E C O N O M I C S   B U L L E T I N

The Lagrange multiplier is not the shadow value of the limiting resource in the presence of strategically interacting agents

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Abstract

In the case of a single net-benefit maximizing agent facing a resource constraint, the economic interpretation of the Lagrange multiplier is that of the shadow value of the constraining resource. The formal justification for this economic interpretation is by way of the classical envelope theorem. Once an environment of strategically interacting agents is contemplated, however, the Lagrange multiplier no longer represents the shadow value of the resource to an agent. A concise proof of this claim and a revised economic interpretation of the Lagrange multiplier are given in this note.

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1. Introduction

Samuelson (1947, p. 66) first established that the value of the Lagrange multiplier associated with the output constraint in the neoclassical model of the cost-minimizing firm has the economic interpretation of marginal cost. He reached this conclusion by proving that the partial derivative of the firm's minimum cost function with respect to the output rate is equal to the value of the Lagrange multiplier at the cost-minimizing solution, i.e., he proved the envelope theorem in the context of said model. Almost three decades later a proof of the envelope theorem for the general class of differentiable, constrained optimization problems was provided by Silberberg (1974) by way of his elegant primal-dual formalism. Numerous research papers and books have subsequently applied the envelope theorem to impart an economic interpretation to the Lagrange multipliers in single agent constrained optimization problems.

When the objective function of a constrained optimization problem represents the monetized net benefits of an agent and the constraint reflects the finite availability of some resource, the envelope theorem implies that the partial derivative of an agent's indirect (or maximized) objective function is equal to the value of the corresponding Lagrange multiplier at the solution. This, in turn, implies that the said value of the Lagrange multiplier has the economic interpretation of the marginal value, or so-called shadow value, of the limiting resource. In contrast, this note shows that the value of the Lagrange multiplier at the solution is not, in general, the shadow value of the limiting resource in models with strategically interacting agents. In particular, it is demonstrated that in such an environment, the partial derivative of an agent's indirect objective function with respect to a constraint parameter is not equal to the value of the agent's Lagrange multiplier associated with that constraint at the Nash equilibrium. Moreover, by studying a given agent's best-response equilibrium and corresponding best-response indirect objective function, an intuitive explanation is provided for this surprising conclusion. Such an approach also leads to the correct economic interpretation of the Lagrange multiplier in an environment consisting of strategically interacting agents. This paper is therefore in the spirit of the works of Dorfman (1969) and Léonard (1987).

In order to emphasize the economics rather than the mathematics, technical details are kept to a minimum. To that end, the next section indicates how the archetype economic interpretation of the Lagrange multiplier is properly determined in the single agent setting. This is followed by a section establishing the aforementioned claim in the context of an oligopoly model characterizing a pair of strategically interacting pollution-regulated profit-maximizing firms. A proof in the general case then follows.

2. The Prototypical Economic Interpretation of the Lagrange Multiplier

In order to demonstrate the main result of the note in an uncomplicated manner, a simple model of a profit maximizing and polluting firm is developed. It is essentially a generalization of one of the models of Besanko (1987) and Helfand (1991), and may be formally stated as follows:

$$\pi^*(E) \stackrel{\text{def}}{=} \max_{a,q} \left\{ R(q) - C(a,q) \;\; \text{s.t.} \;\; e(a,q) \le E \right\}, \tag{1}$$

where $a > 0$ is the rate of pollution abatement by the firm, $q > 0$ is the rate of output produced by the firm, $R(\cdot) \in C^{(1)}$ is the firm's total revenue function, $C(\cdot) \in C^{(1)}$ is the firm's total cost function, $e(\cdot) \in C^{(1)}$ is the firm's pollution production function, i.e., the function that determines the rate of pollution emitted (or emissions rate), and $\pi^*(E)$ is the maximum value of profit that the firm can earn given the maximum allowable emissions rate of $E > 0$ dictated by the regulatory body. It is assumed that $\pi^*(\cdot) \in C^{(1)}$ locally and that a locally $C^{(1)}$ solution to the constrained profit maximization problem (1) exists, denoted by $\left(a^*(E), q^*(E)\right)$, with $\lambda^*(E)$ being the corresponding value of the Lagrange multiplier associated with the pollution constraint. Finally, it is assumed that the emissions constraint binds at the optimum, for otherwise there is essentially no constraint in the problem and thus no Lagrange multiplier to interpret. Note, in passing, that one could contemplate the addition of numerous other parameters and decision variables to, and complications of, the model defined by Eq. (1), but none of these would lend insight to, nor have any material effect on, what follows.

