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A clarifying note on converting to log-deviations from the steady state

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Abstract

The paper discusses the mathematical background and several alternative strategies of converting equations as typically found in dynamic stochastic general equilibrium models into log-deviations from the steady state form. Guidance is provided on when to use which computational strategy. More examples with detailed derivations and a simple Maple program to automate the conversion are made available online.

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1. Introduction

Taking log-deviations around the steady state is a solution to the problem of reducing the computational complexity of highly nonlinear systems of numerically specified, time dependent equations. Such conversion is a rather common practice in macroeconomics for the solution of dynamic stochastic general equilibrium (DSGE) models. The advantages of the log-deviation conversion go beyond simplifying the calculations. For small deviations from the steady state, the log-deviation form also allows for a convenient economic interpretation: the variables are percentage deviations from the steady state and their associated coefficients are elasticities.

This note is motivated by the fact that conversion to log-deviations, although in its core a simple application of first-order Taylor series expansions, is often confusing to beginning students of macroeconomics. Although the material is discussed in some textbooks and papers on macroeconomics (e.g., Romer 2006, Heijdra and van der Ploeg 2002, Uhlig 1999), the presentation often relies on short-cuts that only work in some special cases and little is typically said about the basic logic or advantage of the particular computational strategy that is used. The intent of this paper is to bring together all relevant computational aspects of converting equations to log-deviations form, show their logic, and provide some pertinent examples.[1]

The note is organized as follows. The following section presents the two most commonly used short-cuts of converting an equation to log-deviations form. Next, a less commonly applied conversion strategy is illustrated, which reverses the typical sequence of first transforming an equation and then applying a Taylor series approximation. This latter approach has the advantage of being far easier to automate by computer than the usual conversion strategies.

2. Definition and two common short-cuts

This section defines taking log-deviations around the steady state and introduces two common computational strategies.

2.1 Definition and conversion based on the definition

Converting to log-deviations around the steady state is interpreted here to mean replacing an expression by the difference between its log value and the log value of its steady state. Assume x denotes the steady state value of variable x_t. Then the log-deviation of variable x_t from its steady state x is defined as

$$\widetilde{x}_t \equiv \ln x_t - \ln x. \tag{1}$$

To the extent that x_t is a place holder for any time subscripted expression, equation (1) is a general recipe for converting an expression into log-deviation form: simply take the log of

[1] For anyone interested in reviewing how log-linearization fits into the solution of DSGE models, that is, what steps precede and what steps follow log-linearization, the reader is referred to Uhlig (1999).

the expression and subtract the log of its steady state. In fact, taking the log-deviation of an expression in this manner can be thought of as employing a specialized log operator on the expression.

To fix the idea, consider how to convert a more general algebraic expression, such as βx_t^α, to log-deviation form. Following the definition of equation (1) and employing the rules of logarithm, we can write

$$(\widetilde{\beta x_t^\alpha}) = \ln \beta x_t^\alpha - \ln \beta x^\alpha = \ln \beta + \alpha \ln x_t - \ln \beta - \alpha \ln x = \alpha(\ln x_t - \ln x).$$

Using again the definition of equation (1), this simplifies to

$$(\widetilde{\beta x_t^\alpha}) = \alpha \widetilde{x}_t. \tag{2}$$

The term $\widetilde{x}_t$ can be interpreted as the percentage difference between x_t and its steady state value x. To verify this point, approximate the right-hand side of equation (1) by a first-order Taylor series polynomial at $x_t = x$,

$$\widetilde{x}_t = \ln x_t - \ln x \simeq \frac{1}{x}(x_t - x) = \frac{x_t - x}{x}. \tag{3}$$

The interpretation of $\widetilde{x}_t$ as a percentage deviation of x_t from its steady state value is valid only for small percentage deviations from the steady state because it relies on the rules of calculus. This highlights that log-linearization is a *local* approximation method.

2.2 Conversion by substitution

The method of obtaining the log-deviations of an expression per its definition (equation (1)) can complicate the conversion process if equations involve mainly addition or subtraction, such as in the case of a national income accounting identity,

$$y_t = c_t + i_t.$$

Using the log-deviations operator in this case complicates matters because it generates a term $\left(\widetilde{c_t + i_t}\right)$ on the right side of the equation that itself needs to be converted to obtain expressions in $\widetilde{c}_t$ and $\widetilde{i}_t$.

