E C O N O M I C S B U L L E T I N

Optimal Price Ceilings in a Common Value Auction

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Abstract

A simple common value auction is considered where it is optimal to set a ceiling price in addition to a reserve price. The ceiling price prevents the better informed bidder from outbidding the less informed bidders. This guarantees participation from the less informed bidders raising the seller's revenues. The seller is better off by selling the good in an auction with a price ceiling compared to selling the good at a fixed price.

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1 Introduction

In common value auctions bidders take into account the winner's curse problem and do not bid aggressively, with adverse consequences for seller revenues. Such winner's curse costs become particularly severe when some bidders have inferior information compared to other bidders. In such cases, conditional on winning the auction with any bid, bidders with inferior information realize that the value of the good is likely to be lower than the price paid. As a result, they may not bid at all.

In this note, we provide a simple three bidder example of such an auction. One of the three bidders has superior information to the other two. As a result, the less informed bidders do not participate in the auction. When the better informed bidder buys the object, he does so at the reserve price.

To alleviate this severe winner's curse problem, it is optimal for the seller to set a price ceiling in addition to a price floor (reserve price). Due to the presence of a price ceiling, the better informed bidder cannot outbid the less informed bidders when the value of the good is higher than the ceiling. As a result, if a bidder wins at the ceiling price he knows that with positive probability, the object could have value higher than the ceiling price. This alleviates the winner's curse problem for less informed bidders and all bidders sometimes bid at the ceiling.

If instead of holding an auction, the seller offers a fixed price at which to sell the good, then the better informed bidder is also not able to outbid the less informed bidders when the value of the good exceeds this fixed price. Thus, a fixed price could also generate participation from the less informed bidders. However, a bidder winning at the fixed price necessarily has to pay that price. In contrast, for a second price auction, a bidder bidding at the ceiling does not necessarily have to pay the ceiling price. This implies that the winner's curse problem is less severe for the less informed bidders in an auction with a ceiling price. Consequently, the seller is better off selling the good in a second price auction with a price ceiling compared to offering a fixed price at which to sell the good.

2 The Model

There is one indivisible unit of an object with (common) value $v \in [0, 1]$. There are three bidders. Bidders 1 and 2 each observe different realizations of a noisy informative signal s

of v. Bidder 3 knows v. We will assume that v is distributed uniformly in $[0,1]$ and that for $i \in \{1,2\}$ the signals $s_i \in \{H, L\}$ with $\Pr[s_i = H|v] = v$. Further, conditional on v, the signals s_1 and s_2 are independent.

The seller holds a second price auction. He can choose a reserve price $p_r \in [0,1]$ and the highest bidder wins the object and pays the maximum of the second highest bid and p_r. We allow the seller to also choose a ceiling price $p_c \in [0,1]$, with $p_c \geq p_r$. That is, the seller only accepts bids above p_r and below p_c.

Given a ceiling price, ties will not be zero probability events in this auction, especially because some bidders may bid at the ceiling price. Throughout we will assume that ties are broken uniformly— when n bids tie, each bidder gets the good with probability $\frac{1}{n}$.

Notice that choosing $p_c = p_r = p$ is equivalent to selling the object at the fixed price p. However, it will be shown that a ceiling price of $p_c < 1$ and a reserve price $p_r > 0$, with $p_c > p_r$ is optimal for seller revenues.

Given a choice of a ceiling price and a reserve price, we will look at Bayesian Nash equilibria of the resulting auction. Let $b_i(s)$ be the bid of bidder $i \in \{1,2\}$ and $b_3(v)$ the bidding strategy of bidder 3. We will restrict attention to symmetric equilibria with respect to bidders 1 and 2 where $b_1(s) = b_2(s) = b(s)$ for each s.

2.1 Second Price Auction: No Ceiling Price

We first consider a second price auction with no ceiling price — $p_c = 1$.

Claim 1 *Suppose $p_c = 1$. In any symmetric equilibrium with respect to bidders 1 and 2, $b_i(s) = 0$ for all $s \in \{H, L\}$ and bidder 3 bids v whenever $v \geq p_r$. The optimal reserve price is $p_r = 0.5$ for an expected revenue of* 0.25.

Proof. For any $p_r \in [0,1]$, it can easily be checked that a weakly dominant strategy for bidder 3 is to bid v whenever $v \geq p_r$.

