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Equilibria for circular spatial Cournot markets

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Abstract

This paper investigates a location−quantity model in a circular city. Pal (1998) investigates a duopoly model and finds that an equidistant location pattern appears in equilibrium. Matsushima (2001a) investigates an n−firm oligopoly model and shows that, if the number of firms is even, another equilibrium exists where half of the firms agglomerate at one point and the other firms agglomerate at the opposite point. We find that there exist many other equilibrium patterns that include the above two patterns as special cases.

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1. Introduction

Since the seminal work of Hotelling (1929), a rich and diverse literature on spatial competition has emerged. Location models fall into two categories: those in which firms bear the transport costs are shipping or spatial price discrimination models; those in which consumers pay for transport are shopping or mill pricing models. In each type, one can have either Bertrand-type price setting or Cournot-type quantity-setting.

Most papers on location theory use shopping (mill pricing) models with Bertrand competition. Although Cournot-type and Bertrand-type non-spatial models are equally popular, the body of literature on spatial competition that uses Cournot-type models is relatively small. Economists have recently considered shipping models with Cournot competition. Hamilton, Thisse, and Weskamp (1989) and Anderson and Neven (1991) carry out pioneering works on these models. They use linear-city models and show that all firms agglomerate at the central point. Pal (1998) shows that their result is crucially dependent on the assumption of the linear-city. He investigates a circular city duopoly model and finds that an equidistant location pattern appears in equilibrium, that is, locational dispersion appears.[1] Matsushima (2001a) extends Pal's model to an n-firm oligopoly and shows that another equilibrium exists where half of the firms locate at one point and the other half locate at the opposite side. However, Matsushima (2001a) does not discuss the possibility of other types of equilibria, and says nothing when the number of firms is odd. We take a close look at the shipping spatial Cournot model with circular market discussed by them. We find that many other location patterns appear in equilibria, including those of Pal (1998) and Matsushima (2001a) as special cases. We show that the location-quantity model with a circular city contains very rich locational implications.

The paper is organized as follows. Section 2 formulates the model. Section 3 presents equilibrium outcomes. Section 4 concludes the paper. All proofs of Lemmas and Propositions are presented in Appendix.

2. Model

The model is from Pal (1998) except for the number of the firms. We extend a duopoly model of Pal to an n-firm oligopoly model.

There is a circular market of length 1 where infinitely many consumers lie uniformly. The firms engage in the following location-quantity competition. In the first stage, each

[1] For other circular Cournot models, see, e.g., Shimizu (2002). For applications of this model, see, e.g., Matsushima (2001b).

firm simultaneously decides where on the perimeter to locate. After observing the rivals' locations, in the second stage each firm simultaneously chooses its output level at every point (market) in the continuum $[0,1]$ as to maximize its profit. Let x be a point on the circle located at a distance from 0, measured clockwise. Assume that the demand function at each market is linear, i.e., $p(x) = A - B\,Q(x)$, where A and B are positive constants and $p(x)$ and $Q(x)$ are the price and the total quantity supplied at market x, respectively. Each firm incurs a symmetric constant marginal cost of production, which we normalize to zero without loss of generality. The firms must pay transport costs. To ship a unit of the product from its plant x_i to a market at point x, firm i pays a transport cost $t|x - x_i|$, where t is a positive constant and $|x - x_i|$ is the distance between x and x_i. The norm signifies the shorter distance of the two possible ways to transfer the goods along the perimeter. The consumers are assumed to have a prohibitively costly transport cost, preventing arbitrage.[2] Finally, $A > nt$ is assumed in order to ensure that every firm serves the whole market. All the above assumptions are standard in the literature.

