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Taste Asymmetries and Trade Patterns

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Abstract

We study trade patterns in a pure exchange economy where preferences are symmetric up to taste intensity parameters. In a 2−person, 2−good endowment economy, then all endowments in a particular Edgeworth box rectangle require trading out of that rectangle. Under strictly quasi−concave preferences, trade will occur away from a larger area of initial endowments. The identified area is larger still when preferences are homothetic and identical up to taste intensity parameters. Implications for the factor price equalization theorem are explored.

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1. Introduction

Symmetry and market exchange are intimately related phenomena. Market transactions are motivated by asymmetries in tastes or endowments. Consequently, in a pure exchange economy there should be fundamental structural relationships between the nature of asymmetries among consumers, equilibrium decisions by consumers, and the equilibrium prices that guide these decisions. Our interest is in identifying and characterizing how heterogeneities in tastes over the set of available goods affect general equilibrium. We show how particular forms of heterogeneities among consumers imply that certain consumption bundles, and so certain endowment-dependent trade patterns, cannot be supported regardless of consumer income levels. This is because the terms of trade to support these equilibria are inconsistent with consumer preferences, when considered collectively.

To identify cardinal aspects of general equilibrium relationships in a pure exchange economy it is necessary to employ tools that model similarities and dissimilarities in preferences and endowments. Mathematical science provides a number of related tools, such as the theories of group structures and vector majorization, to model structural symmetries. These tools have been used in diverse fields of economics, as in Samuelson and Swamy (1974), Sato (1981), Balasko (1990), and Chambers and Quiggin (2000). Hennessy and Lapan (2003a, 2003b), who studied firm-level production decisions (2003a) and investor-level portfolio allocation decisions (2003b), provide the most direct links to the approach we will take. Their analyses exploit the symmetries of a functional when a group acts on the functional's arguments. Contradictions then generate bounds on optimal decision vectors.

The present analysis is also built upon the contradictions that symmetries can generate. In a 2-agent, 2-good pure exchange economy, we use invariances to show when some trade pattern, call it E-I, must occur. When monotone utility functions are bilaterally symmetric up to symmetry-breaking scale parameters then conditions exist such that an endowment partition located in one rectangle of the Edgeworth box must support trade pattern E-I. These conditions pertain to the relative strength of preferences and the relative scarcity of endowments. If, in addition to bilateral symmetry, the utility functions are quasi-concave then the region in the allocation box that supports trade pattern E-I may be expanded to include two additional contiguous triangles. If, as well, preferences are homothetic, are identical apart from taste intensity parameters, and possess an elasticity of substitution less than unity, then the region supporting E-I is larger still. When considered separately, neither identical preferences nor homothetic preferences support the larger precluded region.

2. Motivation

In a 2-person pure exchange economy, goods A and B are available in the amounts $\bar{q}_a$ and $\bar{q}_b$. Person 1 has composite utility function $U^1[T^{1,a}(q_{1,a}),T^{1,b}(q_{1,b})]$ while person 2 has utility $U^2[T^{2,a}(q_{2,a}),T^{2,b}(q_{2,b})]$. Functions $T^{1,a}(q_{1,a})$, $T^{1,b}(q_{1,b})$, $T^{2,a}(q_{2,a})$, and $T^{2,b}(q_{2,b})$ are $\mathbb{R}\to\mathbb{R}$ continuously differentiable, monotone increasing maps, while functionals $U^1[\cdot,\cdot]$ and $U^2[\cdot,\cdot]$ are $\mathbb{R}^2\to\mathbb{R}$. All are strictly increasing (i.e., monotone or non-satiated) and once continuously differentiable. There is no waste in allocation and the goods are scarce, so that efficiency requires both $q_{1,a}+q_{2,a}=\bar{q}_a$ and $q_{1,b}+q_{2,b}=\bar{q}_b$. We ask what can be inferred about trade patterns, about Pareto efficient divisions of $\bar{q}_a$ and $\bar{q}_b$, as well as about how prices and quantities relate in general equilibrium.

Further assumptions are clearly necessary. Because Pareto efficiency requires the exhaustion of exchange opportunities due to consumer heterogeneities, the assumptions will involve restrictions on how goods substitute within a consumer's basket of purchases. We place symmetry restrictions on a consumer's iso-utility trade-offs. Pareto improving re-allocations can then be identified by using the symmetries to hold the utility level of one consumer constant while freeing up endowments to make the other consumer better off.

Asymmetries are necessary to model taste differences, while it is necessary to modularize the asymmetries if meaningful insights are to be found. We model these asymmetries through the superscripted $T(\cdot)$ functions. The structure on the symmetries are modeled through the assumption that $U^i[T^{i,a}, T^{i,b}] = U^i[T^{i,b}, T^{i,a}], i \in \Omega_2 = \{1, 2\}$.

To illustrate, we study ray linear $T(\cdot)$ functions; $T^{i,a}(q_{i,a}) = \theta_{i,a} q_{i,a}, i \in \Omega_2$, $T^{i,b}(q_{i,b}) = \theta_{i,b} q_{i,b}, i \in \Omega_2$, where the θ values are strictly positive, symmetry-breaking, taste intensity heterogeneities.[1] Our line of approach exploits the invariances of the two functions $U^1[\theta_{1,a} q_{1,a}, \theta_{1,b} q_{1,b}]$ and $U^2[\theta_{2,a} q_{2,a}, \theta_{2,b} q_{2,b}]$. The ratios $z_1 = \theta_{1,a} / \theta_{1,b}$ and $z_2 = \theta_{2,a} / \theta_{2,b}$ are clearly important in this regard because they gauge a consumer's personal relative intensity of preference for good A. Ratio $z_q = \overline{q}_b / \overline{q}_a$ should also be important because it measures the relative economy-wide scarcity of good A.