Further, in any symmetric equilibrium, we must have that bidders 1 and 2 do not bid, i.e., $\Pr[b(s) > 0] = 0$. For suppose $\Pr[b(s) > 0] > 0$ and let $\max_s b(s) = \overline{b} > 0$. When bidder 1 bids $\overline{b}$ and wins the auction then $v \leq \overline{b}$. If $v \geq b_2$ then bidder 1 pays v and makes zero profits. However if $v < b_2$ bidder 1 pays b_2 and makes negative profits. Since b_2 is strictly positive with strictly positive probability, bidder 1 makes negative expected profits from bidding $\overline{b}$ and so is better off not bidding.

Thus, in any symmetric equilibrium with respect to bidders 1 and 2, only bidder 3 bids a positive amount with positive probability and always pays the price p_r. The seller chooses the reserve price to solve

$$\max \ p_r(1-p_r)$$

Thus the optimal reserve price is $p_r^* = 0.5$ for an expected revenue equal to 0.25.[1] ■

2.2 Ceilings & Floors

A ceiling price will reduce the severe winner's curse problem seen in the symmetric equilibrium above, as long as bidders sometimes hit the ceiling. This is because, conditional on winning at the ceiling price, there is positive probability that the value of the good is also above the ceiling price. The next claim shows the bidding behavior in any symmetric equilibrium, given a ceiling price $p_c < 1$.

Claim 2 *Suppose $0 \leq p_r \leq p_c < 1$. In any symmetric equilibrium with respect to bidders 1 and 2, bidder $i \in \{1,2\}$ either bids p_c or does not bid (bids 0). Bidder 3 bids $\min(p_c, v)$ when $v \geq p_r$.*

Proof. It is straightforward to check that it is a weakly dominant strategy for bidder 3 to behave as specified, given that he cannot bid above p_c.

Suppose bidders $i \in \{1,2\}$ bid in the interval $[p_r, p_c)$ with strictly positive probability, in some symmetric equilibrium. For any such bid, if bidder i wins the auction, then v is not greater than his bid. Bidder i earns zero profits if bidder 3 is the second highest bidder with $v \geq p_r$; and earns strictly negative profits if $v < p_r$ or if bidder $j \in \{1,2\}$, $j \neq i$, is the second highest bidder. Since the last possibility has strictly positive probability in our symmetric equilibrium, it is better for bidder i to bid 0 instead of bidding any amount in the interval $[p_r, p_c)$. ■

In contrast to a bid strictly less than p_c, if bidder $i \in \{1,2\}$ wins at a bid equal to p_c, v might be strictly greater than the price paid with strictly positive probability. Given a

[1] When $p_r = 0$, there are also asymmetric equilibria of the form $b_1(s) = b \in (0,1]$ for all s, $b_2(s) = 0$ for all s. Bidder 3 bids v. The maximum revenue equilibria of this form has $b = 1$ for revenue of $\frac{1}{2}$. However such equilibria do not survive iterated deletion of (weakly) dominated strategies. For any reserve price $p_r > 0$, bidders 1 and 2 will not bid any positive amount in any equilibrium, as they make strictly negative profit when they bid more than p_r and $v < p_r$.

choice of a ceiling price and a floor price, there are thus two possible cases to consider. In the first case, bidders 1 and 2 bid at the ceiling price regardless of their signals s_1 and s_2. In the second case, bidders 1 and 2 bid at the ceiling only when their signal equals H. We consider each of these two cases in succession.

A Low Ceiling (or Fixed) Price Consider a ceiling price p_c such that, in equilibrium, bidders 1 and 2 bid the price p_c regardless of their signal. The object is sold with probability 1 at a price equal to p_c. Thus, setting such a ceiling price is equivalent to setting a fixed price.

Let the density of v given s be denoted by $f(v|s)$:

$$f(v|s) \equiv \begin{cases} 2(1-v) & \text{if } s = L \\ 2v & \text{if } s = H \end{cases} \tag{1}$$

Given the behavior of bidder 3 and bidder $j \in \{1,2\}$, $j \neq i$, when bidder $i \in \{1,2\}$ bids p_c he wins with probability $\frac{1}{3}$ when $v \geq p_c$ and $\frac{1}{2}$ otherwise, regardless of the signal of bidder j. And he always pays the price p_c when he wins. Thus, the expected profit for bidder $i \in \{1,2\}$ from bidding p_c given a signal s, is equal to

$$\frac{1}{2}\int_0^{p_c}(v-p_c)f(v|s)dv + \frac{1}{3}\int_{p_c}^1(v-p_c)f(v|s)dv. \tag{2}$$

Bidder i would be willing to bid at p_c as long as he earns non–negative profits.