The Lagrangian function $L(\cdot)$ for problem (1) is defined as

$$L(a,q,\lambda;E) \stackrel{\text{def}}{=} R(q) - C(a,q) + \lambda\left[E - e(a,q)\right], \tag{2}$$

where λ is the Lagrange multiplier associated with the emissions constraint. The goal of this section is to carefully and rigorously justify the economic interpretation of $\lambda^*(E)$ as the shadow value of pollution. To that end, first recall that $\pi^*(E)$ is the maximum profit that the firm can earn given the maximum allowable rate of pollution E. This definition of $\pi^*(E)$ along with the definition of the partial derivative imply that $\partial\pi^*(E)/\partial E$ may be interpreted as the increase in the maximum value of profit resulting from an increase in the maximum allowable rate of pollution. In other words, $\partial\pi^*(E)/\partial E$ is the most the firm would pay for a marginal increase in the allowable rate of pollution, seeing as it is the maximum additional profit that it could earn when the pollution restriction is relaxed at the margin. Consequently, $\partial\pi^*(E)/\partial E$ has the economic interpretation of the shadow value of pollution. But by the classical envelope theorem [see, e.g., Silberberg and Suen (2001), p. 160], it follows that

$$\frac{\partial\pi^*}{\partial E}(E) = \frac{\partial L}{\partial E}(a,q,\lambda;E)\bigg|_{\substack{a=a^*(E)\\ q=q^*(E)\\ \lambda=\lambda^*(E)}} = \lambda^*(E) > 0, \tag{3}$$

the strict inequality following from the first-order necessary conditions and the assumptions $\partial C(a,q)/\partial a > 0$ and $\partial e(a,q)/\partial a < 0$. Thus, by Eq. (3), $\partial\pi^*(E)/\partial E$ equals $\lambda^*(E)$, *and because of this equality*, $\lambda^*(E)$ may be legitimately interpreted as the shadow value of pollution.

In the ensuing section it is shown that the equality exhibited in Eq. (3) does not generally hold in an oligopoly version of an otherwise identical model. This result therefore implies that the Lagrange multiplier is not the shadow value of the limiting resource in such a setting.

3. The Lagrange Multiplier is not the Shadow Value of Pollution

Consider a generalization of problem (1) that includes a second profit-maximizing firm producing a substitute product and generating a flow of pollution of the same ilk as the first firm. The two firms that make up the oligopoly are assumed to face an industry-wide limit on the rate of pollution emitted. Consequently, they are asserted to solve the following pair of simultaneous constrained optimization problems:

$$\hat{\pi}^1(E) \stackrel{\text{def}}{=} \max_{a^1,q^1}\left\{R^1(q^1,q^2) - C^1(a^1,q^1) \text{ s.t. } e^1(a^1,q^1) + e^2(a^2,q^2) \le E\right\}, \tag{4}$$

$$\hat{\pi}^2(E) \stackrel{\text{def}}{=} \max_{a^2,q^2}\left\{R^2(q^1,q^2) - C^2(a^2,q^2) \text{ s.t. } e^1(a^1,q^1) + e^2(a^2,q^2) \le E\right\}, \tag{5}$$

where a superscript on a variable or function references a particular firm, and where all the functions and variables have their obvious economic interpretation extended from §2. It is assumed that $\hat{\pi}^i(\cdot) \in C^{(1)}$ locally, $i = 1,2$, and that a locally $C^{(1)}$ Nash equilibrium exists to the oligopoly model defined by Eqs. (4) and (5), say $\left(\hat{a}^1(E), \hat{q}^1(E), \hat{a}^2(E), \hat{q}^2(E)\right)$, with $\left(\hat{\lambda}^1(E), \hat{\lambda}^2(E)\right)$ being the corresponding values of the Lagrange multipliers for each firm. Sufficient conditions for the existence of a Nash equilibrium can be found in Besanko (1987). In this setting, $\hat{\pi}^i(E)$, $i = 1,2$, is the maximum value of profit that firm *i* can earn at the Nash equilibrium when the sum of the rate of pollution emitted by both firms is constrained to be less than or equal to the rate $E > 0$. As in §2, it is also assumed that the industry-wide emissions constraint binds at the Nash equilibrium, for otherwise there is essentially no constraint in the game and thus no Lagrange multipliers to interpret. As before, it should be remarked that one may generalize problems (4) and (5) to include any finite number of firms, additional parameters and decision variables, and so forth, but such generalizations haven't any essential bearing on what follows.