An often used short-cut relies on a substitution process in lieu of the log-deviation operator. As part of the substitution process, every occurrence of a time subscripted variable, such as x_t, is replaced by an expression in $\widetilde{x}_t$ and the corresponding steady state value x. The expression that replaces x_t is derived from equation (1) by first isolating $\ln x_t$ on the left

$$\ln x_t = \ln x + \widetilde{x}_t$$

and then taking the exponent on both sides,

$$x_t = e^{\ln x + \widetilde{x}_t} = e^{\ln x} e^{\widetilde{x}_t} = x e^{\widetilde{x}_t}. \tag{4}$$

Up to this point no approximation is involved. By simply replacing all occurrences of x_t in a model with the expression $xe^{\widetilde{x}_t}$, and likewise for all other variables, the model would become more rather than less difficult to solve. Hence, a simple substitution of terms along the lines of equation (4) can not be the end of the conversion process, only the beginning. It is followed by a first-order Taylor series approximation of the expression $e^{\widetilde{x}_t}$ at the point $\widetilde{x}_t = 0$, which yields

$$e^{\widetilde{x}_t} \simeq 1 + e^0 (\widetilde{x}_t - 0) = 1 + \widetilde{x}_t. \tag{5}$$

Applying this approximation to (4) gives rise to a key equation of the substitution method,

$$x_t \simeq x \left(1 + \widetilde{x}_t\right). \tag{6}$$

Equation (6) highlights that the substitution method is about replacing x_t by another level term that contains the percentage deviation from the steady state ($\widetilde{x}_t$) and the corresponding steady state value (x). Equation (6) can be applied to all time subscripted variables with exponent equal to unity. For variables with exponents other than unity a more general substitution equation is required.

To generalize the conversion principle of the substitution method as expressed in equation (6), consider how to convert the algebraic term βx_t^α to log-deviation form. Start again from equation (4), that is, from the equality of x_t and $xe^{\widetilde{x}_t}$ and replace x_t,

$$\beta \left(x_t\right)^\alpha = \beta \left(xe^{\widetilde{x}_t}\right)^\alpha.$$

Simplify the right-hand side using the laws of exponents,

$$\beta \left(xe^{\widetilde{x}_t}\right)^\alpha = \beta x^\alpha e^{\alpha \widetilde{x}_t},$$

and apply the Taylor series approximation along the lines of equation (5) to the exponential term $e^{\alpha \widetilde{x}_t}$ to get

$$\beta x_t^\alpha = \beta x^\alpha e^{\alpha \widetilde{x}_t} \simeq \beta x^\alpha \left(1 + \alpha \widetilde{x}_t\right). \tag{7}$$

Equation (7) is the fundamental equation of the substitution method. Some confusion can arise if the right-hand side of equation (7) is compared to the right-hand side of equation (2). The two methods of converting βx_t^α to log-deviations form generate seemingly different results. The confusion is easily removed if one considers that the right-hand side of equation (7) is an approximate replacement of the level term βx_t^α and, therefore, itself a level term. By contrast, the right-hand side of equation (2) is the percentage deviation of βx_t^α rather than a level term. To make the right-hand side of equation (7) comparable to that of equation (2), subtract the steady state value βx^α and divide the result by βx^α,

$$\frac{\beta x^\alpha \left(1 + \alpha \widetilde{x}_t\right) - \beta x^\alpha}{\beta x^\alpha} = \alpha \widetilde{x}_t.$$

2.3 Some simple applications

Purely linear equations, such as national income accounting identities, are easily handled by both substitution (section 2.2) or the short-cut suggested in section 2.1. The more commonly used substitution method works as follows.

