E C O N O M I C S   B U L L E T I N

The scaling function−based estimator of the long memory parameter: a comparative study

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Abstract

In this paper an original estimator of long memory is considered. It is based on the scaling function directly extracted from multifractal formalism. Monte Carlo simulations show that the scaling function gives interesting results, notably in terms of confidence intervals, which are smaller than the usual methods.

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1 Introduction

The long memory has a long history and remains a topic of active research on economic and financial time series. This property is referred to as Hurst's effect. Two methods are commonly used to measure the long memory: the R/S-analysis (Hurst, 1951)[1] and the GPH method[2] (Geweke and Proter-Huddak, 1983).

In this paper, we compare usual methodologies (R/S-analysis and GPH) to the scaling function method which pays tribute to multifractal formalism. Calvet and Fisher (2002) use this to detect the multifractality of the financial series and to estimate the long memory of the Deutch Mark/US Dollar series. Our study shows that the scaling function presents advantages over other methods, notably this methodology allows the estimation of long memory with a grater precision than GPH and R/S-analysis.

Section 2 presents the definition and the estimation of the scaling function. In section 3, three estimators (R/S-analysis, GPH and scaling function methodologies) are compared in terms of means, variances and confidence intervals of Monte Carlo simulations. Section 4 concludes.

2 Scaling function

To define the scaling function estimator we present briefly ARFIMA models. An ARFIMA$(0, d, 0)$ process x_t is defined as

$$\bigtriangledown^d x_t = z_t, \quad z_t \sim iid\left(0, \sigma_z^2\right) \tag{1}$$

where the fractionally differencing operator $\bigtriangledown^d = (1-L)^d$ is defined by means of the binomial expansion. Baillie (1996) presents the links between ARFIMA$(0, d, 0)$ and Fractional Brownien Motion (FBM) processes. The partial sum of x_t is denoted s_t and behaves as follows

$$\left(\frac{1}{\sigma_s}\right) s_{\lceil rT \rceil} \stackrel{d}{=} B_H(r), \quad \text{for } r \in (1/T, 1) \tag{2}$$

where $\stackrel{d}{=}$ denotes convergence in distribution, T is the length of x_t, σ_s^2 is the variance of s_t, $\lceil \cdot \rceil$ is the integer part operator, and s_t is defined by

$$s_{\lceil rT \rceil} = \sum_{t=1}^{\lceil rT \rceil} x_t, \quad \text{for } r \in (1/T, 1) \tag{3}$$

$B_H(r)$ is a FBM process and H is known as the Hurst exponent, which satisfies: $H = d + 1/2$.

[1] See, for example, Mandelbrot (1972), Lo (1991) and Baillie (1996).

[2] See, for example, Hurvich and Beltrao (1994) or Baillie (1996).

Calvet and Fisher (2002) define the scaling function of a FBM process. The scaling function of x_t is denoted $\tau(q)$ and takes into account the influence of the time t on the moments q according to

$$\mathrm{E}\left(|s_t|^q\right) = t^{\tau(q)+1} c_q \tag{4}$$

where $\mathrm{E}(\cdot)$ is the expectation operator and $c(q)$ is called the prefactor. A FBM process $B_H(r)$ is a self-affine process with a self-affinity index $(d+1/2)$, hence it satisfies $s_t =^d t^{d+1/2} s_1$. From (4) we can deduce that the scaling function $\tau(q) = (d+1/2)\cdot q - 1$ and the prefactor $c_q = \mathrm{E}\left(|s_1|^q\right)$. The scaling function delivers the self-affinity index through the relation

$$\tau\left(\frac{1}{d+1/2}\right) = 0 \tag{5}$$

To estimate the scaling function, Calvet and Fisher (2002) propose a method based on the partition function. The partition function of x_t is denoted $\pi_\delta(x,q)$, defined for each moments q, and obtained by partitioning the series into n subintervals of length δ

$$\pi_\delta(x,q) = \sum_{i=1}^{n} \left|x_{\lceil i\cdot\delta\rceil} - x_{\lceil (i-1)\cdot\delta\rceil}\right|^q \tag{6}$$

using (4) gives us

$$\log\left(\pi_\delta(x,q)\right) = \tau(q)\log(\delta) + \log(c_q) + \log(T) \tag{7}$$

For a given series x_t, computing its partition function according to (6) for various moments q allows us to deduce its scaling function according to (7). Thus, the partition function gives an estimation of the scaling function. It is then straightforward to characterize the long-range dependence of a process.

In the next section, we compare the scaling function methodology to the R/S analysis and GPH methods[3] to estimate the Hurst exponent (H) or the fractional integration (d) of an ARFIMA$(0,d,0)$[4].

3 Monte Carlo Simulations

We simulate 10000 ARFIMA $(0,d,0)$ paths with three sample sizes T ($T = 100$, 500 and 1000). We compare these methodologies for selected values of $d \in \{0, 0.2, 0.4\}$. In Tables (1-2-3) we report some simulation evidence on the performance of the scaling function.