Fully analogous to the economic interpretation developed in §2 is the fact that the shadow value of pollution for firm 1 at the Nash equilibrium is given by the expression $\partial\hat{\pi}^1(E)/\partial E$. This follows from the definition of the partial derivative and the fact that $\hat{\pi}^1(E)$ is by definition the maximum value of profit that firm 1 can earn at the Nash equilibrium when the industry faces a maximum allowable pollution rate of E. In order to justify the interpretation of $\hat{\lambda}^1(E)$ as the shadow value of pollution of firm 1, it must therefore be formally established that $\partial\hat{\pi}^1(E)/\partial E$ *equals* $\hat{\lambda}^1(E)$. As will now be shown, this equality does not generally hold.

In order to establish said claim, first define the Lagrangian function $G^1(\cdot)$ for firm 1 by

$$G^1(a^1,q^1,a^2,q^2,\lambda^1;E) \overset{\text{def}}{=} R^1(q^1,q^2) - C^1(a^1,q^1) + \lambda^1\left[E - e^1(a^1,q^1) - e^2(a^2,q^2)\right], \tag{6}$$

where λ^1 is the Lagrange multiplier for firm 1 associated with the industry wide pollution constraint. Second, an expression for the partial derivative $\partial\hat{\pi}^1(E)/\partial E$ must be derived, analogous to what was done in §2. In other words, an envelope theorem for the game defined by Eqs. (4) and (5) is required. Caputo (1996) has provided just such an envelope theorem that is, moreover, applicable to a much wider class of games than that under consideration here. Stated in terms of the functions and variables defined above, Theorem 1 of Caputo (1996) asserts that

$$\begin{aligned}\frac{\partial\hat{\pi}^1}{\partial E}(E) = {}& \frac{\partial G^1}{\partial E}(a^1,q^1,a^2,q^2,\lambda^1;E)\bigg|_{\substack{a^i=\hat{a}^i(E),\, i=1,2\\ q^i=\hat{q}^i(E),\, i=1,2\\ \lambda^1=\hat{\lambda}^1(E)}} + \frac{\partial G^1}{\partial a^2}(a^1,q^1,a^2,q^2,\lambda^1;E)\bigg|_{\substack{a^i=\hat{a}^i(E),\, i=1,2\\ q^i=\hat{q}^i(E),\, i=1,2\\ \lambda^1=\hat{\lambda}^1(E)}} \frac{\partial\hat{a}^2}{\partial E}(E)\\ &+ \frac{\partial G^1}{\partial q^2}(a^1,q^1,a^2,q^2,\lambda^1;E)\bigg|_{\substack{a^i=\hat{a}^i(E),\, i=1,2\\ q^i=\hat{q}^i(E),\, i=1,2\\ \lambda^1=\hat{\lambda}^1(E)}} \frac{\partial\hat{q}^2}{\partial E}(E).\end{aligned} \tag{7}$$

Using the function $G^1(\cdot)$ defined in Eq. (6), Eq. (7) may be written as

$$\begin{aligned}\frac{\partial\hat{\pi}^1}{\partial E}(E) = {}& \hat{\lambda}^1(E) - \hat{\lambda}^1(E)\frac{\partial e^2}{\partial a^2}\left(\hat{a}^2(E),\hat{q}^2(E)\right)\frac{\partial\hat{a}^2}{\partial E}(E)\\ &+ \left[\frac{\partial R^1}{\partial q^2}\left(\hat{q}^1(E),\hat{q}^2(E)\right) - \hat{\lambda}^1(E)\frac{\partial e^2}{\partial q^2}\left(\hat{a}^2(E),\hat{q}^2(E)\right)\right]\frac{\partial\hat{q}^2}{\partial E}(E).\end{aligned} \tag{8}$$

Equation (8) is the alluded to formal evidence that $\hat{\lambda}^1(E)$ cannot universally be interpreted as the shadow value of pollution for firm 1 at the Nash equilibrium, because it demonstrates that $\partial\hat{\pi}^1(E)/\partial E \neq \hat{\lambda}^1(E)$ in general.

