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E C O N O M I C S B U L L E T I N

Caps in asymmetric all−pay auctions with incomplete information

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Abstract

We study asymmetric all−pay auctions where two privately informed agents bid for a prize. We show that capping the bids is profitable for a designer who wants to maximize the sum of bids (revenue). This finding confims the results of Che and Gale (1998) in the context of incomplete information and completes the analysis of Gavious, Moldovanu and Sela (2002) by analyzing the case of ex−ante asymmetric players.

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1 Introduction

In many competitive situations, we observe constraints imposed on contestants. For instance, in many professional sports, entry is regulated by imposing minimum requirements. Similarly, in auctions, a reserve price or an entry fee is often imposed. These practises have been recognized by the literature to be beneficial. A seller can increase his revenue in an auction, a contest designer can raise the average effort in a contest. The intuition behind the result is to exclude players with low valuation to increase competition among active bidders. Another type of constraint that has been analyzed is to place upper bounds on bids. Salary caps are common practise in US professional sport leagues. The rationale is to level the field and give small cities the ability to compete with larger and richer cities. This leads to a more competitive league in which the average effort is larger. The effect of caps is also debated in the context of political economy. Lobbying and to a lesser extent political campaigns can be understood as contests in which the side which spends most wins. The point has been made that a cap on expenditures can lead to a global increase in expenditures, which in these contests would just be wasteful rent-seeking. At the center of the argument lies the potential asymmetry between players. A cap has a positive effect if it restores competitive balance. Weak players, believing they have a chance to compete, will make more efforts than if they thought the strong player had no limit in the resources he could spend to outbid him. The importance of asymmetry between players is thus crucial for a better understanding of the role of caps.

Che and Gale (1998) in the context of lobbying were the first to show the paradoxical effect that caps can in fact increase average expenditures. They model lobbying as an all-pay auction with complete information but with asymmetric players. Lobbyist value differently the political prize. They derive the equilibrium of such a game and show that a cap can increase lobbying expenditures. Gavious, Moldovanu and Sela (2002) analyze the role of caps in all-pay auctions with incomplete information. They show that the result of Che and Gale does not extend to ex-post asymmetry, when the cost of bidding is linear. If players are ex-ante symmetric (their valuation are drawn from the same distribution function), but ex-post asymmetric (since they have different valuations) a cap can not increase total expenditures. They show that when bidding costs are convex, a cap can increase revenue even with ex-ante symmetric players.

The present note completes these two papers by analyzing the role of caps in all-pay auctions when there is incomplete information and ex-ante asymmetries between bidders. We show that the result of Che and Gale extends to the case of incomplete information with ex-ante asymmetric bidders. For uniform distribution functions, we show that there exists an appropriate choice of cap that increases total expenditures.

The most closely related papers are the ones already mentioned by Che and Gale (1998) and Gavious, Moldovanu and Sela (2002). Another related strand of the literature looks at the role of budget constraints in auctions. See Che and Gale

(1996) and Fang and Parreiras (2001).

In a somewhat related work, Chakraborty (2002) considers a simple example of a common value auction in which setting up a ceiling price in addition to a reserve price increases the revenue.

2 The model

We consider two agents bidding for an indivisible object. Bidder $i's$ valuation for the object, v_i is private information to bidder i. The valuations are independently distributed according to asymmetric distribution functions F_1 and F_2. We assume that F_1 and F_2 are continuously differentiable. We also assume that the density functions F_1' and F_2' are strictly positive on their respective support $[0, \beta]$ and $[0, \alpha]$. Without loss of generality, we assume that $\beta \geq \alpha$.

Each bidder submits a bid $b_i \leq b^c$, where b^c is a commonly known bid cap imposed by the auctioneer. The bidder with the highest bid wins the object. Both bidders pay their bids. If both bidders submit the same bid, then the winner is randomly selected (each bidder has an equal chance to win the object).

