E C O N O M I C S   B U L L E T I N

On the sufficiency of transitive preferences

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Abstract

The assumption that preferences are transitive, or, roughly equivalently, that choice behavior satisfies the Weak Axiom of Revealed Preference, is at the core of much classical normative decision theory. This paper asks to what degree this restricts the possible outcomes of choice behavior: are there objectives that could not be attained by an agent adhering to WARP that could be attained by choices that would be said to be "intransitive"? It is argued that the answer to this question is "no" in one setting of choice under random budget sets; any outcome obtained by intransitive choice methods can also be obtained by transitive ones.

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1 Introduction

The classical, "rational" decision-making agent of economic theory typically is assumed to possess preferences which satisfy transitivity: for any three goods a, b, and c, if a is weakly preferred to b and b is weakly preferred to c, then a must be weakly preferred to c. Under some weak assumptions, this is essentially equivalent to requiring the agent to exhibit choice behavior that satisfies the Weak Axiom of Revealed Preference (WARP): if the agent is willing to choose a in some budget set in which a and b are offered, then, in any other budget set also containing a and b, if the agent is willing to choose b, he must also be willing to choose a.

Fishburn (1991) points out that the arguments against intransitive relations or cyclical choice behavior fall into two categories: one, that intransitivities complicate the analysis of individual behavior, and two, that such preferences are "irrational" and subject to exploitation. In this second category falls the so-called "money pump." An agent who truly adhered to preferences with a violation of the transitivity assumption could be systematically exploited, indefinitely, at no cost to the exploiter. But, in order to begin such an exploitation, it would have to be observed that the agent had such preferences. If one considers the decision-making apparatus of a biological being as a product of evolution, one can imagine that adhering to such "inconsistent" choice procedures would rarely be exploited in nature. Aside from this informational requirement, Fishburn points out that the money pump commingles static and dynamic arguments: in particular, an agent who was exploited once by this argument surely would revise their choice behavior the second time around, or the third.[1] Given these observations, there might be little real cost in following a choice rule at a *given* point in time that would violate transitivity, especially in a complex environment where the cost of evaluation of different outcomes might outweigh its benefits.

Given that the case can be made that this normative argument for transitivity has little bite in practice, this paper asks the complementary question. If the expected "cost," measured by the small possibility of exploitation, to intransitivity is small, could it be the case that there is some positive benefit to using a choice procedure that would violate, or appear to violate, a suitable version of WARP? In the presence of possible intransitivity, defining the word "benefit" can be tricky. For the purposes of this analysis, "benefit" will be operationalized as follows: can the admission of intransitivities expand the feasible set of consumption outcomes?

In static settings, the results of Epstein (1987) and Kim (1987) answer this ques-

[1]Mandler (2005) provides another critique of the money pump, in which apparent violations of revealed preference arise from a distinction between "psychological" and "revealed" preferences.

tion negatively. Both give results stating that any observed demand function can be rationalized using a transitive preference ordering, under some assumptions. The former places a separability assumption on preferences, while the latter uses semi-transitivity or pseudotransitivity conditions.

This paper complements these results in a setting where the choices of an agent are observed over time. In each period, the budget set is determined stochastically, and the agent is observed to make a choice from the realized budget set. This can be interpreted as a low-level model of choice, in which overall consumption is the aggregation of relatively small individual choices, where the frequency with which choices are made is fast relative to any discounting by the agent. In parallel to the existing results noted above, it is shown that the agent can achieve any feasible long-run pattern without resorting to behavior that would appear to the observer to violate WARP. In this environment, any feasible consumption pattern can be achieved entirely through the use of stationary strategies involving choice behavior that satisfy suitable notions of transitivity.

The paper is organized as follows. Section 2 lays out the formal model in more detail. Section 3 presents the results, including a geometric interpretation of choice rules, by which it is seen how the transitive choice rules "span" the intransitive ones. Section 4 concludes with a discussion of the results.

2 Model

Following a model by Piotrowski and Makowski (2005), suppose there are a finite number N of goods. At each point in time $t = 1, 2, 3, \ldots$, the agent is presented with a budget set of two goods drawn from this set, from which the agent may choose exactly one to consume. Consumption within each period is indivisible. The pair of goods available is chosen randomly according to some probability distribution known to the agent. Probabilities of presentation of a budget set will be denoted by suitably subscripted q's. The process is stationary; the presentation probabilities do not change with time, and do not depend on previous choices. The question is whether the agent can attain any long-run distribution of consumption $w = (w_1, \ldots, w_N)$, where $w \geq 0$ and $\sum_{i=1}^{N} w_i = 1$. Thus, the length of each period t can be thought of as being "small" relative to the consumption horizon. This is similar to the use of time-average payoffs in repeated game theory; see, for example, Fudenberg and Tirole (1991, chapter 5) for discussion and cites.