The equilibrium concept used is subgame perfect Nash equilibrium. Thus, we solve the game by backward induction. Because constant marginal cost of production is assumed, each local market can be analyzed independently. Thus the second-stage subgames and the local Cournot competition are examined first. Let the locations of the firms be denoted $x_1,\ x_2,\ \ldots\ ,\ x_n$ respectively, with $x_i \in [0,1]$, $i = \{1,\ \ldots\ ,\ n\}$. Also, let $q_i(x)$ be outputs for firm i at market x and $q_{-i}(x)$ be the total output for firms other than i. Under the above assumptions, at each point $x \in [0,1]$ firm i makes profit given by

$$\pi_i(q_i(x), q_{-i}(x), x) = [A - Bq_i(x) - \sum_{j \neq i} Bq_j(x) - t|x - x_i|]q_i(x).$$

Taking first order conditions to solve for the unique Cournot equilibrium yields,

$$q_i^*(x) = \frac{1}{B(n+1)}\Big[A + \sum_{j \neq i} t|x - x_j| - nt|x - x_i|\Big], \tag{1}$$

$$Q^*(x) = \frac{1}{B(n+1)}\Big[nA - \sum_{j=1}^{n} t|x - x_j|\Big], \tag{2}$$

and the profit function can be rewritten as

$$\pi_i^*(x) = \frac{1}{B(n+1)^2}[A + \sum_{j \neq i} t|x - x_j| - nt|x - x_i|\Big]^2 = Bq_i^*(x)^2. \tag{3}$$

[2] This assumption is not essential. Unless transport costs for consumers are strictly smaller than those of firms, consumer arbitrage plays no role in our model. For this discussion, see Hamilton, Thisse, and Weskamp (1989).

Each firm's total profit function Π_i is,

$$\Pi_i(x_i, x_{-i}) = \int_{x\in[0,1]} \pi_i^*(x; x_i, x_{-i})dx, \tag{4}$$

and each firm chooses its location to maximize it. Let X_i denote the set of markets such that the derivative of $|x - x_i|$ with respect to x_i is non-positive. If $x_i \in [0, 1/2]$, then $X_i = [x_i, x_i + 1/2]$. From (1), (3), and (4) we obtain

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial x_i} &= 2B\int_0^1 q_i^* \frac{\partial q_i^*}{\partial x_i} dx = \frac{2nt}{B(n+1)^2}\Big(\int_{x\in X_i} (A + \sum_{j\neq i} t|x - x_j| - nt|x - x_i|)dx \\
&- \int_{x\notin X_i} (A + \sum_{j\neq i} t|x - x_j| - nt|x - x_i|)dx\Big) \\
&= \frac{2nt^2}{B(n+1)^2}\Big(\int_{x\in X_i} \sum_{j\neq i} |x - x_j|dx - \int_{x\notin X_i} \sum_{j\neq i} |x - x_j|dx\Big),
\end{aligned} \tag{5}$$

where we use

$$\int_{x\in X_i} |x - x_i|dx = \int_{x\notin X_i} |x - x_i|dx = \frac{1}{8}.$$

3. Equilibrium analysis

We now look at the first stage using the solutions derived above. First, we show that the outcome where all firms agglomerate at one point is not an equilibrium.

Proposition 1: $x_1 = x_2 =, \ldots, = x_n$ *is not an equilibrium outcome.*

As is emphasized by Pal (1998), agglomeration at one point never appears in equilibrium in the circular-city model.

We then show that various types of equilibria exist in the game above. Before the actual analysis, let us introduce two related terms. The first is "opposite", which identifies a firm that are located exactly 1/2 away from the original firm. Hence, a firm is located as far away as possible from its opposite. The second is a "pair", which signify two firms that are opposites to each other. Therefore, the result in Pal (1998) shows that a pair is the unique equilibrium of the two firm game. Note that even if there are multiple firms located opposite to a firm, only one of them can be considered as a part of a pair at any one time. Therefore, if firm 1 locates at 0 and firms 2 and 3 locate at 1/2, even though both firms 2 and 3 can form a pair with firm 1, only one of them can be a part of a pair at a time.

Before discussing equilibria, we present one important lemma, from which two of our main results (Propositions 2 and 3) are derived straightforwardly.

Lemma 1: *Suppose that firm g and firm h constitute a pair (i.e., $|x_g - x_h| = 1/2$), then removing this pair will not alter the location incentives for the remaining firms, regarding where to locate, from that before the removal. Similarly, adding a pair will not affect the location incentives for the original firms.*

The following Propositions 2 and 3 are derived straightforwardly from Lemma 1.