3. Symmetric and monotone utility

Denote the set of Pareto efficient allocations as $\tilde{Q}$ with sample elements given by the allocation 2-vector $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$, where "'" is the vector transpose operation. The endowment constraint determines two of the points in this quadruple. Apart from singularities arising from any of the relations $z_q = z_1$, $z_q = z_2$, or $z_1 = z_2$, there are essentially two contexts to be considered. When $z_1 > (<) z_2$ then person 1 (person 2) has a stronger comparative preference for good A. Without further loss of generality, and ignoring equality in relative preference intensities for the moment, we may assign the order $z_1 > z_2$. Also, define $r_1 = z_1 / z_q$ and $r_2 = z_2 / z_q$ to account for the relative abundance of endowments, and so our assignment is equivalent to $r_1 > r_2$.

Upon, again, ignoring singularities that may be studied separately, we may assume that either $z_q \in (z_2, z_1)$ or $z_q \notin [z_2, z_1]$. Defining $\tilde{q}_{1,a} = \delta_a \overline{q}_a$ and $\tilde{q}_{1,b} = \delta_b \overline{q}_b$, the endowment constraints require $\tilde{q}_{2,a} = (1-\delta_a)\overline{q}_a$ and $\tilde{q}_{2,b} = (1-\delta_b)\overline{q}_b$. After exploiting invariances, it is clear that bilateral symmetry in the utility functionals imply $U^1[\theta_{1,a}\delta_a\overline{q}_a, \theta_{1,b}\delta_b\overline{q}_b] = U^1[\theta_{1,a}\delta_b\overline{q}_b / z_1, \theta_{1,b}\delta_a\overline{q}_a z_1]$. Denoting the bundle indifference relation by $\sim$ and identifying vectors $\vec{c}_1 = (\delta_a\overline{q}_a, \delta_b\overline{q}_b)'$, $\vec{c}_1^{\,+} = (\delta_b\overline{q}_b / z_1, \delta_a\overline{q}_a z_1)'$, we have $\vec{c}_1 \sim \vec{c}_1^{\,+}$. Similarly, if we define $\vec{c}_2 = ((1-\delta_a)\overline{q}_a, (1-\delta_b)\overline{q}_b)'$ and $\vec{c}_2^{\,+} = ((1-\delta_b)\overline{q}_b / z_2, (1-\delta_a)\overline{q}_a z_2)'$, then $\vec{c}_2 \sim \vec{c}_2^{\,+}$. However, and this is the foundation of our analysis, if pair $\{\vec{c}_1^{\,+}, \vec{c}_2^{\,+}\}$ frees up resources then pair $\{\vec{c}_1, \vec{c}_2\}$

[1] The approach for general transformations is sketched out in appendix A.

cannot be Pareto efficient.[2]

PROPOSITION 1. *Let the preference functionals $U^1[\cdot,\cdot]$ and $U^2[\cdot,\cdot]$ be monotone and also symmetric functions of the ray linear $T(\cdot)$ functions, while $r_1 > 1 > r_2$. Then any endowment in the Edgeworth box rectangle defined by*

$$\delta_a \le \frac{1-r_2}{r_1 - r_2}, \qquad \delta_b \ge \frac{(1-r_2)r_1}{r_1 - r_2}, \tag{1}$$

with one inequality strict, will require a trade pattern in which Person 1 exports good B and imports good A.

The situation is depicted in Figure 1, where $Q_a = (\overline{q}_a, 0)$ and $Q_b = (0, \overline{q}_b)$. Line L1:$q_{1,b} = \theta_{1,a} q_{1,a} / \theta_{1,b} = z_1 q_{1,a}$ from O1, the consumer 1 origin, gives the locus of consumption bundles that are invariant under the iso-utility symmetry for that consumer. These bundles sit on the consumer 1 axis of symmetry (AS$_1$). Line L2:$q_{2,b} = \theta_{2,a} q_{2,a} / \theta_{2,b} = z_2 q_{2,a}$ performs the same role for consumer 2. Lines L*i* are rays from the origin O*i* because $U^1[0,0]$ is invariant to permutation on the arguments. Conditions $r_1 > 1 > r_2$ require that these two AS lines intersect in the interior of the box and northwest of the diagonal line (L3) between the two origins. The point of intersection of L1 with L2 is

$$Y = (\hat{\delta}_a \overline{q}_a, \hat{\delta}_b \overline{q}_b), \qquad \hat{\delta}_a = \frac{1-r_2}{r_1 - r_2}, \qquad \hat{\delta}_b = \frac{(1-r_2)r_1}{r_1 - r_2}, \tag{2}$$

so that (1) may be interpreted as the pair of requirements $\delta_a \le \hat{\delta}_a, \delta_b \ge \hat{\delta}_b$.

Under these conditions we may rule out all points except the southeastern vertex, *Y*, of the northwestern rectangle in the Edgeworth box. The rectangle is shaded in Figure 1. For any point inside this inadmissible region there is a point somewhere else in the Edgeworth box such that both consumers are as well off while at least one of the resource constraints is slack. Vertex *Y* is special because it is the unique fixed point where the known invariances of both consumer utilities do not even alter the values of either bundle.

For a more detailed version of the argument, pick a candidate equilibrium point $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$ that happens to be in the inadmissible region of the Edgeworth box. There $(\tilde{q}_{1,a}, \tilde{q}_{1,b})$, as measured from O1, coincides with $(\tilde{q}_{2,a}, \tilde{q}_{2,b})$, as measured from O2. The points must coincide for a Pareto efficient equilibrium under strict monotonicity. Map $(\tilde{q}_{1,a}, \tilde{q}_{1,b}) \to (\hat{q}_{1,a}, \hat{q}_{1,b})$, with $\hat{q}_{1,a} = \tilde{q}_{1,b} / z_1$ and $\hat{q}_{1,b} = \tilde{q}_{1,a} z_1$, is also provided in the diagram. Map $(\tilde{q}_{2,a}, \tilde{q}_{2,b}) \to (\hat{q}_{2,a}, \hat{q}_{2,b})$, with $\hat{q}_{2,a} = \tilde{q}_{2,b} / z_2$ and $\hat{q}_{2,b} = \tilde{q}_{2,a} z_2$, is distinct, and so one must be careful that the endowment budgets are not broken. In matrix form, the maps are given as

$$\begin{pmatrix} 0 & z_i^{-1} \\ z_i & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_{i,a} \\ \tilde{q}_{i,b} \end{pmatrix} = \begin{pmatrix} \hat{q}_{i,a} \\ \hat{q}_{i,b} \end{pmatrix}, \quad i \in \Omega_2. \tag{3}$$

The endowment constraints are not broken because the slope (really an arc marginal rate of substitution) for map $(\tilde{q}_{1,a}, \tilde{q}_{1,b}) \to (\hat{q}_{1,a}, \hat{q}_{1,b})$, being $-z_1$, differs from the slope for map

[2] Proofs are provided in appendix B.