It will turn out that the general result will follow from the case with only three goods, so the notation can be specialized. Consider three goods, numbered 1, 2, and

3, and the corresponding three budget sets $B_1 = \{2,3\}$, $B_2 = \{1,3\}$, and $B_3 = \{1,2\}$; let q_1, q_2, and q_3 be their probabilities of presentation, respectively. Note that budget sets are numbered by the good which is absent. Then, there are eight possible choice functions, given in Table 1. Functions f_3 and f_6 exhibit a cyclical behavior that makes it impossible to extend these functions in such a way that would not violate the Weak Axiom of Revealed Preference.[2]

Table 1: The eight choice functions. Note that f_3 and f_6 exhibit a cycle.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
B_1	2	3	2	3	2	3	2	3
B_2	1	1	3	3	1	1	3	3
B_3	1	1	1	1	2	2	2	2

In this environment, in order to obtain a particular long-run w, it will in general be necessary to choose randomly for some budget sets. Let p_k, $k = 1, \ldots, 8$ be the probability that function f_k is adopted at any given time. Again, p_k is assumed to be stationary over time.

Then, the condition that a given vector of probabilities $p = (p_k)_{k=1}^{8}$ achieves the goal of a long-run consumption pattern w is

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} q_2+q_3 & q_2+q_3 & q_3 & q_3 & q_2 & q_2 & 0 & 0 \\ q_1 & 0 & q_1 & 0 & q_1+q_3 & q_3 & q_1+q_3 & q_3 \\ 0 & q_1 & q_2 & q_1+q_2 & 0 & q_1 & q_2 & q_1+q_2 \end{pmatrix} p^T. \tag{1}$$

It is assumed that equation (1) has a solution; that is, that there exists some feasible way to obtain w. Since this linear system has more unknowns than equations, if a solution exists, then infinitely many solutions exist.

3 Results

The goal is to further characterize the solutions p to equation (1). In particular,

[2]WARP has no bite in this environment, since the protasis never holds when considering only two-element budget sets. However, for functions f_3 and f_6, it would not be possible to specify a choice for the budget set $\{1,2,3\}$ that would not violate WARP. Even though three-element budget sets are not considered in this analysis, behavior will be deemed to "violate" WARP if the choice functions could not be extended to satisfy WARP for the three-element case.

it is argued that, while there are solutions p that involve the adoption of choice behavior that exhibit cycles, the long-run consumption obtained by those solutions p can also be obtained by other solutions p' which place zero weight on choice behavior with cycles. Furthermore, p can be chosen such that the agent would not appear to an observer to violate a statistical version of WARP, which will be shown to be a strictly stronger condition. The former result is a statement about choice behavior at any given period, while the latter can be viewed as a statement about consistency of choice behavior over time. The results are presented in turn in the next two subsections.

3.1 The cyclical choice functions add no feasible bundles

The first result shows that the cyclical functions f_3 and f_6 are never required in order to obtain any w. That is to say, there is never any advantage to adopting with positive probability a choice function at any given time that would violate WARP.

Proposition 1 *If $N = 3$, and if the vector w is feasible, there exist distributions p solving equation (1) that give probability zero to the functions f_3 and f_6. In addition, any feasible vector w can be obtained by randomizing over a support of at most three choice functions.*

Proof For ease of exposition, the argument is presented geometrically. The space of consumption bundles w is presented as the simplex in Figure 1. The eight points on the simplex correspond to one of the eight choice functions f_k; the feasible consumption bundles, then, are the points on the simplex which can be attained by taking convex combinations of these points. The six functions which could be extended in such a way as to satisfy WARP are located on the boundaries of the simplex, forming a hexagon. The functions f_3 and f_6 are located interior to this hexagon. Thus, if there is a vector p solving equation (1) such that $p_3 > 0$ or $p_6 > 0$, then there exists another solution such that $p_3 = p_6 = 0$. In other words, whatever choices are being made when using f_3 and f_6 can be "synthesized" using only acyclic choice functions.