Example 1 *Simple substitution for each of the three terms of the national accounting identity*

$$y_t = c_t + i_t$$

yields by application of equation (6).

$$y\left(1+\widetilde{y}_t\right) = c\left(1+\widetilde{c}_t\right) + i\left(1+\widetilde{i}_t\right).$$

To simplify, make use of the steady state relationship

$$y = c + i.$$

In particular, subtract y *on the left and* $(c+i)$ *on the right to obtain*

$$y\widetilde{y}_t = c\widetilde{c}_t + i\widetilde{i}_t.$$

Finally, divide both sides of the equation by y,

$$\widetilde{y}_t = \frac{c}{y}\widetilde{c}_t + \frac{i}{y}\widetilde{i}_t.$$

Example 2 *To convert the national income account identity of Example 1 into log-deviations by the short-cut of section 2.1, take the log of the identity and subtract the log of the steady state*

$$\ln y_t - \ln y = \ln(c_t + i_t) - \ln(c + i).$$

This can be written as

$$\widetilde{y}_t = \widetilde{(c_t + i_t)}.$$

Note that interest centers on the log-deviations of each individual variable rather than on the log-deviations of the sum of c_t *and* i_t*. Equation (3) suggests the following conversion*

$$\begin{aligned}\widetilde{(c_t + i_t)} &\approx \frac{(c_t+i_t)-(c+i)}{(c+i)} = \frac{c_t - c + i_t - i}{c+i} \\ &= \frac{c}{c+i}\left(\frac{c_t - c}{c}\right) + \frac{i}{c+i}\left(\frac{i_t - i}{i}\right),\end{aligned}$$

which yields the same result as in the previous example.

Linear equations that contain nonlinear terms as elements take more effort to convert. A typical example is the state equation of the capital stock often found in DSGE models.[2]

Example 3 *Consider the equation*

$$k_{t+1} = sk_t^{\alpha} - c_t + (1-\delta)k_t,$$

where k stands for capital and c for consumption, and where s identifies the savings rate, α the production elasticity of capital, and δ the depreciation rate. To convert by the substitution method (section 2.2), the first step employs equations (6) and (7),

$$k\left(1+\widetilde{k}_{t+1}\right) = sk^{\alpha}\left(1+\alpha\widetilde{k}_t\right) - c\left(1+\widetilde{c}_t\right) + (1-\delta)k\left(1+\widetilde{k}_t\right).$$

Multiplying out yields

$$k + k\widetilde{k}_{t+1} = sk^{\alpha} + \alpha sk^{\alpha}\widetilde{k}_t - c - c\widetilde{c}_t + (1-\delta)k + (1-\delta)k\widetilde{k}_t.$$

Divide through by k to obtain

$$1 + \widetilde{k}_{t+1} = sk^{\alpha-1} + \alpha sk^{\alpha-1}\widetilde{k}_t - \frac{c}{k} - \frac{c}{k}\widetilde{c}_t + (1-\delta) + (1-\delta)\widetilde{k}_t.$$

Now simplify by making creative use of the steady state equation or its transformation

$$\begin{aligned} k &= sk^{\alpha} - c + (1-\delta)k, \\ 1 &= sk^{\alpha-1} - \frac{c}{k} + (1-\delta) \end{aligned}$$

to get

$$\widetilde{k}_{t+1} = \left[\alpha sk^{\alpha-1} + (1-\delta)\right]\widetilde{k}_t - \frac{c}{k}\widetilde{c}_t.$$

The final equation is linear in $\widetilde{k}_{t+1}$, $\widetilde{k}_t$, and $\widetilde{c}_t$ and depends only on the parameters α, s, and δ and the steady state values of k and c implied by those parameters.

3. Conversion via initial Taylor series approximation

Instead of using algebraic substitutions and a Taylor series approximation on the resulting exponential expressions, the conversion process can be reversed: first employ a Taylor series approximation and only then apply the definition of log-deviations from the steady state.

To see the general applicability of this method, which appears to be less used in the literature, consider an implicit three-variable equation like

$$g(x_t, y_t, z_t) = 0.$$

[2] Further applications can be viewed online (Appendix A.1).