We present our study relative to the sample sizes $T = 500$ (Table 2). For mean values $\overline{d}$, the scaling function and the GPH[5] have quite similar results. These estimations are better than the R/S-analysis estimators. This phenomenon increases when d is large. For example, when $d = 0.4$, the mean values of the R/S estimators is equal

[3] The R/S analysis and the GPH methods are the most commonly used estimations of the long-range dependence. For details see references in the section 1

[4] See Hosking (1981).

[5] We consider the standard value for the periodogram $T^{0.5}$, see Diebold and Rudelusch (1989).

to 0.22 and respectively of 0.38 and 0.41 for the scaling function and GPH methods. On the other hand, in terms of variances $\left(\sigma_{\overline{d}}\right)$ and confidence intervals (CI), the scaling function presents better results than other methods. In fact, the scaling function presents for each value d a confidence interval tighter than GPH and R/S-analysis estimators. Thus, the scaling function allows us to estimate the long memory with a largest precision than GPH and R/S-analysis. The results are the same for smaller ($T = 100-$Table 1) and larger ($T = 1000-$Table 3) sample sizes. Notice that the precision of the estimation obtained by the scaling function is independent of the sample sizes.

4 Conclusion

We use original scaling function's method to estimate the long memory of series like ARFIMA$(0, d, 0)$. Monte Carlo simulations show that estimators obtained by this method are more accurate than usual methods such as R/S-analysis and GPH.

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$T=100$		$\tau(q)$	R/S	GPH
	$\overline{d}$	-0.01	0.024	-0.002
$d=0$	$\sigma_{\overline{d}}$	0.01	0.0025	0.09
	CI	$[-0.19, 0.13]$	$[-0.06; 0.1]$	$[-0.51, 0.45]$
	$\overline{d}$	0.18	0.109	0.21
$d=0.2$	$\sigma_{\overline{d}}$	0.007	0.029	0.08
	CI	$[0.04, 0.3]$	$[0.017; 0.19]$	$[-0.28, 0.66]$
	$\overline{d}$	0.37	0.18	0.42
$d=0.4$	$\sigma_{\overline{d}}$	0.0052	0.003	0.08
	CI	$[0.24, 0.47]$	$[0.08; 0.26]$	$[-0.08, 0.88]$

Table 1: Means, variances and confidence intervals for the three methods, $d \in \{0, 0.2, 0.4\}$ and $T = 100$. $\overline{d}$: mean of the estimator, $\sigma_{\overline{d}}$: variance of the estimator and CI: confidence interval

$T = 500$		$\tau(q)$	R/S	GPH
$d = 0$	$\overline{d}$	-0.008	0.0254	-0.0018
	$\sigma_{\overline{d}}$	0.0013	0.0013	0.029
	CI	$[-0.007, 0.04]$	$[-0.040; 0.057]$	$[-0.29, 0.26]$
$d = 0.2$	$\overline{d}$	0.19	0.1332	0.21
	$\sigma_{\overline{d}}$	0.0015	0.0018	0.029
	CI	$[0.13, 0.25]$	$[0.12; 0.19]$	$[-0.07, 0.47]$
$d = 0.4$	$\overline{d}$	0.38	0.2243	0.41
	$\sigma_{\overline{d}}$	0.0026	0.0020	0.029
	CI	$[0.3, 0.46]$	$[0.15; 0.28]$	$[0.11, 0.67]$

Table 2: Means, variances and confidence intervals for the three methods, $d \in \{0, 0.2, 0.4\}$ and $T = 500$ $\overline{d}$: mean of the estimator, $\sigma_{\overline{d}}$: variance of the estimator and CI: confidence interval

$T = 1000$		$\tau(q)$	R/S	GPH
	$\overline{d}$	-0.0058	0.0251	0.003
$d = 0$	$\sigma_{\overline{d}}$	0.0007	0.0010	0.018
	CI	$[-0.05, 0.035]$	$[-0.048; 0.077]$	$[-0.22, 0.22]$
	$\overline{d}$	0.195	0.14	0.21
$d = 0.2$	$\sigma_{\overline{d}}$	0.001	0.0015	0.01
	CI	$[0.15, 0.24]$	$[0.099; 0.178]$	$[-0.038; 0.41]$
	$\overline{d}$	0.384	0.238	0.407
$d = 0.4$	$\sigma_{\overline{d}}$	0.0021	0.0017	0.019
	CI	$[0.31, 0.46]$	$[0.158; 0.31]$	$[0.17, 0.62]$

Table 3: Means, variances and confidence intervals for the three methodologies, $d \in \{0, 0.2, 0.4\}$ and $T = 1000$ $\overline{d}$: mean of the estimator, $\sigma_{\overline{d}}$: variance of the estimator and CI: confidence interval