3 Equilibrium without caps

Player 1's objective is to maximize $v_1 \cdot F_2\left(b_2^{-1}(x)\right) - x$ over $x \in \mathbb{R}^+$ while player 2's objective is to maximize $v_2 \cdot F_1\left(b_1^{-1}(y)\right) - y$ over $y \in \mathbb{R}^+$. First-order conditions are:

$$F_2'\left(b_2^{-1}(x)\right) \cdot \left(b_2^{-1}\right)'(x) = \frac{1}{b_1^{-1}(x)}, \text{ and } F_1'\left(b_1^{-1}(y)\right) \cdot \left(b_1^{-1}\right)'(y) = \frac{1}{b_2^{-1}(y)}.$$

To determine the equilibrium bidding functions, we use the mapping[1] $h(\cdot) = b_2^{-1} \circ b_1(\cdot)$, mapping player 1's valuation into player 2's valuation making the same bid. The first-order conditions can be rewritten as:

$$\begin{aligned} \left(b_2^{-1}\right)'(b_1(v)) &= \frac{1}{v \cdot F_2'\left(b_2^{-1} \circ b_1(v)\right)} = \frac{1}{v \cdot F_2'(h(v))}, \\ b_1'(v) &= \frac{1}{\left(b_1^{-1}\right)'(b_1(v))} = h(v) \cdot F_1'(v), \end{aligned}$$

[1]This mapping was used in Amann-Leininger (1996) to characterise equilibrium in asymmetric all-pay auctions. See also Parreiras and Rubinchik-Pessach (2006) for a more recent treatment of asymmetric all-pay auctions. In particular, they deal with the issue of existence of equilibria when the support of types are not identical, which is relevant in our set-up.

whenever the density is positive. Finally, since $h'(v) = \left(b_2^{-1}\right)'(b_1(v)) \cdot b_1'(v)$, we obtain the following ordinary differential equation:

$$h'(v) = \frac{h(v) \cdot F_1'(v)}{v \cdot F_2'(h(v))}.$$

The bidding functions b_i are then obtained from $b_1^{'}(v) = F_1'(v) \cdot h(v)$ (with the lowest type making a zero bid) and $b_2(w) = b_1\left(h^{-1}(w)\right)$. In order to be able to analyze the effect of a bid cap on the revenue, we consider some particular form of bidders' asymmetry.

Proposition 1 *Assume that player* 1*'s valuation is distributed uniformly on* $[0, \beta]$ *and that player* 2*'s valuation is distributed uniformly on* $[0, \alpha]$, *with* $\beta > \alpha$. *Then the equilibrium bid functions are:*

$$b_1(v) = \frac{\alpha\beta}{\alpha+\beta} \cdot \left(\frac{v}{\beta}\right)^{\frac{\alpha+\beta}{\beta}}, \quad b_2(w) = \frac{\alpha\beta}{\alpha+\beta} \cdot \left(\frac{w}{\alpha}\right)^{\frac{\alpha+\beta}{\alpha}}.$$

The revenue in the auction is:

$$R = \frac{\alpha}{\alpha+\beta} \cdot \frac{\beta^2}{\alpha+2\beta} + \frac{\beta}{\alpha+\beta}\frac{\alpha^2}{2\alpha+\beta}.$$

4 Price Ceilings

Suppose now that the auctioneer imposes a ceiling b^c on the players' bids. Consider an all-pay auction with a bid cap $b^c \geq b^*$, with b^* being the largest bid submitted in the auction without cap. Then, the bid cap is not effective and the unique equilibrium is the same as in the auction without cap. Suppose now that $b^c < b^*$. In equilibrium, the highest types pool their bids at the cap. Since a lower bid would yield a strictly lower probability of winning (no chance to tie with the atom of types bidding the cap), it must be that the next bid is strictly lower. The pattern of bidding can be seen in figure 1.