$(\tilde{q}_{2,a}, \tilde{q}_{2,b}) \to (\hat{q}_{2,a}, \hat{q}_{2,b})$, $-z_2$. The supermodular nature of the transformations, θq, ensures that resources are freed up. The released resources are represented by the vector between the map image points in Figure 1. An endowment vector located in the inadmissible region requires a trade to a consumption bundle outside the region and southeast of the bundle. Figure 2 identifies the endowment-conditioned admissible region, and we see that person 1 must barter out of good B.

4. Allocation under strictly quasi-concave utility

At this point we require both utility functions to be strictly quasi-concave so that the level sets are strictly convex and any equilibrium is unique. Symmetry, together with continuous differentiability and quasi-concavity imply that the Schur condition holds,[3]

$$(U_1^i[\cdot] - U_2^i[\cdot])(T^{i,a} - T^{i,b}) \leq 0, \quad i \in \Omega_2, \tag{4}$$

where $U_j^i[\cdot]$ represents the derivative with respect to the functional's j^{th} argument. With scaling symmetry breakers we have that efficient equilibria must satisfy

$$(U_1^i[\cdot] - U_2^i[\cdot])(\theta_{i,a}\tilde{q}_{i,a} - \theta_{i,b}\tilde{q}_{i,b}) \leq 0, \quad i \in \Omega_2. \tag{5}$$

Notice that, due to continuity, $U_1^i[\cdot] = U_2^i[\cdot]$ on the respective AS lines under quasi-concavity so that the marginal rates of substitution along the AS lines are given by

$$\frac{U_2^1[\cdot]\theta_{1,b}}{U_1^1[\cdot]\theta_{1,a}}\Big|_{\substack{(q_{1,a},q_{1,b})\in \mathrm{L1} \\ U^1[\cdot]\ \text{fixed}}} = \frac{\theta_{1,b}}{\theta_{1,a}} = z_1^{-1} = -\frac{\partial q_{1,a}}{\partial q_{1,b}}, \qquad \frac{U_2^2[\cdot]\theta_{2,b}}{U_1^2[\cdot]\theta_{2,a}}\Big|_{\substack{(q_{2,a},q_{2,b})\in \mathrm{L2} \\ U^2[\cdot]\ \text{fixed}}} = \frac{\theta_{2,b}}{\theta_{2,a}} = z_2^{-1} = -\frac{\partial q_{2,a}}{\partial q_{2,b}}. \tag{6}$$

The symmetry assumption, together with the scalar structure of the transformation functions, impose a local form of ray homotheticity on preference structures. The importance of strict quasi-concavity lies in the fact that any interior convex combinations of iso-utility points are Pareto improving, if feasible.[4]

PROPOSITION 2. *In addition to the assumptions in Proposition 1, let* $U^1[\cdot,\cdot]$ *and* $U^2[\cdot,\cdot]$ *be quasiconcave in the consumption arguments. Then any endowment in the Edgeworth box rectangle defined by* $(\delta_a, \delta_b) \in [0,1]\times[0,1]$ *with*

$$\delta_b \geq \max\left[\delta_a r_1, \delta_a r_2 + 1 - r_2\right] \tag{7}$$

will require a trade pattern in which Person 1 exports good B and imports good A.

An intuitive approach to confirming (7) is to note that any point such that the marginal rates of substitution across consumers differ can be precluded. From (5) and (6) we have

$$\begin{aligned} -\frac{\partial q_{1,b}}{\partial q_{1,a}}\Big|_{U^1[\cdot]\ \text{fixed}} &= \frac{U_1^1[\cdot]\theta_{1,a}}{U_2^1[\cdot]\theta_{1,b}} \geq (\leq) z_1 \quad \text{whenever} \quad \frac{\theta_{1,a}q_{1,a}}{\theta_{1,b}q_{1,b}} \equiv \frac{z_1}{z_q}\frac{\delta_a}{\delta_b} \leq (\geq) 1, \\ -\frac{\partial q_{2,b}}{\partial q_{2,a}}\Big|_{U^2[\cdot]\ \text{fixed}} &= \frac{U_2^2[\cdot]\theta_{2,a}}{U_1^2[\cdot]\theta_{2,b}} \geq (\leq) z_2 \quad \text{whenever} \quad \frac{\theta_{2,a}q_{2,a}}{\theta_{2,b}q_{2,b}} \equiv \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)} \leq (\geq) 1. \end{aligned} \tag{8}$$

Upon requiring $z_1 > z_2$, the bounds in (8) also deliver (7).

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[3] See Marshall and Olkin (1979, p. 57 and p. 69). See Chambers and Quiggin (2000) for economic applications of the condition.

[4] Notice that $\delta_b / r_1 = (\delta_b + r_2 - 1)/r_2 = \delta_a$ defines point Y as given in Figure 1.

If, instead, we knew that $r_1 = r_2$, i.e., $z_1 = z_2$, then a larger region in the Edgeworth box could be precluded as equilibrium consumption bundles. Then we may rule out points such that

$$\frac{z_1\delta_a}{z_q\delta_b} \geq 1 > \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)}, \tag{9}$$

as well as those such that

$$\frac{z_1\delta_a}{z_q\delta_b} < 1 \leq \frac{z_2}{z_q}\frac{(1-\delta_a)}{(1-\delta_b)}. \tag{10}$$

The geometry of the excluded region depends upon the magnitudes of the r_i relative to unity. The situation for $r_1 > 1 > r_2$ is depicted in Figure 3. We leave it to the interested reader to study the other cases. The dashed parallel lines are tangents to some isoquant along that utility function's AS line. Because the AS lines intersect inside the box, at *Y*, it is clear from (6) above that the tangents on the AS lines must intersect to the north of L1 and west of L2. But any point north of L1 and west of L2 cannot be efficient because the utility indifference curves cannot be tangent there, i.e., $-\theta_{1,a}/\theta_{1,b} < -\theta_{2,a}/\theta_{2,b}$. For any given pair of AS, this trapezoid is larger than the area precluded in Figure 1. Quasi-concavity buys us the difference, namely two right-angled triangles each with a vertex at point *Y*.

