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A comparison of the Dodgson method and the Copeland rule

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Abstract

This paper compares binary versions of two well−known preference aggregation methods designed to overcome problems occurring from voting cycles, Copeland's (1951) and Dodgson's (1876) method. In particular it will first be shown that the Copeland winner can occur at any position in the Dodgson ranking. Second, it will be proved that for some list of individual preferences over the set of alternatives, the Dodgson ranking and the Copeland ranking will be exactly the opposite, i.e. maximally different.

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1 Introduction

Simple majority rule does not always determine a Condorcet winner, i.e. the alternative that beats every other alternative by a simple majority, because of voting cycles. The purpose of this paper is to compare binary versions of Copeland's (1951) and Dodgson's (1876) preference aggregation methods[1] designed to overcome problems occurring from voting cycles. In particular it will be shown that the Copeland winner can occur at any position in the Dodgson ranking. Moreover, it will be proved that for some list of individual preferences over the set of alternatives, the Dodgson ranking and the Copeland ranking will be exactly the opposite, i.e. maximally different. Hence we provide new insight into the relationship of these two methods. This is something that cannot be obtained from the usual axiomatic (Fishburn (1977)) or non-binary (Laffond et al. (1995)) comparisons of different aggregation methods.

The first comparisons of this kind have essentially been undertaken by Ratliff (2001, 2002a,b) who has investigated the relationship between the non-binary Dodgson method and various other binary aggregation procedures. Saari (2000) has provided insight into different methods using a geometric approach. Our paper is also related to Baigent (1983), who discusses the idea of being furthest from majoritarian choice, i.e. furthest from being a Condorcet winner. Some of our results can be seen as a strengthening of those results.

The structure of the paper is as follows. The next section presents the formal framework. Section 3 introduces the Dodgson ranking. In section 4 we introduce the Copeland ranking and provide the main comparisons to the Dodgson ranking. Section 5 concludes the paper.

2 Formal Framework

Let X denote a finite set of n alternatives and H denote a finite set of h individuals. A preference $R \subseteq X \times X$ is a binary relation on X. For all $A_j, A_m \in X$, the weak preference of A_j over A_m will be denoted by $A_j \succsim_R A_m$. The symmetric and asymmetric part of R will be written as $\sim_R$ and $\succ_R$ respectively. Whenever there is no danger of confusion, subscripts will be dropped. Let $\mathcal{B}$ be the set of all complete binary relations on X, $\mathcal{W} \subset \mathcal{B}$ the set of all weak orders (complete and transitive binary relations) on X and $\mathcal{L} \subset \mathcal{W}$ the set of all linear orders (complete, transitive and asymmetric binary relations) on X. The non-empty set $M_A \subset \mathcal{B}$ is the set of all $R \in \mathcal{B}$ such that for all $A' \in X \setminus \{A\}, A \succ_R A'$. Lists of individual (strict) preferences (also called profiles) will be written as $u = (L_1^u, L_2^u, ..., L_h^u) \in \mathcal{L}^h$ where $L_i^u \in \mathcal{L}$ is individual i's preference on X in profile u.

For all $A_j, A_m \in X$, the majority margin of A_j over A_m in profile $u \in \mathcal{L}^h$ is denoted by $a_{j,m}^u = \left|\left\{i \in H : A_j L_i^u A_m\right\}\right| - \left|\left\{i \in H : A_m L_i^u A_j\right\}\right|$.[2] As the discussed methods are procedures to overcome the problems of simple majority rule (SMR) we define SMR as a function $v : \mathcal{L}^h \to \mathcal{B}$ such that for all $u \in \mathcal{L}^h$ and all $A_j, A_m \in X$, $A_j \succsim_{v(u)} A_m$ if and only if $a_{j,m}^u \geq 0$. That is, an alternative A_j is at least as good as alternative A_m if and only if there are not more individuals strictly preferring A_m over A_j than there are individuals strictly preferring A_j

[1] A binary version of Copeland's method can be found in Saari (2000), one of Dodgson's method in Klamler (2002a).

[2] Whenever there is no danger of confusion the superscript will be dropped.

over A_m. The non-empty set $\Gamma(A_j) \subset \mathcal{L}^h$ denotes all profiles for which $A_j \in X$ is the Condorcet winner, i.e. $a_{j,m} > 0$ for all $A_m \in X \setminus \{A_j\}$.

Furthermore, let $\mathbb{Z}$ be the set of all integers, then, using SMR, we can assign to any profile $u \in \mathcal{L}^h$ a point in pairwise space $\mathbb{Z}^{\binom{n}{2}}$ denoted by the vector $w^u = (a^u_{1,2},...,a^u_{j,m},...,a^u_{n-1,n}) \in \mathbb{Z}^{\binom{n}{2}}$ where $j,m \in \{1,2,...,n\}, j < m$.