Let's call $\bar{\beta}$ ($\bar{\alpha}$) the smallest valuation of player 1 (2) which bids b^c. Let b_* be the bid of the largest valuation which does not bid b^c. In equilibrium, $\bar{\beta}$ and $\bar{\alpha}$ must be indifferent between bidding b^c and bidding b_*. This gives two necessary conditions for equilibrium:

$$b^c - b_* = (1 - F_2(\bar{\alpha})) \cdot \frac{1}{2} \cdot \bar{\beta} \tag{1}$$

$$b^c - b_* = \left(1 - F_1\left(\bar{\beta}\right)\right) \cdot \frac{1}{2} \cdot \bar{\alpha} \tag{2}$$

Since the local distribution of valuations does not change with the presence of price ceilings, the *O.D.E.* defining the $h(\cdot)$ function remains unchanged. Only the boundary conditions are changed. We now have $h\left(\bar{\beta}\right) = \bar{\alpha}$. From this modified mapping,

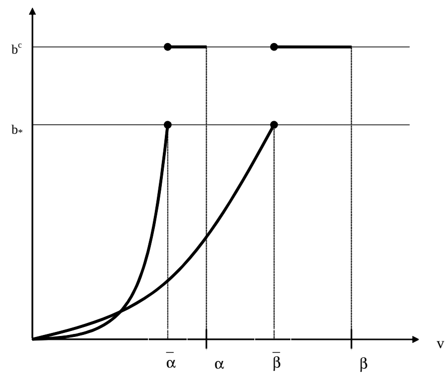

Figure 1: Bid functions with a cap.

we can derive the bidding functions under the bid cap $\bar{b}_1$ and $\bar{b}_2$. This gives us a third equation that characterizes the equilibrium: $\bar{b}_1(\bar{\beta}) = b_*$.

Proposition 2 *Assume that player 1's valuation is distributed uniformly on $[0, \beta]$ and that player 2's valuation is distributed uniformly on $[0, \alpha]$, with $\beta > \alpha$. Then the equilibrium bid functions are :*

$$\bar{b}_1(v) = \frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta} \cdot \left(\frac{v}{\bar{\beta}}\right)^{\frac{\alpha+\beta}{\beta}}, \bar{b}_2(w) = \frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta} \cdot \left(\frac{w}{\bar{\alpha}}\right)^{\frac{\alpha+\beta}{\alpha}}$$

with $\bar{\alpha}$ and $\bar{\beta}$ being defined as follows.

$$\bar{\alpha} = \rho,\ \bar{\beta} = \frac{\beta(\alpha+\beta)}{(\beta-\alpha)} \frac{2b^c - \rho}{\rho}.$$

with

$$\rho = \frac{\alpha+\beta}{\beta} b^c + \frac{\alpha}{2}\left(\frac{\alpha+\beta}{\beta-\alpha}\right) - \frac{\alpha+\beta}{\beta}\sqrt{(b^c)^2 - \frac{\alpha\beta}{\alpha+\beta} b^c + \left(\frac{\alpha\beta}{2(\beta-\alpha)}\right)^2}.$$

Proof. See Appendix. ■

We want to show the optimality of a price ceiling. We show that the derivative of the revenue with respect to b^c at $b^c = \frac{\alpha\beta}{\alpha+\beta}$ is equal to zero and that the second derivative is positive at that point, which demonstrates that the revenue increasing when the caps decreases and becomes binding.

Proposition 3 *The revenue in an all-pay auction with a bid cap $b^c < \frac{\alpha\beta}{\alpha+\beta}$ is*

$$R^c = b^c\left(\frac{\beta-\bar{\beta}}{\beta}+\frac{\alpha-\bar{\alpha}}{\alpha}\right)+\frac{\bar{\alpha}^2\bar{\beta}}{(\alpha+\beta)(\beta+2\alpha)}+\frac{\bar{\alpha}\bar{\beta}^2}{(\alpha+\beta)(2\beta+\alpha)}$$

There exists an auction with a binding cap b^c that generates a revenue R^c strictly larger than the revenue in the auction without cap.