A geometric interpretation of quasi-concavity is that, to exclude a point as an equilibrium consumption point, we only need to know that some point on the line segment connecting the original consumption point for person 2 with its reflection lies northeast of some point on the comparable line segment for person 1. Suppose point K in Figure 3 is posited as being Pareto efficient. It lies outside the excluded rectangle given in Figure 1, but it satisfies condition (7). To see why it can be excluded, observe the point's reflections through the two axes. For person 1 the reflected point is K', while for person 2 it is K''. Although K' and K'' are not comparable, a point on segment KK'' is northeast of a point on segment KK'. This means that by giving each person some convex combination of his original point and its reflection, a surplus of goods can be created. But at these same convex combinations the respective consumers are at least weakly better off. If K is the initial endowment point then consumers trade south and east.

5. Allocation under strictly quasi-concave and homothetic utility

As one might expect, the imposition of homotheticity can further expand the endowment set for which the trade pattern is certain. The argument concerns a comparison of slopes away from the axes of symmetry. We have

$$-\frac{dq_{1,b}}{dq_{1,a}}\Big|_{U^1[\cdot]\text{ fixed}} = \frac{\theta_{1,a}U_1^1[\theta_{1,a}q_{1,a},\theta_{1,b}q_{1,b}]}{\theta_{1,b}U_2^1[\theta_{1,a}q_{1,a},\theta_{1,b}q_{1,b}]} = z_1\phi^1(x_1), \quad x_1 \equiv \frac{\theta_{1,a}q_{1,a}}{\theta_{1,b}q_{1,b}} = \frac{\delta_a r_1}{\delta_b}, \tag{11}$$

where $\phi^1(\cdot)$ is the marginal rate of substitution function with respect to the transformed 'goods' $\theta_{1,a}q_{1,a}$ and $\theta_{1,b}q_{1,b}$, and where homotheticity has been used to express the ratio in terms of relative consumption. If person 2 also has a homothetic utility function, then

$$-\frac{dq_{2,b}}{dq_{2,a}}\Big|_{U^2[\cdot]\text{ fixed}} = \frac{\theta_{2,a}U_1^2[\theta_{2,a}q_{2,a},\theta_{2,b}q_{2,b}]}{\theta_{2,b}U_2^2[\theta_{2,a}q_{2,a},\theta_{2,b}q_{2,b}]} = z_2\phi^2(x_2), \quad x_2 \equiv \frac{\theta_{2,a}q_{2,a}}{\theta_{2,b}q_{2,b}} = \frac{(1-\delta_a)r_2}{(1-\delta_b)}, \tag{12}$$

with $\phi^2(\cdot)$ as the marginal rate of substitution function with respect to transformed 'goods'

$\theta_{2,a}q_{2,a}$ and $\theta_{2,b}q_{2,b}$.

Following the earlier analysis we would like to identify a domain on which the marginal rates of substitution cannot be common across the consumers. In this regard identical preferences, where we mean that $U^1(T',T'') \equiv U^2(T',T'')$, by itself does not help. Similarly, homotheticity by itself does not help. However under identical and homothetic preferences, so that bilateral symmetry then implies $\phi^1(1) = \phi^2(1) = 1$, we can conclude:

$$z_1 > z_2 \quad \text{implies} \quad -\frac{dq_{1,b}}{dq_{1,a}}|_{U^1[\cdot]\text{ fixed}} > -\frac{dq_{2,b}}{dq_{2,a}}|_{U^2[\cdot]\text{ fixed}} \quad \text{whenever} \quad x_1 \le x_2. \tag{13}$$

We cannot extend the deduction to the half-space $x_1 > x_2$ because we do not know how rapidly the marginal rate of substitution declines.

To summarize the contradiction that (13) provides, insert (11) and (12) and infer

PROPOSITION 3. *In addition to the assumptions in Proposition 2, let $U^1[\cdot,\cdot]$ and $U^2[\cdot,\cdot]$ be homothetic in the consumption arguments and identical, i.e., $U^1(T',T'') \equiv U^2(T',T'')$.*
A) *If $z_1 = z_2$, then the diagonal is the Pareto efficient set so that agents will trade to the diagonal.*
B) *If $z_1 > z_2$ then any endowment in the Edgeworth box rectangle defined by $(\delta_a, \delta_b) \in [0,1] \times [0,1]$ with*

$$\delta_b \ge \delta_b^* \equiv \frac{\delta_a r_1}{\delta_a r_1 + (1-\delta_a) r_2}. \tag{14}$$

will require a trade pattern in which Person 1 exports good B and imports good A.

Part A) is well-known. It arises because (13) then provides $x_1 \le x_2$ and $x_1 \ge x_2$ where $x_1 = x_2$ defines the main diagonal. As for part B), an inspection of (2) reveals that $\delta_b^* \ge (\le)$ $\delta_a r_1$ according as $\delta_a \le (\ge) \hat{\delta}_a$, while we also have that $\delta_b^* \ge (\le) \delta_a r_2 + 1 - r_2$ according as $\delta_a \ge$ $(\le) \hat{\delta}_a$. Put another way, we can write $\delta_b^* = \delta_b^*(\delta_a)$ and make the following observations. The function passes through the point $(\hat{\delta}_a, \hat{\delta}_b)$. It crosses L1 just once on the interior, and from above as δ_a increases. The function also crosses L2 just once (again at $(\hat{\delta}_a, \hat{\delta}_b)$) on the interior, but from below as δ_a increases. Partitioning the decision space, these observations require

$$(\delta_a \le \hat{\delta}_a) \Rightarrow (\delta_a r_2 + 1 - r_2 \ge \delta_b^* \ge \delta_a r_1), \qquad (\delta_a > \hat{\delta}_a) \Rightarrow (\delta_a r_2 + 1 - r_2 \le \delta_b^* \le \delta_a r_1). \tag{15}$$