The key initial step consists of linearly approximating this function at the steady state values of all variables (x, y, z) by a multivariate Taylor series expansion,

$$g(x,y,z) + g_x'(x,y,z)(x_t - x) + g_y'(x,y,z)(y_t - y) + g_z'(x,y,z)(z_t - z) = 0. \tag{8}$$

Because the equality

$$g(x,y,z) = 0$$

holds in steady state, equation (8) simplifies to

$$g_x'(x,y,z)(x_t - x) + g_y'(x,y,z)(y_t - y) + g_z'(x,y,z)(z_t - z) = 0.$$

The second step consists of changing the equation to incorporate percentage deviations from the steady state. For that purpose, multiply and divide each term by its associated steady state value to obtain

$$g_x'(x,y,z)x\frac{(x_t - x)}{x} + g_y'(x,y,z)y\frac{(y_t - y)}{y} + g_z'(x,y,z)z\frac{(z_t - z)}{z} = 0.$$

By the definitional equation (3), the last equation can be written as

$$g_x'(x,y,z)x\widetilde{x}_t + g_y'(x,y,z)y\widetilde{y}_t + g_z'(x,y,z)z\widetilde{z}_t = 0. \tag{9}$$

Equation (9) follows an easy-to-remember pattern, which extends to any number of variables. It is straightforward to implement on most equations, not only by hand, but also by computer.[3] Note that all terms other than those that represent percentage deviations from the steady state are functions only of steady state values. This has an important practical implication: as long as the problem is numerically specified, which is typically the case, the above equation converts directly to the simple linear form

$$\lambda_1\widetilde{x}_t + \lambda_2\widetilde{y}_t + \lambda_3\widetilde{z}_t = 0,$$

where the $\lambda_i, i = 1, 2, 3$, are numbers that depend on the assumed parameters and implied steady state values.

Example 4 *Consider the conversion of the national income accounting identity,*

$$y_t = c_t + i_t + g_t,$$

into log-deviations form by equation (9). First, write the equation into equal-to-zero format,

$$y_t - c_t - i_t - g_t = 0.$$

Second, apply equation (9) to get

$$y\widetilde{y}_t - c\widetilde{c}_t - i\widetilde{i}_t - g\widetilde{g}_t = 0.$$

If so desired, divide by y to obtain,

$$\widetilde{y}_t - \frac{c}{y}\widetilde{c}_t - \frac{i}{y}\widetilde{i}_t - \frac{g}{y}\widetilde{g}_t = 0.$$

[3] To help make the conversion process less burdensome than it is often perceived (e.g., Gong and Semmler 2006, p. 20 and p. 45), a simple *Maple* program along with some examples is availabl online (Appendix A.2)

It is apparent that the log-deviation form of a linear equation is very easy to obtain with the help of equation (9) because all derivatives are either plus or minus unity. The derivation is somewhat more complicated when the equation consists of a sum of non-linear terms, as in the next example.

Example 5 *To convert the state equation for the capital stock*

$$k_{t+1} = sz_t k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$$

transform to implicit form,

$$k_{t+1} - sz_t k_t^{\alpha} n_t^{1-\alpha} - (1-\delta)k_t = 0.$$

The equation contains four variables in t if one treats k_{t+1} and k_t as separate variables. Employing equation (9) generates

$$k\widetilde{k}_{t+1} - \left[\alpha szk^{\alpha-1}n^{1-\alpha} + (1-\delta)\right] k\widetilde{k}_t - \left(sk^{\alpha}n^{1-\alpha}\right) z\widetilde{z}_t - (1-\alpha)szk^{\alpha}n^{-\alpha}n\widetilde{n}_t = 0. \tag{10}$$

Dividing by k and rearranging the last two terms simplifies (10) to

$$\widetilde{k}_{t+1} - \left[\alpha szk^{\alpha-1}n^{1-\alpha} + (1-\delta)\right] \widetilde{k}_t - \left(szk^{\alpha-1}n^{1-\alpha}\right) \widetilde{z}_t - (1-\alpha)szk^{\alpha-1}n^{1-\alpha}\widetilde{n}_t = 0.$$

If so desired, this equation can be further simplified with the help of the steady state relationship

$$k = szk^{\alpha}n^{1-\alpha} + (1-\delta)k$$

and its transformations. These simplifications eventually yield

$$\widetilde{k}_{t+1} - [1-\delta(1-\alpha)]\,\widetilde{k}_t - \delta\widetilde{z}_t - \delta(1-\alpha)\widetilde{n}_t = 0.$$

4. Conclusion

In discussing the practical issues of converting equations into log-deviations from the steady state form, this note brings together in one place a number of related computational approaches, illustrates their relationship, and their relative advantages. Presenting the various approaches in one consistent notation and illustrating their use on a small set of examples helps to remove the confusion that surrounds the various computational short-cuts.