Proof. See Appendix. ■

5 Conclusion

We have analyzed the role of caps in a model of an all-pay auction under incomplete information with two asymmetric players. We have shown in the case of uniform distributions that the designer of the auction can increase the revenue by imposing an appropriate cap on the bids. A cap on bids lowers the bids of high valuation types but increases the bids of types with a low valuation. It also increases the competitiveness of the contest and leads to more aggressive bidding. This leads to a higher revenue. This result has interesting implications for asymmetric auctions. Bid caps can be an effective way of increasing competition when bidders are asymmetric. Our results also have implications for auctions with budget constrained buyers when the seller can provide financing. It is not obvious that providing financing to asymmetric bidders would increase revenue. Possible extensions to other auction mechanisms and to more general distribution functions are left for future research.

6 Appendix

Proof of proposition 1

When the distributions are uniform, $F_1'(v)$ and $F_2'(h(v))$ are constant. The ordinary differential equation becomes $\frac{h'(v)}{h(v)} = \frac{1}{v}\cdot\frac{\alpha}{\beta}$ with unknown h, which yields $h(v) = K\cdot v^{\frac{\alpha}{\beta}}$. The constant K is easily calculated using the boundary condition $h(\beta)=\alpha$. This yields $K = \alpha/\beta^{\alpha/\beta}$. The bidding functions b_i are then obtained from $b_1'(v) = \frac{1}{\beta}\cdot h(v)$ (with the lowest type making a zero bid) and $b_2(w) = b_1(h^{-1}(w))$.We get :

$$b_1(v) = \frac{\alpha\beta}{\alpha+\beta}\cdot\left(\frac{v}{\beta}\right)^{\frac{\alpha+\beta}{\beta}},\ b_2(w) = \frac{\alpha\beta}{\alpha+\beta}\cdot\left(\frac{w}{\alpha}\right)^{\frac{\alpha+\beta}{\alpha}}.$$

The revenue generated

$$R = \int_0^\beta b_1(v)\cdot\frac{1}{\beta}dv + \int_0^\alpha b_2(v)\cdot\frac{1}{\alpha}dv = \frac{\alpha\beta}{\alpha+\beta}\left(\frac{\beta}{\alpha+2\beta}+\frac{\alpha}{2\alpha+\beta}\right).$$

Proof of proposition 2

The boundary condition is now $h\left(\bar{\beta}\right)=\bar{\alpha}$. This yields:

$$K\cdot\bar{\beta}^{\frac{\alpha}{\beta}}=\bar{\alpha},\ h\left(v\right)=\bar{\alpha}\cdot\left(\frac{v}{\bar{\beta}}\right)^{\frac{\alpha}{\beta}}$$

We get the bid functions:

$$\bar{b}_1(v)=\frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta}\cdot\left(\frac{v}{\bar{\beta}}\right)^{\frac{\alpha+\beta}{\beta}},\ \bar{b}_2\left(w\right)=\frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta}\cdot\left(\frac{w}{\bar{\alpha}}\right)^{\frac{\alpha+\beta}{\alpha}}.$$

Hence, we have the following equation defining b_* :

$$b_*=\bar{b}_1(\bar{\beta})=\bar{b}_2\left(\bar{\alpha}\right)=\frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta}. \tag{3}$$

Along with $(1-2)$, we have a system of 3 equations in three unknowns $b_*,\bar{\alpha}$ and $\bar{\beta}$. We get:

$$\bar{\alpha}=\rho,\ \bar{\beta}=\frac{\beta\left(\alpha+\beta\right)}{\left(\beta-\alpha\right)}\frac{2b-\rho}{\rho},\ b_*=\frac{\beta}{\beta-\alpha}\left(2b-\rho\right).$$

with ρ is the positive root of $\left(\beta^2-\alpha\beta\right)Z^2+\left(2\alpha^2b^c-\alpha^2\beta-\alpha\beta^2-2b^c\beta^2\right)Z+2\alpha^2\beta b^c+2\alpha\beta^2b^c$:

$$\rho=\frac{\alpha+\beta}{\beta}b^c+\frac{\alpha}{2}\left(\frac{\alpha+\beta}{\beta-\alpha}\right)-\frac{\alpha+\beta}{\beta}\sqrt{\left(b^c\right)^2-\frac{\alpha\beta}{\alpha+\beta}b^c+\left(\frac{\alpha\beta}{2\left(\beta-\alpha\right)}\right)^2}.$$