Upon imposing the weaker of the two inequalities in either direction we have

$$\max[\delta_a r_1, \delta_a r_2 + 1 - r_2] \ge \delta_b^* \ge \min[\delta_a r_1, \delta_a r_2 + 1 - r_2] \tag{16}$$

regardless of the evaluation of δ_a. Comparing with the bound in (7), $\delta_b \ge$ $\max[\delta_a r_1, \delta_a r_2 + 1 - r_2]$, the joint impositions of identical homothetic $U^i(\cdot)$ does (weakly at any rate) extend the set of excludable points on $(\delta_a, \delta_b) \in [0,1] \times [0,1]$.

The extent to which the additional assumptions lead to the ruling out of a strictly larger area depends upon where the efficiency locus occurs relative to the principal diagonal. As is well-known, with homothetic preferences the efficiency locus cannot cut the diagonal, i.e., it either coincides with the diagonal or only the end points are common. If $z_1 = z_2$ then the efficiency locus is the main diagonal. Relative to this benchmark and for a given pair of allocation vectors, suppose we then increase the value of z_1. The effect is to increase the marginal rate of substitution for person 1 whenever the elasticity of substitution exceeds unity, $\sigma > 1$, and to decrease the marginal rate whenever $\sigma < 1$. For $z_1 > z_2$, the efficiency locus must be above the principal diagonal whenever $\sigma < 1$. In that case, (14) combines with the diagonal to provide tight bounds on the Pareto efficient set. However, when the elasticity of substitution exceeds unity then the efficiency locus will be below the main diagonal and (14) bears no information beyond that given by homotheticity and this knowledge on the elasticity of substitution.

Astute readers will note that this analysis can be applied to a two good, two factor model. In the re-interpreted model, consumers represent output, e.g., Person 1 becomes Good M and Person 2 becomes Good C, while the utility functions become the production functions. Comparably, the goods of the exchange model become the inputs $\left(A \to K, B \to L\right)$ of the production model. The basic technological assumption is that the two sectors have identical and symmetric production functions, but the degree of factor-augmenting technical progress, $\theta_{i,j}$, may differ across inputs and sectors. In this reinterpretation, $z_1 > z_2$ implies that "technical progress" is relatively capital-augmenting in sector 1 (M) as compared to sector 2 (C),[5] while $r^i > 1$ implies that, in sector *i* efficiency units, the economy is relatively "capital-rich" (i.e., in sector *i* units, the effective supply of capital exceeds that of labor).

If we assume, as is common, that production functions exhibit constant returns to scale (and hence are homothetic), the analysis underlying Proposition 3 rules out certain input allocations as being efficient. In particular, assuming $r^1 > 1 > r^2$, then no input allocation in the region defined by (14) can yield production efficiency. If we know that $\sigma < 1$, then points on or below the main diagonal can be also excluded, so the efficient allocation must satisfy

$$\delta_b^*\left(\delta_a\right) \geq \delta_b \geq \delta_a. \qquad (17)$$

If $\sigma > 1$ then all allocations above the diagonal can be excluded.[6]

Finally, the question arises as to whether this technique allows one to say for which initial allocations of resources free trade cannot lead to factor price equalization (FPE) between two countries, say the US and the EU. First, to allow for the possibility of FPE, in addition to the preceding assumptions we must assume that the US and EU have identical technologies. Given this assumption, the preceding analysis identifies where an efficient world input allocation - when factor mobility is allowed - may (or may not) occur. To see whether this integrated equilibrium (i.e., one with factor mobility) can be supported by free trade alone entails identifying the actual equilibrium (where factor and goods markets clear), and then considering whether national factor endowments lie within the "cone of diversification."[7]

[5] Given the assumed identical production function across sectors, $z_1 > z_2$ implies that sector 1 (M) is capital-intensive compared to sector C if $\sigma > 1$, while sector 1 is the labor-intensive sector if $\sigma < 1$.

[6] If $\sigma = 1$, i.e., if technology is Cobb-Douglas, then the assumptions imply that factor intensities will be the same in the two sectors, and the efficiency locus is the main diagonal.

[7] This region is found by locating the specific efficient world allocation of capital and labor

To be precise, let Φ be the set of input allocations that may correspond to production efficiency[8] and let $\phi \in \Phi$. Suppose ϕ is the actual equilibrium allocation under free trade and factor mobility; corresponding to ϕ there is a set of input allocations between the two countries that support this efficient allocation through free trade alone; call this set $\mathrm{K}(\phi)$.[9] If we knew ϕ were the actual equilibrium, then we would know the set of input allocations across countries that supported FPE (and hence the set of allocations that do not support FPE). However, if our knowledge is limited to the technology assumptions given above, then we must allow for the possibility the integrated equilibrium may be any element of Φ. Thus, define: $\mathrm{X} = \bigcup_{\phi \in \mathrm{K}} \mathrm{K}(\phi)$; if the input allocation between countries is not in this set, then free trade cannot lead to FPE; however, whether free trade does lead to FPE cannot be ascertained without knowing which equilibrium occurs as well as knowing the input allocation.[10]

Finally, what does set X (or its complement) look like? For $\sigma > 1$, since any point below the main diagonal can represent production efficiency, set X includes the whole (Edgeworth) box, and its complement is empty. Hence we cannot a priori identify allocations that cannot lead to FPE. On the other hand, if $\sigma < 1$, then only points which satisfy condition (17) (i.e., points above the main diagonal and below the δ_b^* locus) are candidates for efficient production, and hence the set of points which are candidates for FPE (the set X) lies within the set determined by the δ_b^* locus and its reflection around the diagonal. All input allocations outside this set cannot support FPE through free trade.