For the reason that $\hat{\lambda}^1(E)$ cannot, in general, be interpreted as the shadow value of pollution of firm 1 at the Nash equilibrium, it is important to determine the correct economic interpretation of $\hat{\lambda}^1(E)$. This is achieved by focusing on the constrained optimization problem that defines the best-response equilibrium of firm 1, videlicet, problem (4).

To begin, first observe that firm 1 determines its best-response functions by considering problem (4) in isolation of problem (5). That is to say, in deriving its best response functions, firm 1 solves problem (4) for the profit-maximizing value of its decision variables, taking as given the decision variables of firm 2. In other words, the best-response functions $\tilde{a}^1(\cdot)$ and $\tilde{q}^1(\cdot)$ of firm 1 are defined by

$$\left(\tilde{a}^1(a^2,q^2;E),\tilde{q}^1(a^2,q^2;E)\right) \stackrel{\text{def}}{=} \underset{a^1,q^1}{\arg\max}\left\{R^1(q^1,q^2)-C^1(a^1,q^1) \text{ s.t. } e^1(a^1,q^1)+e^2(a^2,q^2)\le E\right\}, \quad (9)$$

with corresponding value of the best-response Lagrange multiplier given by $\tilde{\lambda}^1(a^2,q^2;E)$, best-response profit function $\tilde{\pi}^1(\cdot)$ defined by

$$\tilde{\pi}^1(a^2,q^2;E) \stackrel{\text{def}}{=} \max_{a^1,q^1}\left\{R^1(q^1,q^2)-C^1(a^1,q^1) \text{ s.t. } e^1(a^1,q^1)+e^2(a^2,q^2)\le E\right\}, \quad (10)$$

and Lagrangian function $G^1(\cdot)$ defined in Eq. (6).

From Eq. (10), it follows that $\tilde{\pi}^1(a^2,q^2;E)$ is the maximum value of profit that firm 1 can earn when facing the industry-wide pollution limit E, holding constant the actions of firm 2. In turn, this and the definition of the partial derivative imply that $\partial\tilde{\pi}^1(a^2,q^2;E)/\partial E$ is the *nonstrategic shadow value of pollution*, the adjective “nonstrategic” being necessitated by the facts that, in general, the actions of firm 2 change when E undergoes a change and thus have a nonzero impact on the profits of firm 1, whereas the actions of firm 2 are held fixed in the calculation of $\partial\tilde{\pi}^1(a^2,q^2;E)/\partial E$ by the definition of a partial derivative.

Inasmuch as the determination of the best-response functions for firm 1 is formally a single-agent constrained optimization problem, the classical envelope theorem [see, e.g., Silberberg and Suen (2001), p. 160] is fully applicable to it. Hence, by the classical envelope theorem applied to problem (10), it follows that

$$\frac{\partial\tilde{\pi}^1}{\partial E}(a^2,q^2;E)=\frac{\partial G^1}{\partial E}(a^1,q^1,a^2,q^2,\lambda^1;E)\Bigg|_{\substack{a^1=\tilde{a}^1(a^2,q^2;E)\\ q^1=\tilde{q}^1(a^2,q^2;E)\\ \lambda^1=\tilde{\lambda}^1(a^2,q^2;E)}}=\tilde{\lambda}^1(a^2,q^2;E). \quad (11)$$

Equation (11) therefore formally establishes that at the best-response equilibrium of firm 1, the value of its Lagrange multiplier corresponding to the industry-wide pollution constraint can be legitimately interpreted as the nonstrategic shadow value of pollution.