■

Proof of proposition 2

$$\begin{aligned}
R^c &= b^c\left(\frac{\beta-\bar{\beta}}{\beta}+\frac{\alpha-\bar{\alpha}}{\alpha}\right)+\int_0^{\bar{\beta}}b_1\left(v\right)\cdot\frac{1}{\beta}dv+\int_0^{\bar{\alpha}}b_2\left(v\right)\cdot\frac{1}{\alpha}dv,\\
&= b^c\left(\frac{\beta-\bar{\beta}}{\beta}+\frac{\alpha-\bar{\alpha}}{\alpha}\right)+\int_0^{\bar{\alpha}}\frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta}\cdot\left(\frac{w}{\bar{\alpha}}\right)^{\frac{\alpha+\beta}{\alpha}}\cdot\frac{1}{\alpha}dv+\int_0^{\bar{\beta}}\frac{\bar{\alpha}\bar{\beta}}{\alpha+\beta}\cdot\left(\frac{v}{\bar{\beta}}\right)^{\frac{\alpha+\beta}{\beta}}\cdot\frac{1}{\beta}dv,\\
&= b^c\left(\frac{\beta-\bar{\beta}}{\beta}+\frac{\alpha-\bar{\alpha}}{\alpha}\right)+\frac{\bar{\alpha}^2\bar{\beta}}{\left(\alpha+\beta\right)\left(\beta+2\alpha\right)}+\frac{\bar{\alpha}\bar{\beta}^2}{\left(\alpha+\beta\right)\left(2\beta+\alpha\right)}.
\end{aligned}$$

■

Proof of proposition 3

For $b^c=\frac{\alpha\beta}{\alpha+\beta}$, we have $\bar{\alpha}=\rho=\alpha$ and $\bar{\beta}=\beta$.

Let's evaluate the following derivatives $\frac{d\bar{\alpha}}{db^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}}, \frac{d\bar{\beta}}{db^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}}, \frac{d\rho}{db^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}}$:

$$
\begin{aligned}
\frac{\partial\rho}{\partial b^c} &= \frac{\alpha+\beta}{\beta} - \frac{\alpha+\beta}{\beta}\left(\frac{1}{2}\frac{2b^c - \frac{\alpha\beta}{\alpha+\beta}}{\sqrt{\left(b^c\right)^2 - \frac{\alpha\beta}{\alpha+\beta}b^c + \left(\frac{\alpha\beta}{2(\beta-\alpha)}\right)^2}}\right), \\
\frac{\partial\rho}{\partial b^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}} &= \frac{\alpha+\beta}{\beta}\left(1 - \frac{1}{2}\frac{2\frac{\alpha\beta}{\alpha+\beta} - \frac{\alpha\beta}{\alpha+\beta}}{\frac{\alpha\beta}{2(\beta-\alpha)}}\right) = \frac{2\alpha}{\beta}, \\
\frac{\partial\bar{\beta}}{\partial b^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}} &= \frac{2\beta\left(\alpha+\beta\right)}{\left(\beta-\alpha\right)}\frac{\rho - b^c\cdot\frac{d\rho}{db^c}}{\rho^2} = \frac{2\beta\left(\alpha+\beta\right)}{\left(\beta-\alpha\right)}\frac{\alpha - \frac{\alpha\beta}{\alpha+\beta}\cdot\left(\frac{2\alpha}{\beta}\right)}{\alpha^2} = 2\frac{\beta}{\alpha}, \\
\frac{\partial\bar{\alpha}}{\partial b^c}|_{b^c=\frac{\alpha\beta}{\alpha+\beta}} &= \frac{\partial\rho}{\partial b^c} = \frac{2\alpha}{\beta}.
\end{aligned}
$$

We now show that the derivative of the revenue with respect to b^c is zero at $b^c = \frac{\alpha\beta}{\alpha+\beta}$.