The hexagon formed by the points corresponding to the six choice functions satisfying WARP are exactly the vectors w which are feasible. By inspection, all such points lie within at least one triangle generated by exactly three of the extreme points. Thus, any w can be obtained by using at most three choice functions with positive probability. □

Corollary 1 *For all finite N, if the vector w is feasible, there exist distributions p over the choice functions that give probability zero to the functions that violate WARP.*

Proof Observe first that if a cycle exists over four or more goods and budget sets, then there must be a cycle within those goods consisting of exactly three of the goods over three budget sets. Without loss of generality, call those goods 1, 2, and 3. Then, apply the argument of Proposition 1, using the eight choice functions which prescribe behavior on the budget sets $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ as in Table 1, and which agree on all other budget sets. □

3.2 A stronger dynamic consistency result

The previous result illuminates geometrically why the cyclical choice functions f_3 and f_6 add nothing to the feasible set of long-run consumption bundles w. Yet, it could be argued that this is of little use in practice, since what one would observe in this setting is not the choice functions adopted by the agent, but rather the choices that result from those functions.

This section strengthens the result by showing that the agent can achieve any feasible w by adopting a stationary choice strategy that would satisfy, in a stochastic sense, the self-consistency ideas embodied in WARP. Let $P(x; B)$ be the probability some good $x \in B$ is chosen from a budget set B.

Definition 1 *Observed choice behavior satisfies **stochastic WARP** if, for any three goods A, B, C, and any three choice sets $\alpha \supseteq \{A, B\}$, $\beta \supseteq \{B, C\}$, and $\gamma \supseteq \{A, C\}$, the conditions*

$$P(A; \alpha) > P(B; \alpha) \text{ and } P(B; \beta) > P(C; \beta) \text{ and } P(C; \gamma) > P(A; \gamma)$$

are not jointly satisfied.

Then, the result of Proposition 1 can be strengthened as follows.

Proposition 2 *For any feasible w, there exist distributions p over the choice functions that satisfy both equation (1) and stochastic WARP. Furthermore, stochastic WARP is strictly stronger than just requiring $p_k = 0$ for all choice functions f_k that do not individually satisfy WARP.*

Proof By symmetry, there are only three types of support for p to consider:

1. The extreme points are all adjacent on the hexagon;

2. Two extreme points are adjacent;

3. None of the extreme points are adjacent.

These are investigated in turn.

1. *All adjacent:* Suppose $w = \alpha f_4 + \beta f_7 + \gamma f_8$, for some $\alpha + \beta + \gamma = 1$, $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$.[3] Then, $P(1|B_3) = \alpha$; $P(3|B_2) = 1$, and $P(2|B_1) = \beta$. Stochastic WARP fails iff all three probabilities are greater than one-half or less than one-half. They cannot all be less (since $1 > 1/2$), and they cannot be all greater, since $\alpha > 1/2$ and $\beta > 1/2$ cannot both hold.

2. *Two adjacent:* Suppose $w = \alpha f_4 + \beta f_5 + \gamma f_8$. Then $P(1|B_3) = \alpha$; $P(3|B_2) = \alpha + \gamma = 1 - \beta$; $P(2|B_1) = \beta$. Since both β and $1 - \beta$ cannot simultaneously be less than or greater than $1/2$, stochastic WARP holds.

3. *None adjacent:* Suppose $w = \alpha f_2 + \beta f_5 + \gamma f_8$. Then $P(1|B_3) = \alpha$; $P(3|B_2) = \gamma$; $P(2|B_1) = \beta$. If, for example, $\alpha = \beta = \gamma = 1/3$, stochastic WARP would fail.

□

The intuition for why stochastic WARP is not necessarily satisfied when using choice functions that correspond to extreme points that are not adjacent can be obtained by examining those functions. There are two such groups: $\{f_1, f_4, f_7\}$ and $\{f_2, f_5, f_8\}$. Taken together, each group forms a "cycle" in that there is a one-to-one mapping between a function in the group and a good which is never selected by that function. So randomizing over those functions is itself a form of inconsistency, which is revealed not at any point in time, but by viewing the choice behavior over time as a whole.

4 Conclusion

The standard normative justification for the use of transitive preferences is that intransitivities in preferences make an agent subject to systematic exploitation. However, the conditions for that systematic exploitation are, in some sense, relatively unlikely to occur. It is possible to create suitable conditions once such a cycle is

[3]The notation for choice functions is abused slightly to also represent the points on the simplex.

identified; but this assumes a level of detailed information about the preferences or behavior of an individual that seems implausible.