Finally, use will be made of concepts measuring the distance between binary relations and profiles, respectively. Let $\mathbb{R}$ be the set of all real numbers. The Kemeny distance function on $\mathcal{B}$ will be defined as $\delta : \mathcal{B} \times \mathcal{B} \to \mathbb{R}_+$ such that for all $R, R' \in \mathcal{B}, \delta(R,R') = \frac{|(R-R') \cup (R'-R)|}{2}$.[3] Distance on the set of profiles will be measured by distance function $d : \mathcal{L}^h \times \mathcal{L}^h \to \mathbb{R}_+$ such that for all $u,u' \in \mathcal{L}^h, d(u,u') = \sum_{i=1}^{h} \delta(L_i^u, L_i^{u'})$.

3 The Dodgson Ranking

Dodgson (1876) devised a non-binary procedure to overcome the problem of voting cycles. He suggested that one should always choose the alternative in X that is "closest" from being a Condorcet winner. His concept of distance was based on the number of inversions of pairs in the individual preferences. Hence, profile information is essential for applying Dodgson's method. A natural binary extension of Dodgson's non-binary method is to rank the alternatives with respect to the minimal number of inversions necessary to make the alternatives Condorcet winners.

For a formal statement of the Dodgson ranking, let, for all $u \in \mathcal{L}^h$ and all $T \subseteq \mathcal{L}^h$, the Dodgson distance of an alternative be determined by the function $\Delta^u : X \to \mathbb{R}_+$ such that for all $A \in X$, $\Delta^u(A) = \min_{u' \in \Gamma(A)} d(u,u')$.

Definition 3.1: For all $u \in \mathcal{L}^h$, the Dodgson ranking $D \in \mathcal{W}$ is such that for all $A_j, A_m \in X, A_j \succsim_D A_m$ if and only if $\Delta^u(A_j) \leq \Delta^u(A_m)$.

Intuitively the Dodgson ranking seems to be a very attractive solution to the problem of voting cycles. The method insures that alternatives are higher ranked in the Dodgson ranking whenever they are of smaller distance from being a Condorcet winner. Moreover there will always be an alternative that is considered furthest away from being a Condorcet winner.[4] For obvious reasons such an alternative can be seen as a bad alternative. The question arises how other procedures devised to overcome Condorcet's paradox compare to the Dodgson ranking. As those methods are extensions of SMR this could be seen as undesirable because it goes against the majoritarian legitimacy inherited from SMR.

The rest of this section will introduce some prerequisites necessary to derive the later results. Saari (1995) has shown that for any vector $z \in \mathbb{Z}^{\binom{n}{2}}$ such that all entries are either even or odd, there exists, for some $h \in \mathbb{Z}_+$, a profile $u \in \mathcal{L}^h$ such that $w^u = z$, i.e. any point in pairwise space can be obtained using SMR. This has been extended by Ratliff (2001, 2002b) in the sense that for any point in pairwise space there always exists a profile such that only switches

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[3] As we are exclusively concerned with linear orders, the division by 2 is for the convenience of being able to talk about distance values and numbers of pairwise switches interchangeably.
[4] For further discussion see also Baigent (1983).

in adjacent alternatives in the individual rankings are necessary to determine the Dodgson winner.[5]

Lemma 3.2 (Ratliff (2002b)): Given any profile *u*, there exists a profile *u'* with the same pairwise election margins as *u* where all of the pairwise outcomes can be reversed via adjacency switches in *u'*.

4 The Copeland Ranking

Fishburn (1977, p. 473) said about Copeland's method that it "extends Condorcet's Principle under the philosophy that an alternative with the greatest number of simple majority wins minus losses deserves to be in the choice set. If a simple majority win is a good thing for an alternative, then the more the better." Hence, the Copeland ranking can be seen as an ordering of the alternatives according to the stated principle that the more majority wins the better. However, it can also be shown that this is equivalent to a ranking of the alternatives according to their distances from being Condorcet winners relative to the Kemeny metric (Klamler 2002b). This is indeed close to the definition of the Dodgson ranking with the difference that the informational basis for determining the Copeland ranking is far more restricted. There is no information about individual preferences or pairwise margins needed to determine the Copeland ranking.

For all $R \in \mathcal{B}$, let the Copeland value of an alternative be determined by the function $c_R : X \to \mathbb{R}_+$ such that for all $A \in X$ and all $R' \in M_A$, $c_R(A) = \min_{R' \in M_A} \delta(R, R')$.