Now observe that the envelope result in Eq. (11) establishes the economic interpretation of $\tilde{\lambda}^1(a^2,q^2;E)$, not of $\hat{\lambda}^1(E)$. This is a relatively easy matter to correct, however, for the definitions of a Nash equilibrium and best-response equilibrium imply that the value $\tilde{\lambda}^1(a^2,q^2;E)$ coincides with the value $\hat{\lambda}^1(E)$ when the former is evaluated at the Nash equilibrium solution of firm 2, i.e., when $\tilde{\lambda}^1(a^2,q^2;E)$ is evaluated at $a^2=\hat{a}^2(E)$ and $q^2=\hat{q}^2(E)$. In other words, the definitions of a Nash equilibrium and best-response equilibrium imply that

$$\hat{\lambda}^1(E) \stackrel{\text{def}}{=} \tilde{\lambda}^1\left(\hat{a}^2(E),\hat{q}^2(E);E\right). \quad (12)$$

Upon evaluating Eq. (11) at $(a^2,q^2)=\left(\hat{a}^2(E),\hat{q}^2(E)\right)$ and using Eq. (12), it follows that $\hat{\lambda}^1(E)$ has the economic interpretation of the *nonstrategic shadow value of pollution* of firm 1 at the Nash equilibrium. Nonetheless, recalling that $\partial\hat{\pi}^1(E)/\partial E \ne \hat{\lambda}^1(E)$ by Eq. (8), it is clear that $\hat{\lambda}^1(E)$ does not represent the increment to the maximum value of profit to firm 1 when the industry-wide pollution limit is relaxed at the Nash equilibrium and both firms are permitted to adjust to the relaxation in E, for by Eqs. (11) and (12), $\hat{\lambda}^1(E)$ fails to capture the response of firm 2 to the perturbation in E. As such, $\hat{\lambda}^1(E)$ is not the shadow value of pollution to firm 1 at the Nash equilibrium.

Alternatively, the nonstrategic shadow value interpretation of $\hat{\lambda}^1(E)$ can be justified by relating the value of the indirect profit function of firm 1 in the best-response equilibrium, scilicet $\tilde{\pi}^1(a^2,q^2;E)$, to the value of the indirect profit function of firm 1 in the Nash equilibrium, namely $\hat{\pi}^1(E)$. As was done in establishing the relationship between the values of the Lagrange multipliers in Eq. (12), it follows from the definitions of a Nash equilibrium and a best-response equilibrium that

$$\hat{\pi}^1(E) \stackrel{\text{def}}{=} \tilde{\pi}^1\left(\hat{a}^2(E),\hat{q}^2(E);E\right). \tag{13}$$

This relationship asserts that the maximum value of profit that firm 1 can earn at the Nash equilibrium is by definition equal to the maximum value of profit that it can earn at its best-response equilibrium when the actions of firm 2 are evaluated at its Nash equilibrium solution. The real insight from Eq. (13) regarding the Lagrange multiplier, however, is gleaned by differentiating it with respect to E using the chain rule. Doing just that yields

$$\underbrace{\frac{\partial \hat{\pi}^1}{\partial E}(E)}_{\substack{\text{shadow value}\\ \text{of pollution}}} = \underbrace{\frac{\partial \tilde{\pi}^1}{\partial a^2}\left(\hat{a}^2(E),\hat{q}^2(E);E\right)\frac{\partial \hat{a}^2}{\partial E}(E) + \frac{\partial \tilde{\pi}^1}{\partial q^2}\left(\hat{a}^2(E),\hat{q}^2(E);E\right)\frac{\partial \hat{q}^2}{\partial E}(E)}_{\text{strategic effect}} + \underbrace{\hat{\lambda}^1(E)}_{\substack{\text{nonstrategic shadow}\\ \text{value of pollution}}}, \tag{14}$$