6. Conclusion

By way of the notion of exchange, we have developed several sets of relationships that symmetries and controlled heterogeneities in the primitives underlying a pure endowment economy imply for trade patterns in an efficient equilibrium. While we have confined the analysis to invariances in a 2-agent, 2-good economy, the framework naturally extends to more general sets of invariances. Then group theory and majorization theory, together with work in Eaton and Perlman (1977), may be useful.

Because symmetry structures have such strong implications for the nature of an efficient equilibrium, they should also have implications for how a failure in conditions underlying efficiency affects equilibrium. A number of extensions to the present work then arise naturally. What, for example, can symmetries in technologies and preferences convey when market power leads to strategic interaction? Symmetry structures on consumer preferences

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between the two goods, drawing rays from each origin through this point (representing factor-intensities), and then drawing parallel rays from the opposite origin. Upon reinterpreting the box as representing the division of the total world's resources between two countries, if the distribution of inputs between these countries lies within this constructed parallelogram, then for this particular equilibrium free trade will support FPE; if the allocation lies outside this area, then free trade without factor movements cannot support FPE for this equilibrium.

[8] That is, the complement of Φ are the allocations we can exclude as candidates for efficiency.

[9] Footnote 7 explains how the set $\mathrm{K}(\phi)$ is constructed.

[10] Of course, any input division between the two countries that lies on the main diagonal leads to FPE since, under constant returns to scale, the countries are identical and differ only in scale. No net trade in goods will occur in this case.

should provide further insights on bundling, and other price discrimination strategies by imperfectly competitive producers of differentiated goods. The present framework could also be expanded to accommodate Arrow-Debreu state-contingent equilibria, perhaps even when markets are incomplete.[11] Hopefully, efforts on related topics would also point to ways through which the insights provided in this paper can be sharpened.

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Appendix A

While scale transformations are convenient for describing symmetry structures, one can readily generalize beyond scale transformations. Define $\mu_{i,j} = T^{i,j}(q_{i,j})$ and label the inverse relation as $S^{i,j}(\mu_{i,j}) = (T^{i,j})^{-1}(\mu_{i,j}) = q_{i,j}$. Then bilateral symmetry in an utility functional can be described as $U^i[T^{i,a}(q_{i,a}), T^{i,b}(q_{i,b})] = U^i[T^{i,b}(q_{i,b}), T^{i,a}(q_{i,a})]$, $i \in \Omega_2$, or

$$U^i[T^{i,a}(q_{i,a}), T^{i,b}(q_{i,b})] = U^i[T^{i,a}(\hat{q}_{i,a}), T^{i,b}(\hat{q}_{i,b})], \quad i \in \Omega_2,$$

[11] Balasko (1990) has pointed to the role of temporal, rather than taste, asymmetries in identifying the nature of stationary general equilibrium under extrinsic uncertainty.

where $\hat{q}_{i,a} = S^{i,a}[T^{i,b}(q_{i,b})]$ and $\hat{q}_{i,b} = S^{i,b}[T^{i,a}(q_{i,a})]$. In this way we may conclude, for points not on both $\mu_{1,a}(q_{1,a}) = \mu_{1,b}(q_{1,b})$ and $\mu_{2,a}(q_{2,a}) = \mu_{2,b}(q_{2,b})$, that at least one of

$$\overline{q}_a \geq S^{1,a}[T^{1,b}(q_{1,b})] + S^{2,a}[T^{2,b}(\overline{q}_b - q_{1,b})], \qquad \overline{q}_b \geq S^{1,b}[T^{1,a}(q_{1,a})] + S^{2,b}[T^{2,a}(\overline{q}_a - q_{1,a})],$$

fails because otherwise an utility-preserving endowment surplus would exist.

To identify Pareto inefficient points under symmetry and monotonicity only, let equations

$$K^a(q'_{1,b}) = S^{1,a}[T^{1,b}(q'_{1,b})] + S^{2,a}[T^{2,b}(\overline{q}_b - q'_{1,b})],$$
$$K^b(q'_{1,a}) = S^{1,b}[T^{1,a}(q'_{1,a})] + S^{2,b}[T^{2,a}(\overline{q}_a - q'_{1,a})],$$

implicitly define a vector-valued function of $(q'_{1,a}, q'_{1,b})$ on the non-empty, compact, convex set $[\overline{q}_a, 0] \times [0, \overline{q}_b]$. If the function, and for both arguments, is continuous and into then Brouwer's fixed point theorem is satisfied (MasColell, Whinston, and Green (1995, p. 952)) and a fixed point exists in the Edgeworth box. If, in addition, one function is strictly increasing and the other is strictly decreasing then there exists a rectangle interior to the Edgeworth box where equilibrium consumption points cannot be located. These monotonicity conditions are satisfied whenever one of

$$\frac{t^{1,a}(q)}{t^{1,b}(q)} > \frac{t^{2,a}(\overline{q}_a - q)}{t^{2,b}(\overline{q}_b - q)}; \qquad \frac{t^{1,a}(q)}{t^{1,b}(q)} < \frac{t^{2,a}(\overline{q}_a - q)}{t^{2,b}(\overline{q}_b - q)};$$

holds on $\forall q \in [0, \min[\overline{q}_a, \overline{q}_b]]$, where $t^{i,j}(q_{i,j}) = dT^{i,j}(q_{i,j}) / dq_{i,j}$. The rectangle has $(q'_{1,a}, q'_{1,b})$ as one vertex, is bounded by the axes, and cannot contain a consumer origin.

The analysis in Section 4 can also be extended to the more general context. Since $U_1^i[\cdot] = U_2^i[\cdot]$ on the AS lines, the marginal rates of substitution along the AS lines are given by

$$\begin{aligned}
&\frac{U_2^1[\cdot]t^{1,b}(q_{1,b})}{U_1^1[\cdot]t^{1,a}(q_{1,a})}\Big|_{\substack{(q_{1,a},q_{1,b})\in \mathrm{L}1 \\ U^1[\cdot]\text{ fixed}}} = \frac{t^{1,b}(q_{1,b})}{t^{1,a}(q_{1,a})} = -\frac{\partial q_{1,a}}{\partial q_{1,b}},\\
&\frac{U_2^2[\cdot]t^{2,b}(q_{2,b})}{U_1^2[\cdot]t^{2,a}(q_{2,a})}\Big|_{\substack{(q_{2,a},q_{2,b})\in \mathrm{L}2 \\ U^2[\cdot]\text{ fixed}}} = \frac{t^{2,b}(q_{2,b})}{t^{2,a}(q_{2,a})} = -\frac{\partial q_{2,a}}{\partial q_{2,b}},
\end{aligned} \tag{A1}$$

where Li refers to the line identified by the equation $T^{i,a}(q_{i,a}) = T^{i,b}(q_{i,b})$. Section 4 may now be adapted, except that $t^{i,a}(q_{i,a}) / t^{i,b}(q_{i,b})$ replaces z_i.