Proposition 2: *If a situation is such that all firms can be paired at one time, then the situation is an equilibrium.*

Proposition 3: *Suppose that the number of firms is $2m+1$, where m is a positive integer. Then the situation (and others differing by symmetry) where $m+1$ firms locate at 0 and m firms locate at $1/2$ is an equilibrium.*

Note that the equilibrium location patterns described in Proposition 2 include both of the results of Pal (1998) and Matsushima (2001a) as special cases.

We now discuss another type of equilibria where agglomeration appears at several points. Proposition 2 implies that the following location pattern constitutes an equilibrium: There are even number of locations that splits the whole market into equidistant parts; at each location, some firms are located, where the number of firms at each location is equal. Then a natural question arises: is it true when the number of locations is odd? The answer is yes if and only if the number of the location is no larger than the number of firms at each location.

Proposition 4: *Let k be the number of potential locations, where the circular market is divided up equidistantly apart by the locations. Let m be the number of firms locating at each of these above locations. (i) Let k be even. Then this situation is an equilibrium for any $k \geq 2$ and $m \geq 1$; (ii) Let k be odd. Then this situation is an equilibrium if and only if $k \geq m$.*

Note that this result implies that, for any number of firms (even or odd), the situation where firms separate themselves equidistantly apart to the closest neighbors is an equilibrium outcome. Thus, the conjecture of the existence of this type of equilibrium in the n-firm model of Pal (1998) is correct. Propositions 2–4 also indicate that many other equilibrium location patterns exist.

We explain the intuition behind Proposition 4 (ii). Consider the situation where firms agglomerate at three points. The situation where m firms locate at 0, m firms locate at 1/3, and m firms locate at 2/3 constitutes an equilibrium only if $m \leq 3$. Suppose that initially firm 1 locates at 0, and it relocates from 0 to 1/2. The relocation increases the firm's distance from 0 by 1/2, and it mitigates competition between firms locating at 0. At the same time the relocation decreases the distance from 1/3 and 2/3 by 1/6, and it accelerates competition between firms locating at 1/3 and 2/3. After the relocation the number of firms locating at 0 is $m-1$ and the number of firms locating at 1/3 or 2/3 is $2m$. The former competition restricting effect becomes relatively strong when m is large. Thus, each firm has an incentive for relocating if m is large.

4. Concluding remarks

In this paper we reexamine the claim in Pal (1998) that the equilibrium outcome of the circular Cournot game is when the firms are located symmetrically apart (therefore no agglomeration). We show that such an equilibrium always exists regardless of the number of firms. We also find that many other equilibria exist. We show that three types of equilibrium structures exist, where any of them can have agglomeration at multiple locations.

Finally, we make a remark on the applicability of the shipping spatial model. The most natural interpretation of the model is that each firm chooses where it builds a plant in the model. There is another important interpretation. We can interpret "space" as product varieties. Each firm's location indicates the product or sector in which it has an advantage. Distant locations are the products the firm is in a disadvantage and to produce them it incurs additional costs. In short, the location choice corresponds to the technology choice and transportation costs correspond to the additional production costs. Hence, the shipping model is a suitable analytical tool for both spatial and non-spatial competitions.[3] Following this interpretation, our results indicate the existence of various equilibrium patterns of technological choice. In other words, our model can explain various partial 'herd behavior' of firms. It is never the case that all firms choose the technology, while it is possible that some of them choose the same one. Such partial 'herd behavior' is widely observed in many industries. Our model can explain such a situation without assuming any informational externality or network externality, which is assumed in standard models of herd behavior.

[3] See Eaton and Schmitt (1994), and Norman and Thisse (1999).

Appendix

Proof of Proposition 1: We prove it by contradiction. Suppose that all firms agglomerate at one point in an equilibrium. Without loss of generality we assume that all firms locate at zero. We now show that, if firm i deviates from the strategy above and chooses $x_i \in (0, 1/2)$, then its profit increases. Substituting $x_j = 0 \; \forall \; j \neq i$ into (5) we obtain that, for all $x_i \in (0, \frac{1}{2})$,

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{2nt^2}{B(n+1)^2}\Big(\int_{x\in X_i}\sum_{j\neq i}|x-0|dx-\int_{x\notin X_i}\sum_{j\neq i}|x-0|dx\Big) = \frac{2nt}{B(n+1)^2}(x_i-x_i^2) > 0. \quad (6)$$