The results in this paper complement previous results of Epstein and Kim in affirming that incorporating the use of "intransitive" behaviors does not add to the set of possible consumption choices in a setting where apparent preference reversals might seem to be potentially useful. There is no feasible pattern of consumption in this model that cannot be obtained with behavior that the observer would classify as being transitive, at least in a stochastic sense. Thus, there is no sense in which a decision-maker would strictly benefit, in terms of the resulting choice outcomes, by appearing to behave "intransitively." Consistency in this setting is not the hobgoblin of the rational agent's mind.

The focus of this analysis is on the feasibility of attaining any given outcome; thus, information about the environment does not play a significant role. Relaxing the informational assumptions is one route to justifying why observed behavior might appear inconsistent. Vega-Redondo (1995) presents a dynamic model of decision-making in which agents have imprecise impressions of the outcomes possible from the available actions, and discusses the extent to which a rationally learning agent will appear to be consistent.

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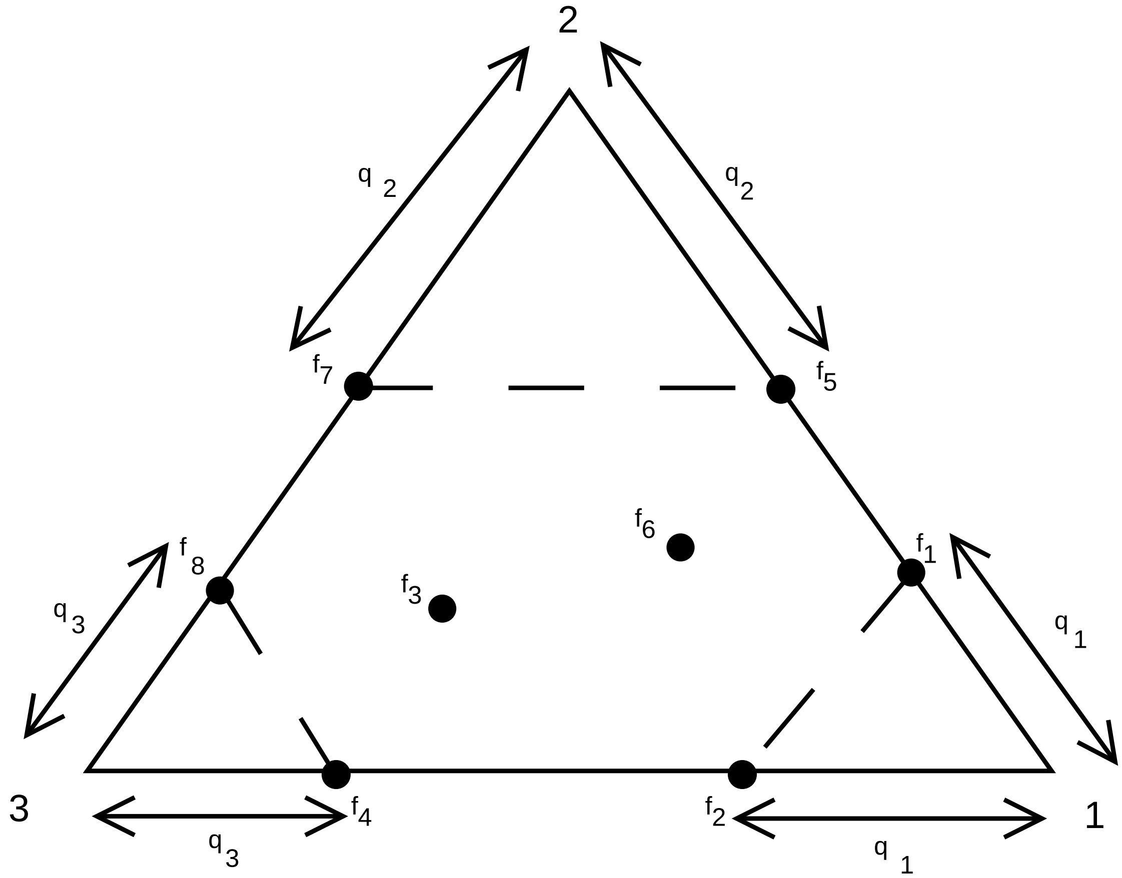

Figure 1: Feasible consumption bundles as a function of the budget set probabilities.