_2^2)$ which is negative for all $b > 0$. This implies that the corresponding first-order condition does not give an equilibrium strategy. Note, however, that this does not rule out the possibility of bundle-bidding equilibrium strategy with a complex discontinuous structure, for instance, or an asymmetric equilibrium.

4. Concluding Remarks

The exposure problem in multi-object auctions with synergies generally makes bidding strategies difficult to describe. Moreover, in multi-unit auctions often equilibrium separating strategies cannot be described as closed form expressions. However, with synergies possibility of separating strategy arises only if there are pressures of demand reduction in the auction. Thus for a large number of situations the equilibria involve the relatively simpler bundle-bidding strategy. In many cases bundle-bidding effectively collapses the auction into one where bidders have unit-demands for the bundle, and a finite number of bundles are offered on sale. However, in other situations a bundle-bidding equilibrium strategy can become less trivial. As we demonstrated, a well-behaved bundle-bidding equilibrium strategy may not even exist under the exposure problem. Nonetheless, equilibria in bundle-bidding strategies are easier to study. In fact, to what extent the existence and nonexistence of equilibria hold in multi-unit auctions under the exposure problem appears to be a more tractable question to examine with bundle-bidding strategies.

[3] See Engelbrecht-Wiggans and Kahn (1998a) for a similar construction.

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6. Appendix

Proof of Proposition 1. *Vickrey auction rule.* Suppose bidders $2, ..., n$ are bundle-bidding. If bidder 1 bids $b_1 \geq b_2$ his expected payoff is

$$\int_0^{b_2} (v_1 + v_2 - 2x)h(x)dx + \int_{b_2}^{b_1} (v_1 - x)h(x)dx$$

where $h(\cdot)$ is the density of the highest of the rival bidders' bids. The derivative with respect to b_1, $(v_1 - b_1)h(b_1)$, is positive if $b_1 < v_1$, and similarly, the derivative with respect to b_2 is positive if $b_2 < v_2$. It follows that bundle-bidding is the best response for bidder 1.

Discriminatory-price rule. Suppose bidders $2, ..., n$ are bundle-bidding. Denote by $H(b)$ the probability that the highest of these bidders' bids is lower than or equal to b.

At a valuation pair (v_1, v_2), with $v_1 < v_2$, the expected payoff to bidder 1 bidding $b_1 \geq b_2$ is

$$\begin{aligned} & (v_1 - b_1)(H(b_1) - H(b_2)) + (v_1 + v_2 - b_1 - b_2)H(b_2) \\ = \; & (v_1 - b_1)H(b_1) + (v_2 - b_2)H(b_2). \end{aligned}$$

The first-order conditions[4] for maximizing this with respect to b_1 and b_2 are given by

$$\begin{aligned} (v_1 - b_1)h(b_1) - H(b_1) &= 0 \\ (v_2 - b_2)h(b_2) - H(b_2) &= 0. \end{aligned}$$

[4] Note that there is no corner solution to the problem.

If at (v_1, v_2) the best response b_1^* is larger than the best response b_2^* then at (v_1, v_1) the derivative of the expected payoff with respect to b_2 is

$$\begin{aligned} & (v_1 - b_2^*)h(b_2^*) - H(b_2^*) \\ = \;& (v_2 - b_2^*)h(b_2^*) - H(b_2^*) - (v_2 - v_1)h(b_2^*) \\ < \;& 0. \end{aligned}$$

This implies that at (v_1, v_1) bidder 1's best response is to bid (b_1^*, b_2) with $b_2^* > b_2$.

Hence it is enough to prove that for any valuation pair (v, v) bidder 1's best response is to bid equal amounts. The first-order conditions at (v, v) are given by

$$\begin{aligned} (v - b_1)h(b_1) - H(b_1) &= 0 \\ (v - b_2)h(b_2) - H(b_2) &= 0. \end{aligned}$$

Thus bidding equal amounts is a best response at (v, v) and hence for all value pairs.

High-bid uniform-price auction. Suppose bidders 2, ..., n are bundle-bidding. Let bidder 1 with valuations (v_1, v_2) consider bids $b_1 \geq b_2$. His expected payoff is given by

$$\int_0^{b_2} (v_1 + v_2 - 2x)h(x)dx + \int_{b_2}^{b_1} (v_1 - x)h(x)dx$$

where $h(x)$ is the density of the highest of the rival bidders' bids, and the first-order conditions for maximizing this expression are given by

$$(v_1 - b_1)h(b_1) = 0$$

and

$$(v_1 + v_2 - 2b_2)h(b_2) - (v_1 - b_2)h(b_2) = 0$$

or,

$$(v_2 - b_2)h(b_2) = 0,$$

and the result follows.

Low-bid uniform-price auction. Suppose bidders $2, ..., n$ are bidding the same amount on both units. Consider a valuation pair (v, v) for bidder 1 when v is less than the highest joint bid made by the other bidders.