Definition 4.1: For all $u \in \mathcal{L}^h$, $C \in \mathcal{W}$ is the Copeland ranking if and only if for all $A, A' \in X, A \succsim_C A' \Leftrightarrow c_{v(u)}(A) \leq c_{v(u)}(A')$.

The following three theorems show precisely how different Copeland's and Dodgson's methods can be. First, this is of particular interest because of the same distance idea underlying both of these methods. Second, it gives insight into the relation of the two methods in a way that cannot be provided by the usual axiomatic comparison of different aggregation procedures.

Theorem 4.2: Let $|X| \geq 4$. Then for some profile $u \in \mathcal{L}^h$, $2 < |H| = h < \infty$, there exists an alternative $A^* \in X$ such that for all $A \in X \setminus \{A^*\}, A^* \succ_C A$ and $A \succ_D A^*$.

(This is a corollary of both theorems 4.4 and 4.5)

Example 4.3: Consider $|X| = 4$, $|H| = 20$, and the following profile $u \in \mathcal{L}^{20}$ given in Table 1, where numbers determine how many voters have each ranking.

Nr.	Ranking	Nr.	Ranking
4	$A_1 \succ A_3 \succ A_2 \succ A_4$	5	$A_3 \succ A_2 \succ A_4 \succ A_1$
1	$A_1 \succ A_4 \succ A_3 \succ A_2$	8	$A_4 \succ A_1 \succ A_2 \succ A_3$
2	$A_2 \succ A_3 \succ A_4 \succ A_1$		

Table 1

[5] This is sufficient for the results in this paper. However, it might not go far enough for comparisons of the Dodgson ranking with other procedures. See Klamler (2002a).

From Table 1 we obtain, by using SMR, the pairwise tallies and margins which are presented in Table 2. As can be clearly seen, for the above profile there is no Condorcet winner. From Table 2 we calculate the Dodgson-distances, i.e. the distance (or necessary number of switches) of each alternative from becoming the Condorcet winner. E.g., to make A_1 the Condorcet winner it has to be moved above A_4 in at least 6 individual rankings. For A_2 we need 4 switches above A_1 and one switch above A_3. The Dodgson distance (number of

	Tallies	Margins		Tallies	Margins
$A_1 \succ A_2$	13,7	6	$A_2 \succ A_3$	10,10	0
$A_1 \succ A_3$	13,7	6	$A_2 \succ A_4$	11,9	2
$A_1 \succ A_4$	5,15	-10	$A_3 \succ A_4$	11,9	2

Table 2

pairwise switches) for every $A \in X$, denoted by $\Delta^u(A)$, is given in Table 3. To make A_1 the Condorcet winner measured in the Copeland sense, it has to be moved above A_4 in the simple majority relation, what means that one switch is necessary. For A_2 we need 1 switch above A_1 and half a switch above A_3. The Copeland value for every $A \in X$, denoted by $c_{v(u)}(A)$ is also given in Table 3.

$\Delta^u(A_1)$	6	$c_{v(u)}(A_1)$	1
$\Delta^u(A_2)$	5	$c_{v(u)}(A_2)$	$\frac{3}{2}$
$\Delta^u(A_3)$	5	$c_{v(u)}(A_3)$	$\frac{3}{2}$
$\Delta^u(A_4)$	4	$c_{v(u)}(A_4)$	2

Table 3

Hence, the Dodgson ranking is $A_4 \succ_D A_3 \sim_D A_2 \succ_D A_1$ with A_1 being bottom in that ranking. The Copeland ranking is $A_1 \succ_C A_2 \sim_C A_3 \succ_C A_4$. This shows that for some preference profiles the two methods lead to maximally different results.

Theorem 4.4: Let $|X| \geq 4$. Then there exists a profile $u \in \mathcal{L}^h$, $2 < |H| = h < \infty$, such that for all $A, A' \in X$, $A \succsim_C A' \Leftrightarrow A' \succsim_D A$.

Proof. Let $a_{1,n} = -k$. Let $a_{1,j} = m$ and $a_{j,n} = l$, $j = 2,3,...,n-1$. Let $a_{p,q} = 0$ for $p,q \in \{2,3,...,n-1\}, p < q$. From lemma 3.2 there exists a profile $u \in \mathcal{L}^h$ such that $w^u \in \mathbb{Z}^{\binom{n}{2}}$ has exactly those margins. From the definition of the Copeland value, $c_{v(u)}(A_1) = 1$, $c_{v(u)}(A') = 1 + \frac{n-3}{2}$ for all $A' \in \{A_2, ..., A_{n-1}\}$ and $c_{v(u)}(A_n) = n-2$. Hence the Copeland ranking is $A_1 \succ_C A'$ for all $A' \neq A_1$ and $A' \succ_C A_n$ for all $A' \neq A_n$ and $A' \sim_C A''$ for all $A', A'' \in \{A_2, A_3, ..., A_{n-1}\}$. From the definition of the Dodgson number and the assumed pairwise margins we get (for k,m,l even) $\Delta^u(A_1) = \frac{k}{2} + 1$, $\Delta^u(A') = \frac{m}{2} + 1 + (n-3)$ for