We have:

$$
\begin{aligned}
\frac{\partial R^c}{\partial b^c} &= b^c\left(\frac{-\partial\bar{\beta}/\partial b^c}{\beta} - \frac{\partial\bar{\alpha}/\partial b^c}{\alpha}\right) + \left(\frac{\beta-\bar{\beta}}{\beta} + \frac{\alpha-\bar{\alpha}}{\alpha}\right) \\
&\quad + \frac{\left[\bar{\alpha}^2\cdot\partial\bar{\beta}/\partial b^c + 2\bar{\alpha}\bar{\beta}\cdot\partial\bar{\alpha}/\partial b^c\right]}{\left(\alpha+\beta\right)\left(\beta+2\alpha\right)} + \frac{\left[\bar{\beta}^2\cdot\partial\bar{\alpha}/\partial b^c + 2\bar{\alpha}\bar{\beta}\cdot\partial\bar{\beta}/\partial b^c\right]}{\left(\alpha+\beta\right)\left(2\beta+\alpha\right)}. \\
\frac{\partial R^c}{\partial b^c}|_{\frac{\alpha\beta}{\alpha+\beta}} &= -2 + \frac{\left[2\alpha\beta + 4\alpha^2\right]}{\left(\alpha+\beta\right)\left(\beta+2\alpha\right)} + \frac{\left[2\alpha\beta + 4\beta^2\right]}{\left(\alpha+\beta\right)\left(2\beta+\alpha\right)} = 0
\end{aligned}
$$

We now need to evaluate the sign of the second derivative of R^c at $b^c = \frac{\alpha\beta}{\alpha+\beta}$. We have:

$$
\frac{\partial^2\rho}{\partial\left(b^c\right)^2} = -\frac{\alpha+\beta}{2\beta}\left(\frac{2\sqrt{b^{c2} - \frac{\alpha\beta}{\alpha+\beta}b^c + \left(\frac{\alpha\beta}{2(\beta-\alpha)}\right)^2} - \frac{1}{2}\frac{\left(2b^c - \frac{\alpha\beta}{\alpha+\beta}\right)^2}{\sqrt{\left(b^c\right)^2 - \frac{\alpha\beta}{\alpha+\beta}b^c + \left(\frac{\alpha\beta}{2(\beta-\alpha)}\right)^2}}}{\left(b^c\right)^2 - \frac{\alpha\beta}{\alpha+\beta}b^c + \left(\frac{\alpha\beta}{2(\beta-\alpha)}\right)^2}\right)
$$

$$
\frac{\partial^2\rho}{\partial\left(b^c\right)^2}|_{\frac{\alpha\beta}{\alpha+\beta}} = -\frac{\alpha+\beta}{2\beta}\frac{\frac{\alpha\beta}{(\beta-\alpha)} - \frac{(\beta-\alpha)\alpha\beta}{(\alpha+\beta)^2}}{\left(\frac{\alpha\beta}{(\beta-\alpha)}\right)^2} = \frac{-2\left(\alpha+\beta\right)}{\beta}\left(\frac{\left(\beta-\alpha\right)}{\alpha\beta} - \frac{\left(\beta-\alpha\right)^3}{\left(\alpha+\beta\right)^2\left(\alpha\beta\right)}\right) = -\frac{8\left(\beta-\alpha\right)}{\beta\left(\alpha+\beta\right)},
$$