The reader needs to be cautioned that the conversion to log-deviation form, although a convenient tool, is not an economically sensible simplification for all models. For example, if the variability of a random variable is important, such as in the modeling of risk, log-deviations may not be appropriate because only the mean of a random variable is considered by equations converted to log-deviations not its variance. Other methods of making equations computationally tractable need to be employed in such cases. It may also become apparent that the conversion to log-deviations is difficult to fully automate. If one wants to avoid all manual intervention, some other approximation method has to be chosen. Perturbation methods and other techniques that make use of higher order terms have gained popularity also for this reason (Judd 1998, Miranda and Fackler 2002).

References

Gong, G., and W. Semmler (2006) *Stochastic Dynamic Macroeconomics.* Oxford University Press: Oxford and New York.

Heijdra, B.J., and F. van der Ploeg (2002) *Foundations of Modern Macroeconomics,* Oxford University Press: Oxford and New York.

Judd, K.L. (1998) *Numerical Methods in Economics*, MIT Press: Cambridge, Massachusetts.

Miranda, M.J., and P.L. Fackler (2002) *Applied Computational Economics and Finance,* MIT Press: Cambridge and London.

Romer, D. (2006) *Advanced Macroeconomics,* 3rd edition, McGraw-Hill/Irwin: New York.

Uhlig, H. (1999) "A toolkit for analysing nonlinear dynamic stochastic models easily" in *Computational Methods for the Study of Dynamic Economics by* R. Marimon and A. Scott, Eds., Oxford University Press: Oxford and New York, 30-61.

Appendix (available online)

A.1 Applications of the two short-cut methods

This section of the appendix provides several examples of the conversion methodology discussed in sections 2.1 and 2.2. The chosen examples are typical of those encountered in DSGE models. The examples are organized to highlight key mathematical properties of the equations that help with the selection of the most appropriate method of converting to log-deviations.

Linear or semi-linear equations These types of equations are fully discussed in section 2.3.

Multiplicative equations Consider the investment equation

$$i_t = sz_t k_t^{\alpha}, \tag{11}$$

where i is investment and z a technology variable. Purely multiplicative equations of this type are best handled with the methodology of section 2.1. This methodology includes the following sequence of steps,

$$\begin{aligned} \ln i_t - \ln i &= \ln sz_t k_t^{\alpha} - \ln szk^{\alpha} \\ \widetilde{i}_t &= \ln s + \ln z_t + \alpha \ln k_t - \ln s - \ln z - \alpha \ln k \\ \widetilde{i}_t &= \widetilde{z}_t + \alpha \widetilde{k}_t. \end{aligned}$$

Alternatively, the investment equation can be converted into log-deviations form along the lines of section 2.2. To do that, apply the approximations given by equations (6) and (7),

$$i\left(1 + \widetilde{i}_t\right) = sz\left(1 + \widetilde{z}_t\right) k^{\alpha}(1 + \alpha \widetilde{k}_t). \tag{12}$$

Next, utilize the steady state,

$$i = szk^{\alpha},$$

to simplify equation (12). Dividing the left-hand side of equation (12) by i and the right-hand side by szk^{α} yields

$$\left(1 + \widetilde{i}_t\right) = \left(1 + \widetilde{z}_t\right)(1 + \alpha \widetilde{k}_t),$$

which can be solved for $\widetilde{i}_t$,

$$\widetilde{i}_t = 1 + \widetilde{z}_t + \alpha \widetilde{k}_t + \alpha \widetilde{z}_t \widetilde{k}_t - 1.$$