The optimal such price should be set such that bidder $i \in \{1,2\}$ makes zero expected profits from bidding p_c when $s = L$. Using (1) in (2) for $s = L$, and performing the integrations we obtain that p_c must solve:

$$\frac{1}{36}p_c^3 - \frac{1}{12}p_c^2 + \frac{1}{18} - \frac{1}{6}p_c = 0$$

The solution is $p_c = 0.29428$. Using this value of p_c in (2) for $s = H$, we obtain that the expected profit for bidder $i \in \{1,2\}$ from bidding p_c, given a signal $s = H$, is equal to 0.12271, so that bidder $i \in \{1,2\}$ is also willing to bid p_c when $s_i = H$. Since the good is sold with probability 1 for such a ceiling (fixed) price, the expected revenue for the seller is equal to 0.29428. Thus, the seller is better off from charging this fixed price compared to the auction above.

A High Ceiling We now turn to the second case. Suppose that the seller sets p_c and p_r such that, in equilibrium, bidders 1 and 2 do not bid when their signal equals L, and bid p_c when their signal equals H. As usual, bidder 3 bids p_c if $v \geq p_c$ and bids v when $v \in [p_r, p_c)$ and does not bid otherwise.

Bidding p_c is optimal for bidders 1 and 2 given s if they earn non–negative expected profits from doing so. Let $\pi(v, s'; p_c, p_r)$ be the expected profit for bidder $i \in \{1, 2\}$ from bidding p_c, for each v and bidder $j \neq i$'s signal s', given that all the other bidders are bidding as above. The table below describes $\pi(v, s'; p_c, p_r)$ for different values of v and s':

	$s' = H$	$s' = L$
$v \geq p_c$	$(v - p_c)/3$	$(v - p_c)/2$
$p_c > v \geq p_r$	$(v - p_c)/2$	0
$p_r > v$	$(v - p_c)/2$	$v - p_r$

(3)

Let the joint density of v and s' given s be denoted by $f(v, s'|s)$. Then

$$f(v, s'|H) \equiv \begin{cases} 2(1-v)v & \text{if } s' = L \\ 2v^2 & \text{if } s' = H \end{cases} \tag{4}$$

and

$$f(v, s'|L) \equiv \begin{cases} 2(1-v)^2 & \text{if } s' = L \\ 2v(1-v) & \text{if } s' = H \end{cases} \tag{5}$$

Let $\pi_s(p_c, p_r)$ be the expected profit from bidding p_c given a signal $s \in \{H, L\}$, and given p_c and p_r:

$$\pi_s(p_c, p_r) \equiv \int_0^1 \sum_{s'} \pi(v, s'; p_c, p_r) f(v, s'|s) dv \tag{6}$$

The specified behavior above is an equilibrium if the expression in (6) is non–negative when $s = H$ is non–positive when $s = L$.

The price obtained by the seller is the ceiling price if $v \geq p_c$ and either s_1 or s_2 equals H or if $v < p_c$ and both s_1 and s_2 equal H. If $v \in [p_r, p_c)$ and exactly one of s_1 and s_2 equals H, then the price obtained is v. The price is equal to p_r otherwise, as long as the good is sold. The good is not sold if $v < p_r$ and both s_1 and s_2 equal L. Thus the expected revenue for the seller is:

$$\begin{aligned}
\Pi(p_c,p_r) &= \int_0^{p_r}(p_c v^2 + 2p_r v(1-v))dv \\
&+ \int_{p_r}^{p_c}(p_c v^2 + 2v^2(1-v) + p_r(1-v)^2)dv \\
&+ \int_{p_c}^{1}(p_c v^2 + 2p_c v(1-v) + p_r(1-v)^2)dv \qquad (7)
\end{aligned}$$