$$
\frac{\partial^2\bar{\beta}}{\partial\left(b^c\right)^2} = \frac{2\beta\left(\alpha+\beta\right)}{\left(\beta-\alpha\right)}\left(-\frac{1}{\rho^2}\frac{\partial\rho}{\partial b^c} - \frac{\rho^2\left(\frac{d\rho}{db^c} + b^c\cdot\frac{\partial^2\rho}{\partial b^c}\right) - 2\rho\frac{d\rho}{db^c}\left(b^c\frac{d\rho}{db^c}\right)}{\rho^4}\right),
$$

$$
\frac{\partial^2\bar{\beta}}{\partial\left(b^c\right)^2}|_{\frac{\alpha\beta}{\alpha+\beta}} = \frac{2\beta\left(\alpha+\beta\right)}{\left(\beta-\alpha\right)}\left(-\frac{1}{\alpha^2}\frac{2\alpha}{\beta} - \frac{\alpha^2\left(2\frac{\alpha}{\beta} - 8\frac{(\beta-\alpha)}{\beta(\alpha+\beta)}\frac{\alpha\beta}{\alpha+\beta}\right) - 2\alpha\frac{2\alpha}{\beta}\left(\frac{\alpha\beta}{\alpha+\beta}\frac{2\alpha}{\beta}\right)}{\alpha^4}\right) = \frac{8}{\alpha}\frac{\left(\beta-\alpha\right)}{\alpha+\beta},
$$

$$\frac{\partial^2 \bar{\alpha}}{\partial (b^c)^2}|_{\frac{\alpha\beta}{\alpha+\beta}} = \frac{\partial^2 \rho}{\partial (b^c)^2}|_{\frac{\alpha\beta}{\alpha+\beta}} = -8\frac{(\beta-\alpha)}{\beta(\alpha+\beta)}.$$

Now we have:

$$\frac{\partial^2 R^c}{\partial (b^c)^2} = -2\left(\frac{\partial\bar{\beta}/\partial b^c}{\beta} + \frac{\partial\bar{\alpha}/\partial b^c}{\alpha}\right) - b^c\left(\frac{\partial^2\bar{\beta}/\partial b^{c2}}{\beta} + \frac{\partial^2\bar{\alpha}/\partial b^{c2}}{\alpha}\right)$$
$$+\frac{\left[\bar{\alpha}\frac{{}^2\partial^2\bar{\beta}}{\partial(b^c)^2} + 2\bar{\alpha}\frac{\partial\bar{\alpha}}{\partial b^c}\frac{\partial\bar{\beta}}{\partial b^c} + 2\bar{\alpha}\bar{\beta}\frac{\partial^2\bar{\alpha}}{\partial b^{c2}} + 2\frac{\partial\bar{\alpha}}{\partial b^c}\left(\bar{\alpha}\frac{\partial\bar{\beta}}{\partial b^c} + \bar{\beta}\frac{\partial\bar{\alpha}}{\partial b^c}\right)\right]}{(\alpha+\beta)(\beta+2\alpha)}$$
$$+\frac{\left[\bar{\beta}^2\frac{\partial^2\bar{\alpha}}{\partial(b^c)^2} + 2\bar{\beta}\frac{\partial\bar{\beta}}{\partial b^c}\frac{\partial\bar{\alpha}}{\partial b^c} + 2\bar{\alpha}\bar{\beta}\frac{\partial^2\bar{\beta}}{\partial b^{c2}} + 2\frac{\partial\bar{\beta}}{\partial b^c}\left(\bar{\alpha}\frac{\partial\bar{\beta}}{\partial b^c} + \bar{\beta}\frac{\partial\bar{\alpha}}{\partial b^c}\right)\right]}{(\alpha+\beta)(2\beta+\alpha)}$$
$$= \frac{(\alpha-\beta)^2}{2\alpha^3 + 7\alpha^2\beta + 7\alpha\beta^2 + 2\beta^3} > 0.$$

Hence, the revenue function R^c is convex in the cap b^c at $b^c = \frac{\alpha\beta}{\alpha+\beta}$. This proves that introducing a binding cap $b^c < \frac{\alpha\beta}{\alpha+\beta}$ increases the revenue.■

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