The properties of strict quasi-concavity, strict monotonicity, differentiability, and bilateral symmetry (on the functional) also allow us to make deductions about equilibrium prices. In general equilibrium, (A1) implies

$$\frac{U_2^1[\cdot]t^{1,b}(\tilde{q}_{1,b})}{U_1^1[\cdot]t^{1,a}(\tilde{q}_{1,a})} = \frac{U_2^2[\cdot]t^{2,b}(\tilde{q}_{2,b})}{U_1^2[\cdot]t^{2,a}(\tilde{q}_{2,a})} = \frac{P_b}{P_a},$$

for Pareto efficient points so that (5) modifies to

$$[P_a t^{1,b}(\tilde{q}_{1,b}) - P_b t^{1,a}(\tilde{q}_{1,a})][T^{1,a}(\tilde{q}_{1,a}) - T^{1,b}(\tilde{q}_{1,b})] \leq 0,$$
$$[P_a t^{2,b}(\overline{q}_b - \tilde{q}_{1,b}) - P_b t^{2,a}(\overline{q}_a - \tilde{q}_{2,a})][T^{2,a}(\overline{q}_a - \tilde{q}_{1,a}) - T^{2,b}(\overline{q}_b - \tilde{q}_{1,b})] \leq 0,$$

upon imposing general equilibrium efficiency conditions. Thus, attending any solution $(\tilde{q}_{i,a}, \tilde{q}_{i,b})$ are constraints on the equilibrium price ratio P_b / P_a.

Appendix B

Proof of Proposition 1: With the end of identifying contradictions, we specify convex combinations of iso-utility bundles. Restrict $\lambda_i \in [0,1], i \in \Omega_2$, and define

$$D(\lambda_1, \lambda_2, \vec{c}_1, \vec{c}_2, \vec{c}_1^{\,+}, \vec{c}_2^{\,+}) = 1 \quad \text{whenever} \quad \vec{v} \geq \vec{0},\ \vec{v} \neq \vec{0}, \qquad \text{(B1)}$$
$$= 0 \quad \text{otherwise},$$

where $\vec{v} = (\bar{q}_a, \bar{q}_b)' - \lambda_1 \vec{c}_1 - (1-\lambda_1)\vec{c}_1^{\,+} - \lambda_2 \vec{c}_2 - (1-\lambda_2)\vec{c}_2^{\,+}$. If $D(\cdot) > 0$ for some reallocation of endowments that weakly increases all utility levels, then the candidate allocation $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$ cannot be Pareto efficient as a consumption bundle for utilities that are strictly monotone.

When all we know of the utility functions are that they are symmetric and monotone, then invariance only allows us to make deductions for the lattice points of the unit square, $(\lambda_1, \lambda_2) \in \Lambda^{lp} = \{(0,0), (0,1), (1,0), (1,1)\}$. When, in addition, strict quasi-concavity is known to hold then we may seek violations on any $(\lambda_1, \lambda_2) \in ch(\Lambda^{lp}) = [0,1] \times [0,1]$ where $ch(\cdot)$ is the convex hull set operation.

The comparisons in (B1) reduce to the assertion that $D(\cdot) = 1$ whenever

$$\lambda_1 \delta_a + (1-\lambda_1)\delta_b \frac{z_q}{z_1} + \lambda_2 (1-\delta_a) + (1-\lambda_2)(1-\delta_b) \frac{z_q}{z_2} \leq 1, \qquad \text{(B2)}$$
$$\lambda_1 \delta_b + (1-\lambda_1)\delta_a \frac{z_1}{z_q} + \lambda_2 (1-\delta_b) + (1-\lambda_2)(1-\delta_a) \frac{z_2}{z_q} \leq 1,$$

and one does not bind. On Λ^{lp}, i.e., for monotone, symmetric utilities only, then $D(\cdot) = 1$ if

$$\frac{\delta_b}{r_1} + \frac{(1-\delta_b)}{r_2} \leq 1, \qquad \delta_a r_1 + (1-\delta_a) r_2 \leq 1, \qquad \text{(B3)}$$

where one inequality is strict. The solution interval is non-degenerate only if $r_i > 1, r_j < 1,\ i \neq j; i, j \in \Omega_2$. Rearrange (B3) to obtain (1). For person 1, the value of δ_b is too high while that of δ_a is too low. If (1) represents the endowment of person 1, then she will barter out of B and into A.

Proof of Proposition 2: Returning to the program provided in (B2) and now choosing over any $(\lambda_1, \lambda_2) \in ch(\Lambda^{lp}) = [0,1] \times [0,1]$, some manipulation of (B2) shows that consumption bundles adhering to

$$(1-\lambda_2) r_1 M_1 \leq (1-\lambda_1) r_2 M_2 \leq (1-\lambda_2) r_2 M_1, \qquad \text{(B4)}$$
$$M_1 \equiv (1-\delta_b) - (1-\delta_a) r_2, \quad M_2 \equiv \delta_a r_1 - \delta_b,$$

with one inequality strict, constitute a violation of Pareto efficiency on the part of candidate optimum $\{(\tilde{q}_{1,a}, \tilde{q}_{1,b})', (\tilde{q}_{2,a}, \tilde{q}_{2,b})'\}$. Obviously the pair of inequalities is always satisfied when $\lambda_1 = \lambda_2 = 1$, i.e., when there is zero displacement along either arc between an initial consumption bundle and an iso-utility bundle.