As both $\widetilde{z}_t$ and $\widetilde{k}_t$ are by assumption close to zero, the product of the two terms will be negligibly different from zero. Setting the product zero and simplifying yields the result

$$\widetilde{i}_t = \widetilde{z}_t + \alpha \widetilde{k}_t. \tag{13}$$

Example 6 *To convert the Cobb-Douglas production function*

$$y_t = A_t k_t^\alpha n_t^\beta$$

to log-deviations by the methodology of section 2.1, take the log on both sides of the equation and then subtract the log of the steady state value on both sides,

$$\ln y_t - \ln y = \ln A_t k_t^\alpha n_t^\beta - \ln A k^\alpha n^\beta.$$

Employing the rules of logarithm, the above equation can be rewritten as

$$\begin{aligned} \widetilde{y}_t &= \ln A_t + \alpha \ln k_t + \beta \ln n_t - \ln A - \alpha \ln k - \beta \ln n \\ &= \widetilde{A}_t + \alpha \widetilde{k}_t + \beta \widetilde{n}_t. \end{aligned}$$

The conversion method of section 2.1 works very efficiently whenever taking the log greatly simplifies algebraic expressions. Here is another example.

Example 7 *To convert the first-order profit maximizing condition of a competitive firm with production function as in Example 6,*

$$\frac{w_t}{p_t} = \frac{\partial y_t}{\partial n_t} = \beta A_t k_t^\alpha n_t^{\beta - 1},$$

take the log on both sides of the equation and subtract the log of the steady state to get

$$\begin{aligned} \ln w_t - \ln p_t - (\ln w - \ln p) &= \ln \beta + \ln A_t + \alpha \ln k_t + (\beta - 1) \ln n_t \\ &\quad - [\ln \beta + \ln A + \alpha \ln k + (\beta - 1) \ln n]. \end{aligned}$$

Rearranging terms and employing the definition of log-deviations the above equation simplifies to

$$\widetilde{w}_t - \widetilde{p}_t = \widetilde{A}_t + \alpha \widetilde{k}_t + (\beta - 1) \widetilde{n}_t.$$

Although the procedure of section 2.2 also works on ratios of variables, the short-cut of section 2.1 is significantly faster.

Example 8 *To convert the labor productivity term y_t/n_t into log-deviations form by the substitution method (section 2.2), first convert the ratio to a product,*

$$\frac{y_t}{n_t} = y_t n^{-1}.$$

Next, apply the approximations of equations (6) and (7) to obtain

$$y_t n^{-1} \simeq y \left(1 + \widetilde{y}_t\right) n^{-1} \left(1 - \widetilde{n}_t\right).$$

Multiplying out yields

$$\frac{y}{n} \left(1 + \widetilde{y}_t - \widetilde{n}_t - \widetilde{y}_t \widetilde{n}_t\right),$$

which condenses to

$$\frac{y}{n}\left(1+\widetilde{y}_t-\widetilde{n}_t\right)$$

because the term $\widetilde{y}_t\widetilde{n}_t$ is the product of two small numbers and, hence, negligible. The result is an approximate replacement for the level term y_t/n_t and, therefore, itself a level term. To express it in percentage terms, subtract and then divide by the steady state expression y/n. This yields

$$\widetilde{y}_t-\widetilde{n}_t.$$

The same result can be obtained significantly faster by the method of section 2.1,

$$\widetilde{\left(\frac{y_t}{n_t}\right)}=\ln y_t-\ln y-(\ln n_t-\ln n)=\widetilde{y}_t-\widetilde{n}_t.$$

Equations in logs DSGE models often contain log equations for exogenous variables, such as stochastic technology shocks,

$$\ln z_t=z_0+\rho\ln z_{t-1}+\epsilon_t,$$

where ϵ_t is a disturbance term with mean zero and constant variance. It is apparent that the above equation can be rewritten in multiplicative form as

$$z_t=e^{z_0}z_{t-1}^{\rho}e^{\epsilon_t}.$$

As suggested above, multiplicative equations are best converted into log-deviations form by the short-cut of section 2.1, that is, by taking the log and then subtracting the log of the steady state. As the equation to be converted is already given in log form, one only needs to subtract the log of the steady state from the given equation to obtain the log-deviation form,

$$\begin{aligned}\ln z_t-\ln z &= z_0-z_0+\rho\ln z_{t-1}-\rho\ln z+\epsilon_t\\ \widetilde{z}_t &= \rho\widetilde{z}_{t-1}+\epsilon_t.\end{aligned}$$