where Eqs. (11) and (12) were used on the last term on the right-hand side. Equation (14) shows the decomposition of the shadow value of pollution in the Nash equilibrium into its intrinsic parts, to wit, (i) a nonstrategic portion given by the value of the Lagrange multiplier at the Nash equilibrium, and (ii) a strategic part given by the effect that a change in the industry-wide pollution limit has on the Nash equilibrium values of the decision variables of firm 2, and the resulting effect the change in these decision variables of firm 2 has on the maximum profit of firm 1 at the Nash equilibrium. It is this latter strategic (or cross-firm) effect exhibited in Eq. (14) that the Lagrange multiplier $\hat{\lambda}^1(E)$ fails to capture in a strategic setting, and is precisely why the Lagrange multiplier does not represent the shadow value of pollution to firm 1. In passing, note that upon applying the classical envelope theorem to Eq. (10) and evaluating the results at the Nash equilibrium, and then using Eq. (12) and the fact that $\hat{q}^1(E) \stackrel{\text{def}}{=} \tilde{q}^1\left(\hat{a}^2(E),\hat{q}^2(E);E\right)$, it can be shown that Eqs. (8) and (14) are identical.

4. A General Result on the Economic Interpretation of the Lagrange Multiplier

The goal of this section is to provide a compact but general proof of the fact that the partial derivative of an agent's maximized objective function with respect to a constraint parameter is not equal to the value of the agent's corresponding Lagrange multiplier at the Nash equilibrium. This implies that, in general, the Lagrange multiplier corresponding to a given constraint in a static game does not have the economic interpretation of the shadow value of the limiting resource represented by that constraint, for it ignores the strategic response of the other agents to the perturbation in the constraint parameter.

As discussed by Caputo (1996, pp. 205–206), the following set of P simultaneous constrained optimization problems may be mapped into a normal form static game:

$$\phi^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{x}^p \in \mathbb{R}^{N^p}} \left\{ f^p(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^P) \text{ s.t. } \mathbf{g}(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^P) = \boldsymbol{\beta} \right\},\ p = 1,2,\ldots,P. \tag{15}$$

The corresponding Lagrangian functions $G^p(\cdot)$ are defined by

$$G^p(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^P,\boldsymbol{\lambda}^p;\boldsymbol{\beta}) \stackrel{\text{def}}{=} f^p(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^P) + \boldsymbol{\lambda}^{p\dagger}\left[\boldsymbol{\beta} - \mathbf{g}(\mathbf{x}^1,\mathbf{x}^2,\ldots,\mathbf{x}^P)\right],\ p = 1,2,\ldots,P, \tag{16}$$

where $\boldsymbol{\lambda}^p \in \mathbb{R}^K$, $p = 1,2,\ldots,P$, is the pth player's vector of Lagrange multipliers, $\boldsymbol{\beta} \in \mathbb{R}^K$ is the vector of constraint parameters, and "†" denotes transposition. The following assumptions are imposed on the game defined by Eq. (15):

(A.1) $f^p(\cdot): \mathbb{R}^{N^1} \times \mathbb{R}^{N^2} \times \cdots \times \mathbb{R}^{N^P} \to \mathbb{R}$, $f^p(\cdot) \in C^{(1)}$, $p = 1,2,\ldots,P$.

(A.2) $\mathbf{g}(\cdot): \mathbb{R}^{N^1} \times \mathbb{R}^{N^2} \times \cdots \times \mathbb{R}^{N^P} \to \mathbb{R}^K$, $\mathbf{g}(\cdot) \in C^{(1)}$, $K < N^p$, $p = 1,2,\ldots,P$.

(A.3) There exists a unique Nash equilibrium to the game defined by Eq. (15) for each value of $\boldsymbol{\beta}$ in an open neighborhood of $\boldsymbol{\beta}^\circ \in \mathbb{R}^K$, denoted by $\hat{\mathbf{x}}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \left(\hat{\mathbf{x}}^1(\boldsymbol{\beta}),\hat{\mathbf{x}}^2(\boldsymbol{\beta}),\ldots,\hat{\mathbf{x}}^P(\boldsymbol{\beta})\right)$, where

$$\hat{\mathbf{x}}^p(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \arg\max_{\mathbf{x}^p \in \mathbb{R}^{N^p}} \left\{ f^p\left(\mathbf{x}^p,\hat{\mathbf{x}}^{-p}(\boldsymbol{\beta})\right) \text{ s.t. } \mathbf{g}\left(\mathbf{x}^p,\hat{\mathbf{x}}^{-p}(\boldsymbol{\beta})\right) = \boldsymbol{\beta} \right\},\ p = 1,2,\ldots,P,$$

and where $\hat{\mathbf{x}}^{-p}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \left(\hat{\mathbf{x}}^1(\boldsymbol{\beta}),\hat{\mathbf{x}}^2(\boldsymbol{\beta}),\ldots,\hat{\mathbf{x}}^{p-1}(\boldsymbol{\beta}),\hat{\mathbf{x}}^{p+1}(\boldsymbol{\beta}),\ldots,\hat{\mathbf{x}}^P(\boldsymbol{\beta})\right)$, $p = 1,2,\ldots,P$.