If we assume that $r_1 \geq r_2$, then we need only consider two scenarios: $r_1 > r_2$ and $r_1 = r_2$ where we identify the common ratio as r. For the latter case the interval in (B4) that r_2 must satisfy is degenerate. In that case, it is clear that the set of points (δ_a, δ_b) satisfying $D(\cdot)=1$ in (B1) has positive measure if and only if M_1 and M_2 have the same sign, i.e., the only possible Pareto efficient solutions are such that $M_1 M_2 < 0$ or $M_1 = M_2 = 0$. Thus, with $r_1 = r_2 = r$, the following allocations are the only potential Pareto efficient allocations:

$$\begin{aligned} &(a) \quad r<1: \quad \text{all allocations such that} \quad (\delta_b/r) > \delta_a > (\delta_b/r) + ((r-1)/r), \\ &(b) \quad r=1: \quad \text{all allocations such that} \quad \delta_a = (\delta_b/r), \\ &(c) \quad r>1: \quad \text{all allocations such that} \quad (\delta_b/r) < \delta_a < (\delta_b/r) + ((r-1)/r). \end{aligned} \tag{B5}$$

All allocations not satisfying these bounds may be excluded. The most informative situation arises in case (b) when both taste intensity indices equal the index of relative scarcity, i.e., $r_1 = r_2 = 1$. Then the Pareto efficient locus must be the main diagonal. In both cases (a) and (c) the remaining admissible region is a parallelogram between the axes of symmetry.[12] Turning to scenario $r_1 > r_2$, the set of solutions ruled out by (B4) under strict quasi-concavity is empty if $M_1 > 0$ or $M_2 > 0 > M_1$. We may, however exclude as consumption bundles all points such that both $M_1 \le 0$ and $M_2 \le 0$. These exclusions are represented in (7).

[12] To conserve on space we have not drawn the associated regions. However, we encourage the reader to sketch them out.

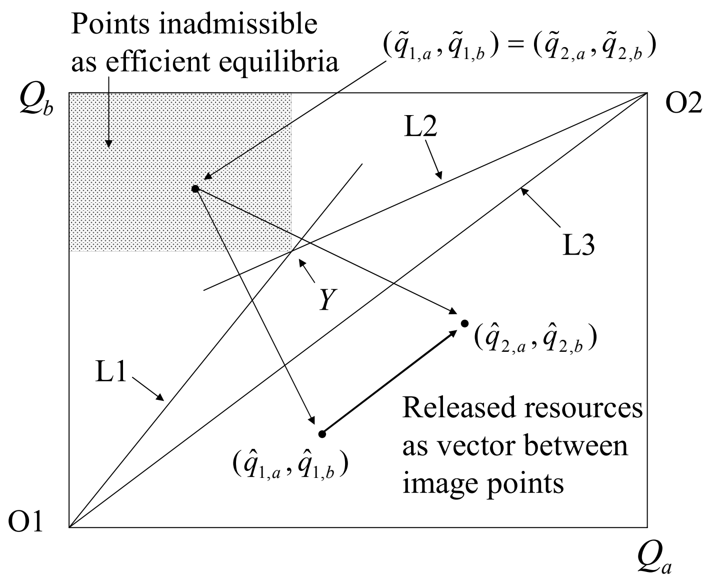

FIGURE 1.–Inadmissible equilibria under symmetry, strict monotonicity, and $r_1 > 1 > r_2$

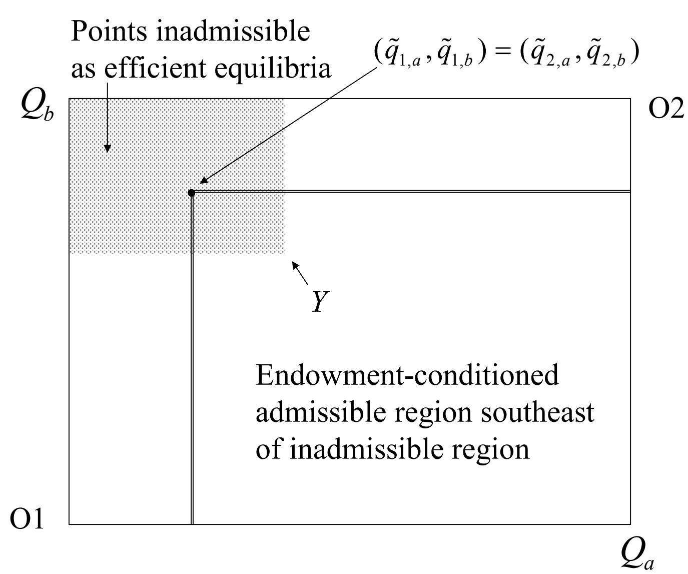

FIGURE 2.–Endowment-conditioned admissible equilibria under symmetry, strict monotonicity, and $r_1 > 1 > r_2$

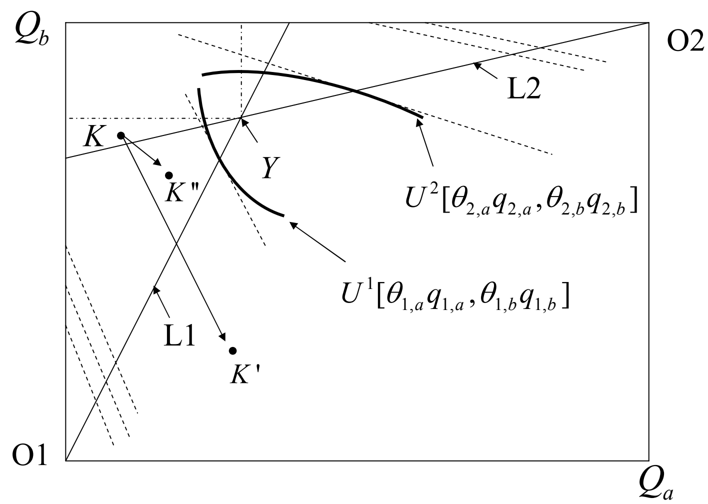

FIGURE 3.–Implications of bilateral symmetry for marginal rates of substitution under quasi-concave preferences.