Conversion to log-deviations with the substitution methodology (section 2.2) takes one additional step. First, replace the time subscripted variables per equation (4) and apply the logarithm,

$$\begin{aligned}\ln ze^{\widetilde{z}_t} &= z_0+\rho\ln ze^{\widetilde{z}_{t-1}}+\epsilon_t\\ \ln z+\widetilde{z}_t &= z_0+\rho\left(\ln z+\widetilde{z}_{t-1}\right)+\epsilon_t.\end{aligned}$$

Use the steady state equation

$$\ln z=z_0+\rho\ln z$$

to subtract $\ln z$ on the left and $(z_0+\rho\ln z)$ on the right to simplify the log equation to

$$\widetilde{z}_t=\rho\widetilde{z}_{t-1}+\epsilon_t.$$

Equations with expectations terms Consider the Euler equation that connects present and future consumption for an intertemporal utility maximization problem,

$$\frac{1}{c_t} = \beta E_t \left(\frac{1 + r_{t+1}}{c_{t+1}} \right), \tag{14}$$

where E_t denotes an expectations operator and β a discount factor.

The conversion short-cut of section 2.1 is inappropriate if expectation terms are present because taking the expectation of a log term is not the same as taking the log of an expectation term.[4] The conversion should employ equations (6) and (7) of the substitution method.

To convert the Euler equation rewrite all ratios in product form,

$$1 = \beta E_t \left[c_t \left(1 + r_{t+1}\right) \left(c_{t+1}\right)^{-1} \right].$$

Application of equations (6) and (7) to the time-subscripted variables yields

$$1 = \beta E_t \left\{ c \left(1 + \widetilde{c}_t\right) \left[1 + r \left(1 + \widetilde{r}_{t+1}\right)\right] c^{-1} \left(1 - \widetilde{c}_{t+1}\right) \right\}.$$

Eliminate c, multiply out, and drop all products of log-deviation terms,

$$1 = \beta E_t \left\{ 1 + \widetilde{c}_t - \widetilde{c}_{t+1} + r + r\widetilde{c}_t - r\widetilde{c}_{t+1} + r\widetilde{r}_{t+1} \right\}.$$

Factor $(1 + r)$ to obtain

$$1 = \beta E_t \left\{ (1 + r) \left(\widetilde{c}_t - \widetilde{c}_{t+1}\right) + (1 + r) + r\widetilde{r}_{t+1} \right\}.$$

Finally, consider that the Euler equation implies the steady state

$$\beta = \frac{1}{1 + r}.$$

Using this steady state relationship the log-deviation form of the Euler equation simplifies to

$$0 = E_t \left[\left(\widetilde{c}_t - \widetilde{c}_{t+1}\right) + \left(\frac{r}{1 + r} \right) \widetilde{r}_{t+1} \right]. \tag{15}$$

A.2 Automating the conversion process

Converting equations into log-deviations form is often considered burdensome because, in contrast to alternatives such as simple first-order Taylor approximations, it is difficult to fully automate by computer.

The purpose of this section of the appendix is to demonstrate that conversion to log-deviations form can be reasonably well automated if one makes use of the conversion method discussed in section 3. A simple *Maple* routine is provided that can be adapted to convert any

[4] This results from Jensen's inequality, which implies $\ln(Ex) > E \ln x$ for the log function. Only for a linear function $f(x)$ is $f(Ex) = Ef(x)$.

equation to log-deviations form. This works very efficiently if model parameters and steady state values are numerically specified before the conversion process starts. If that is not the case, the program will provide a final equation that requires some algebraic simplifications by hand involving the equation's steady state equivalent.