(A.4) $\hat{\mathbf{x}}^p(\cdot) \in C^{(1)}$ for all $\boldsymbol{\beta}$ in an open neighborhood of $\boldsymbol{\beta}^\circ \in \mathbb{R}^K$, $p = 1,2,\ldots,P$.

Seeing as these assumptions were thoroughly discussed by Caputo (1996, p. 207), there is no reason to discuss them here. With these technical assumptions in place, the main result of the paper may now be stated and proven.

Theorem 1: *Under assumptions (A.1)–(A.4), the static game defined by the P simultaneous constrained optimization problems in Eq. (15) has the envelope property*

$$\frac{\partial \phi^p}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}) = \hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})^\dagger + \sum_{\substack{j=1 \\ j \neq p}}^{P} \left[\frac{\partial G^p}{\partial \mathbf{x}^j}\left(\hat{\mathbf{x}}(\boldsymbol{\beta}),\hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta});\boldsymbol{\beta}\right) \frac{\partial \hat{\mathbf{x}}^j}{\partial \boldsymbol{\beta}}(\boldsymbol{\beta}) \right],\ p = 1,2,\ldots,P, \tag{17}$$

where $\hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})$, $p = 1,2,\ldots,P$, is the corresponding Nash equilibrium value of the Lagrange multiplier vector for player p.

Proof: The proof follows by applying Theorem 1 of Caputo (1996) to the Lagrangian function $G^p(\cdot)$ defined in Eq. (16). *Q.E.D.*

Theorem 1 demonstrates the central claim of the note, namely, that inasmuch as $\partial \phi^p(\boldsymbol{\beta})/\partial \boldsymbol{\beta} \neq \hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})^\dagger$, the Lagrange multiplier $\hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})$ is not the shadow value of the constraint parameter $\boldsymbol{\beta}$ of agent p at the Nash equilibrium. Moreover, by mimicking the mathematics and logic used in the last four paragraphs of §3, it follows that $\hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})$ has the economic interpretation of the nonstrategic shadow value of the constraint parameter $\boldsymbol{\beta}$ of agent p at the Nash equilibrium. That is to say, $\hat{\boldsymbol{\lambda}}^p(\boldsymbol{\beta})$ is the incremental measure of the value of the constraint parameter $\boldsymbol{\beta}$ to agent p when the actions of the $P-1$ other agents are held fixed at their Nash equilibrium values.

5. Concluding Remarks

One view of the results derived here is that the Lagrange multiplier from single-agent and multiple-agent strategic models have analogous economic interpretations, videlicet, as the nonstrategic shadow value of the limiting resource. In the single agent case the adjective “nonstrategic” is fully appropriate in view of the fact that there is literally no other agent to be concerned about when a decision is made. That is to say, “nonstrategic” is an appropriate adjective to apply to the shadow value interpretation of the Lagrange multiplier in the single agent setting because of the degenerate nature of the single agent “game,” for it, by construction, rules out strategic considerations. In other words, what has been shown is that the economic interpretation of the Lagrange multiplier in the single-agent setting essentially carries over to the multiple-agent strategic setting. This economic interpretation notwithstanding, Theorem 1 establishes that, in general, the Lagrange multiplier associated with a constraint *cannot* legitimately be interpreted as the correct measure of the marginal value of the limiting resource associated with that constraint as soon as one moves to a multiple-agent strategic setting, for it ignores the strategic considerations of the other agents. Consequently, said Lagrange multiplier is not the shadow value of a limiting resource to an agent in a static game. The legitimate shadow value of a limiting resource in a multiple-agent strategic setting is given by the partial derivative of an agent’s indirect objective function with respect to that limiting resource, just as it is in the single-agent constrained optimization framework.

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