We look for the ceiling and floor prices that solve:

$$\begin{aligned}
&\max_{p_c,p_r} \Pi(p_c,p_r) \\
&\text{s.t.} \\
&\text{(i) } \pi_H(p_c,p_r) \geq 0, \\
&\text{(ii) } \pi_L(p_c,p_r) \leq 0 \\
&\text{(iii) } p_r \leq p_c
\end{aligned} \qquad (8)$$

We solve the problem by ignoring the last two constraints (ii) and (iii) and then checking that they are satisfied at the optimum. Performing the integrations in (6) and (7), the relaxed problem is

$$\begin{aligned}
&\max_{p_c,p_r} \tfrac{1}{6}p_c^4 - \tfrac{1}{3}p_c^3 + \tfrac{2}{3}p_c - \tfrac{1}{2}p_r^4 + \tfrac{4}{3}p_r^3 - p_r^2 + \tfrac{1}{3}p_r \\
&\text{s.t.} \\
&-4p_c^4 - 14p_c + 6p_c^3 + 6p_r^4 - 12p_r^3 + 9 \geq 0
\end{aligned} \qquad \text{(P)}$$

The constraint in (P) must bind. Computations yield that the optimal prices are $p_c^{**} = 0.7158$, $p_r^{**} = 0.23$ for an expected revenue of 0.43729. At the optimum, $\pi_L(p_c,p_r) = -0.08772$ so that all the omitted constraints are satisfied. Thus, choosing a ceiling price strictly less than 1 and strictly greater than a positive reserve price is optimal for the seller.

Consider now the solution to (8) with the additional constraint that the seller charges no reserve price so that $p_r = 0$. The corresponding optimal ceiling price is obtained by solving $\pi_H(p_c, 0) = 0$ for p_c. The solution is $p_c^* = 0.727947$ for an expected revenue of 0.40352. Starting from this point, if the seller raises the reserve price from 0, he increases the winner's curse problem for bidders 1 and 2 from bidding p_c. This is because when $s' = L$ and $v < p_r$ they have to now pay p_r when they win, instead of v, and so earn negative profits. Thus, to get bidders $i \in \{1, 2\}$ to participate, the seller has to lower the ceiling price as he raises the reserve price.

At the other extreme, consider the case where the seller is constrained to charge a fixed price of $p_c = p_r = p$. Solving $\pi_H(p, p) = 0$, we obtain that the solution is $p^* = 0.576516$, for an expected revenue of 0.39894. Comparing with the low ceiling case, we see that this is the optimal choice of a *fixed* price for the seller. However, for such a fixed price, bidders have to pay the fixed price whenever they win. In contrast, for a ceiling price of $p_c^* = 0.727947$ and a reserve price of 0, bidders pay v and earn 0 profits when bidder 3's bid is the second highest bid. This lowers the winner's curse problem for bidders 1 and 2. Holding a second price auction with a ceiling price of 0.727947 and no reserve price yields the seller higher revenues than selling at a fixed price of 0.576516.

The table below summarizes the results. It provides the ranking of seller revenues from setting different ceiling and floor prices

	p_c	p_r	**Revenue**
1. Optimal Ceiling and Floor	**0.7158**	**0.23**	**0.43729**
2. High Ceiling, No Floor	0.727947	0	0.40352
3. Optimal Fixed Price	0.576516	0.576516	0.39894
4. Low Ceiling/Fixed Price	0.29428	0.29428	0.29428
5. Only Floor	1	0.5	0.25

3 Conclusion

While the example above is admittedly simple, the intuition that price ceilings raise seller revenues by generating participation from less informed bidders is quite robust and should extend to richer environments. For example, it is possible to construct examples with more than one better–informed bidder (in addition to multiple less–informed bidders), where holding a second price auction with a price ceiling yields higher revenues compared to either holding a second price auction with a reserve price or from setting a fixed price at which to sell the good. Thus the result above does not hinge on the existence of one better–informed bidder. It is also of interest to characterize the nature of the gains from imposing a price ceiling as a function of the relative sizes of the two classes of bidders and to investigate the possibility of extending the results to other auction formats.

References

[1] Klemperer, P. (1999), "Auction Theory: A Guide to the Literature," *Journal of Economic Surveys*, **13** (3), July, 227-86.

[2] Milgrom P. and R. Weber (1982), “The Theory of Auctions and Competitive Bidding,” *Econometrica*, **50** (5), September, 1089–122.