The *Maple* program code below presents the conversion for Example 5 if variable n_t is set equal to unity for all t.

```
restart:
#>>>>>> change the four input lines below <<<<<<<<
eq:=kt1-s*zt*kt^alpha-(1-delta)*kt; #define equation
L1:=[kt1,kt,zt]:  #define names of variables
L2:=[kd1,kd,zd]:  #define names of log-deviations
L3:=[k,k,z]:  #define names of steady state values
#>>>>> no changes needed below this line <<<<<<<
n:= nops(L2):
# the line below generates the steady state equation
eq0:=unapply(eq,L1):  sstate:=eq0(seq(L3[i],i=1..n))=0;
# the line below is a simple transformation of sstate
collect(sstate, L3[1]);
# equation (9) is implemented and simplified
eqt1:=[seq(L3[i]*L2[i]*diff(eq,L1[i]),i=1..n)]:
eqt2:=sum(eqt1[i],i=1..n);
eqt3:=subs(seq(L1[i]=L3[i],i=1..n),eqt2)=0;
simplify(%,power,symbolic):  collect(%,L2);
```

The *Maple* program code provides the final results line

$$k * kd1 + k * kd * (-s * z * k^{(-1+\alpha)} * \alpha - 1 + \delta) - z * zd * sk^{\alpha} = 0.$$

Converted back into the format used elsewhere in this paper, this result can be written as

$$k\widetilde{k}_{t+1} - k\widetilde{k}_t \left[\alpha szk^{\alpha-1} + (1-\delta)\right] - szk^{\alpha}\widetilde{z}_t = 0, \tag{16}$$

which is equal to equation (10) if variable n_t is set equal to unity for all t.

Simplifications from here on need to be done by hand. In this case, they would be similar to those of Example 5. They rely on the use of the steady state relationship. To provide some help with the derivations, the *Maple* program prints out the steady state relationship along with a simple transformation of it. These intermediate results typically help in simplifying the final equation of interest. In the case above, the *Maple* program reports as an intermediate result the following simplification of the steady state,

$$\delta k - szk^{\alpha} = 0.$$

This helps to simplify the coefficients of both $\widetilde{k}_t$ and $\widetilde{z}_t$ in equation (16),

$$k\widetilde{k}_{t+1} - k\widetilde{k}_t \left[\alpha\delta + (1-\delta)\right] - \delta k\widetilde{z}_t = 0.$$

Division by k generates the final equation

$$\widetilde{k}_{t+1} - \widetilde{k}_t\,[1 - \delta + \alpha\delta] - \delta\widetilde{z}_t = 0.$$

To provide some more hints on the use of the *Maple* program, consider how to change the four input lines for a couple of the other examples that are worked out in detail in the previous sections. Take, for example, the log-linear equation

$$\ln z_t = z_0 + \rho \ln z_{t-1} + \epsilon_t.$$

The four input lines for this case can be specified as

```
eq:=ln(zt)-z0-rho*ln(ztl); #define equation
L1:=[zt,ztl]:  #define names of variables
L2:=[zd,ztd]:  #define names of log-deviations
L3:=[z,z]:  #define names of steady state values
```

Note that the error term ϵ_t is not included in the *Maple* program. It needs to be added to the results equation that is output by the program. Also note that a steady state name needs to be provided for each variable name and that this name has to be the same for variables that only differ by their time subscript, such as z_t and z_{t-1} in the above case.

The input lines for the Euler equation,

$$\frac{1}{c_t} = \beta E_t \left[\frac{(1 + r_{t+1})}{c_{t+1}} \right],$$

can be written as

```
eq:=1/ct-beta*((1+rt1)/ct1); #define equation
L1:=[rt1,ct1,ct]:  #define names of variables
L2:=[rtd,ctd,cd]:  #define names of log-deviations
L3:=[r,c,c]:  #define names of steady state values
```

The expectations operator is left out. It needs to be added back into the results equation of the program. The *Maple* output for this case is given by

$$-\frac{\beta r}{c} rtd + \frac{\beta(1+r)}{c} ctd - \frac{1}{c} cd = 0.$$

In the notation of this paper the equation can be written as

$$-\frac{\beta r}{c}\widetilde{r}_{t+1} + \frac{\beta(1+r)}{c}\widetilde{c}_{t+1} - \frac{1}{c}\widetilde{c}_t = 0.$$

Adding the expectations operator to all $(t+1)$ terms and multiplying through by the constant term c yields

$$-\beta r E_t \widetilde{r}_{t+1} + \beta(1+r) E_t \widetilde{c}_{t+1} - \widetilde{c}_t = 0.$$

Replacing β by $1/(1+r)$ will make the equation equivalent to (15).