Bidder 1's bid for the first unit can only affect his probability of winning this unit and does not influence his price when he wins this unit.[5] Hence he must bid v on the first unit. For the second unit he must bid v or less. He cannot have a second bid v since that will give him zero expected payoff whenever he wins. Hence his best response on the second unit must be less than v. Thus we have proved that there is no equilibrium where for every pair of values the bidder must bid equal amounts on both units. ■

Proof of Proposition 2. *Vickrey auction.* Suppose bidders $2, .., n$ are bundle-bidding. Let $h(x, y)$ denote the joint density of the highest bidder's and the second highest bidders' bids (as opposed to the highest and the second highest bids that are equal in this case) from bidders $2, ..., n$. The expected payoff to bidder 1 from bidding $b_1 \geq b_2$ is

$$\int_0^{b_2}\int_0^{x}(v_1 + v_2 - x - y)h(x, y)dydx + \int_{b_2}^{b_1}\int_0^{x}(v_1 - y)h(x, y)dydx$$
$$+ \int_{b_1}^{\bar{b}}\int_0^{b_1}(v_1 - y)h(x, y)dydx$$

The result then follows upon observing that the derivative of this expression with respect to b_i is positive if $b_i < v_i$, $i = 1, 2$.

Discriminatory-price auction. Suppose that bidders $2, ..., n$ are bundle-bidding. Let $H_i(x)$ denote the distribution of the i-th highest bidder's joint bid from these bidders $(i = 1, 2)$. If b_1^* and b_2^* with $b_1^* > b_2^*$ are bidder 1's best response bids for $v_1 > v_2$ his expected payoff is

$$(v_1 - b_1^*)H_2(b_1^*) + (v_2 - b_2^*)H_1(b_2^*).$$

Now consider the partial derivatives at (b_1^*, b_2^*). If $(v_2 - b_2^*)h_1(b_2^*) - H_1(b_2^*) = 0$ then $(v_1 - b_2^*)h_1(b_2^*) - H_1(b_2^*) < 0$ so that with value (v_1, v_1) the best response second bid b_2

[5] Since the other bidders are bidding the same amount on both units, bidder 1's highest bid cannot be the second highest bid in the auction.

is strictly less than b_2^*. Thus it is enough to show that the best response bids at (v,v) are always equal.

Suppose not, and that for some (v,v) the best response bids b_1 and b_2 satisfy $b_1 > b_2$. Then it must be the case that

$$\frac{d}{db}(v-b)H_2(b)|_{b=b_1} = 0 \text{ and } \frac{d}{db}(v-b)H_1(b)|_{b=b_1} \leq 0$$

or,

$$\frac{H_1(b)}{h_1(b)} \geq \frac{H_2(b)}{h_2(b)} \tag{1}$$

for some $b > 0$, or, equivalently $H_1(b)h_2(b) - H_2(b)h_1(b) \geq 0$ for some $b > 0$.

Let $G(\cdot)$ denote the joint (bundle-) bid for a unit by bidders $2,..,n$, and $g(\cdot)$ the corresponding density. Then $H_1(b) = G(b)^{n-1}$ and $H_2(b) = G(b)^{n-1} + (n-1)G(b)^{n-2}(1-G(b))$ implies that

$$\begin{aligned} & H_1(b)h_2(b) - H_2(b)h_1(b) \\ = \; & G(b)^{n-1}(n-1)(n-2)G(b)^{n-3}(1-G(b))g(b) \\ & -(G(b)^{n-1} + (n-1)G(b)^{n-2}(1-G(b)))(n-1)G(b)^{n-2}g(b) \end{aligned}$$

Clearly, for $n = 2$ the expression is negative for all $b > 0$. If $n \geq 3$ it is equal to

$$\begin{aligned} & (n-1)G(b)^{2n-4}[(n-2)-(n-2)G(b)-G(b)-(n-1)+(n-1)G(b)]g(b) \\ = \; & -(n-1)G(b)^{2n-4}g(b) \\ < \; & 0 \end{aligned}$$

for all $b > 0$. In other words, inequality (1) cannot be satisfied for any $b > 0$ giving a contradiction to our hypothesis. Therefore, for all value pairs (v,v) the bids must be equal which implies that for all values the best response bids must be equal.

Low-bid uniform-price auction. Again, suppose that bidders $2,...,n$ are bundle-bidding. Let $H_1(x)$ and $H_2(x)$ denote the the probability distribution of the bids of the highest and the second highest of these bidders. If bidder 1 bids b_1 and b_2 satisfying $b_1 \geq b_2$, the expected payoff is equal to

$$\int_0^{b_2}(v_1+v_2-2x)h_1(x)dx+\int_{b_2}^{b_1}(v_1-x)h_1(x)dx+(v_1-b_1)(H_2(b_1)-H_1(b_1))$$

The derivative with respect to b_1 is $(v_1-b_1)h_2(b_1)-(H_2(b_1)-H_1(b_1))$ which is negative at $b_1=v_1$. The derivative with respect to b_2 is $(v_2-b_2)h_1(b_2)$ which is positive for $b_2<v_2$. This implies that a bundle-bid is a best response to other bidders bundle-bidding.

High-bid uniform-price auction. If bidders $2,..,n$ bundle-bid then at a value (v,v) bidder 1's expected payoff is

$$\int_0^{b_2}(v+v-2x)h_1(x)dx+\int_{b_2}^{b_1}(v-x)h_2(x)dx+(v-b_2)(H_2(b_2)-H_1(b_2))$$

where H_1 and H_2 are defined the same way as in the low-bid uniform-price auction. The derivative with respect to b_1 is $(v-b_1)h_2(b_1)$ which is positive for all $b_1<v$. The derivative with respect to b_2 is $(v-b_2)h_1(b_2)-(H_2(b_2)-H_1(b_2))$ which is negative at $b_2=v$. Thus when the other bidders are bundle-bidding, at value (v,v) the best response is not a bundle-bid. Hence there is no bundle-bidding equilibrium. ■