$A' \neq A_1, A_n$ and $\Delta^u(A_n) = (n-2)(\frac{l}{2}+1)$. For A_1 to be bottom in the Dodgson ranking, $\frac{k}{2}+1 > (n-2)(\frac{l}{2}+1)$ which leads to $k > (n-2)(l+2)-2$ and $\frac{k}{2}+1 > \frac{m}{2}+n-2$ which leads to $k > m+2n-6$. For A_n to be top in the Dodgson ranking, $\frac{m}{2}+1+(n-3) > (n-2)(\frac{l}{2}+1)$ which leads to $m > (n-2)(l+2)-2n+4$ which can be simplified to $m > (n-2)l$. From that and the two inequalities containing *k* as determined above, those reduce to one inequality containing *k*, namely $k > (n-2)(l+2)$. To show that there exists a solution to this system of inequalities for all $n \geq 4$, choose $l = 2$. This implies that $m > 2n-4$. Hence, let $m = 2n-2$. Then from the above inequality containing *k*, we get $k > 4n-8$ which exists for all finite $n < \infty$. □

Theorem 4.5: Let $|X| \geq 4$. The Copeland winner can occur at any position in the Dodgson ranking.

Proof. Let $u \in \mathcal{L}^h, 2 < h < \infty$ be a profile such that $w^u = (l, l, ..., l, 0, l, ..., l, -k, l, ..., l)$, k, l both even, i.e. $a_{1,n} = -k, a_{2,3} = 0$ and all other pairwise margins are equal to *l*. From lemma 3.2 we know that such a lemma exists. The Dodgson and Copeland values for the alternatives in such a profile can be found in Table 4.

Alt.	$\Delta^u(.)$	$c_{v(u)}(.)$
A_1	$\frac{k}{2}+1$	*1*
A_2	$\frac{l}{2}+2$	$\frac{3}{2}$
A_3	$\frac{l}{2}+2$	$\frac{3}{2}$
A_4	$\frac{3l}{2}+3$	*3*
$\vdots$		
A_j	$\frac{(j-1)l}{2}+(j-1)$	$j-1$
A_n	$\frac{(n-2)l}{2}+(n-2)$	$n-2$

Table 4

Obviously, $\Delta(A_{j+1}) \geq \Delta(A_j)$ for $j = 2, 3, ..., n-1$. Hence for A_1 to be bottom in the Dodgson ranking, $\frac{k}{2}+1 > \frac{(n-2)l}{2}+(n-2)$ which leads to $k > (n-2)l+2n-6 = (n-2)(l+2)-2$. Also, as the Copeland ranking (and therefore the Copeland winner) is determined by the values of $c_{v(u)}$, from table 1 we see that A_1 is the Copeland winner as it has the lowest Copeland value.

From the definition of $c_{v(u)}$ it is clear that the Copeland ranking does not change as long as the signs in the vector of pairwise margins do not change. To change A_1's position in the Dodgson ranking, change the pairwise margins without changing the sign. Let $a_{1,2} = m > l$. For $\Delta(A_2) > \Delta(A_1)$ we need $\frac{k}{2}+1 < \frac{m}{2}+2$ which implies $m > k-2$. For every alternative $A_j, j \in \{3, 4, ..., n-1\}$ let $a_{i,j} = m, i \in \{1, 2, ..., j-1\}$ with $m > k-2$ and $k > (n-2)(l+2)-2$. Doing so will move one alternative after the other below A_1 in the Dodgson ranking. Finally, let $a_{i,n} = m, i \in \{2, 3, ..., n-1\}$, $m > k-2$. This, in addition to the steps taken before implies that

A_1 can also be the Dodgson winner and hence the Copeland winner can take any position in the Dodgson ranking. □

5 Conclusion

This paper has provided a comparison of two famous rules to overcome Condorcet's problem, Dodgson's method and Copeland's method. Essentially we have shown that Dodgson's and Copeland's ranking will be maximally different for some preference profiles. First, this is of interest because such comparisons are not feasible in the usual axiomatic framework and have been initiated only recently by Ratliff (2001, 2000a,b). Second, it is of interest because, as we showed, the Copeland ranking can be derived from a very attractive distance minimization concept. This indicates that the main underlying idea of both Dodgson's and Copeland's method, namely the closer an alternative from being a Condorcet winner the better, is actually identical. The essential difference lies in the informational basis of the two methods.

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