(6) implies the deviation increases its profit, a contradiction. ∎

Proof of Lemma 1: Consider an m-firm game. From (5) we obtain

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{2mt^2}{B(m+1)^2}\Big(\int_{x\in X_i}\sum_{j\neq i}|x-x_j|dx-\int_{x\notin X_i}\sum_{j\neq i}|x-x_j|dx\Big). \quad (7)$$

We then add two firms (firm $m+1$ and firm $m+2$) to the m-firm game above. Suppose that $|x_{m+1} - x_{m+2}| = 1/2$ and $x_1, x_2, ..., x_m$ are the same as the m-firm game. From (5) we obtain

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{2(m+2)t^2}{B(m+3)^2}\Big(\int_{x\in X_i}\sum_{j\neq i}|x-x_j|dx-\int_{x\notin X_i}\sum_{j\neq i}|x-x_j|dx\Big). \quad (8)$$

Obviously the sign of (7) is equal to that of (8). It implies that adding a pair does not affect the derivative of each firm's profit with respect to its own location; thus adding a pair does not affect the optimal location for firm i ($i = 1, 2, ..., m$). A similar principle can be applied when considering removing a pair. ∎

Proof of Proposition 2: A firm looking to optimize faces one firm to its opposite and other firms being paired. However, using Lemma 1 repeatedly, the situation after removing all pairs must have the same solution as before as to whether it is an equilibrium or not. Thus the firm needs to worry only about itself and the opposite firm. As shown in Pal (1998), the unique best response of a firm in a two firm game is to locate at the opposite of the other firm. Therefore the original location is optimal for the firm. Since every firm in this situation can apply the same process, no firm has an incentive to deviate and this is an equilibrium. To reiterate, the "pairs" equilibria include as special cases situations with even number of firms locating equidistantly apart. ∎

Proof of Proposition 3: From Lemma 1, removing pairs has no effect on the location incentives for the remaining firms. Since removing all pairs create a situation where there is only one firm, this firm can locate anywhere. Thus the firm can choose to locate at 0, as every point is a best response. The same idea can be used for all $m+1$ firms located at 0 to justify their locating at 0.

As for the firms at 1/2, we consider a situation after removing all but one pair from the original situation. Thus there are two firms at 0, and one at 1/2. The best response for a firm given two firms at 0 is indeed to locate at 1/2. Thus the firms at 1/2 are willing to locate at 1/2 in the original situation, also. Therefore the given situation is an equilibrium. ∎

Proof of Proposition 4: Suppose that k is even. Then Proposition 2 implies Proposition 4 (i). We now suppose that k is odd.

Suppose that firms are located so that there are m firms each at $0, 1/k, 2/k, \dots,$ and $(k-1)/k$. Then we discuss whether or not $x_1 = 0$ is optimal for firm 1 given $x_2 =, ..., x_m = 0$, $x_{m+1} = x_{m+2} =, ..., = x_{2m} = 1/k$, ..., $x_{(k-1)m+1} =, ..., = x_n = (k-1)/k$. Without loss of generality we assume that $x_1 \le 1/2$. Substituting $x_1 = 0$ (i.e., substituting $X_i = [0, 1/2]$) into (5), we obtain that the derivative is zero. Thus the first order condition is satisfied when $x_1 = 0$. We then discuss the second order condition.[4] In (5), only X_i depends on x_i. Differentiating (5) with respect to x_i yields

$$\frac{\partial^2 \Pi_i}{\partial x_i^2} = \frac{4nt^2}{B(n+1)^2}\Big(\sum_{j \neq i} |x_i + \frac{1}{2} - x_j| - \sum_{j \neq i} |x_i - x_j|\Big). \tag{9}$$

We now substitute $i = 1$, $x_1 = x_2 =, ..., = x_m = 0$, $x_{m+1} = x_{m+2} =, ..., = x_{2m} = 1/k$, ..., and $x_{(k-1)m+1} =, ..., = x_n = (k-1)/k$ into (9). Since $|0 - h/k| = |0 - (k-h)/k| = h/k$ for $h \le (k-1)/2$, we have

$$\sum_{j \neq i} |0 - x_j| = \frac{2m}{k}(1 + 2 + 3 +, ..., + \frac{k-1}{2}).$$

Since

$$|\frac{1}{2} - \frac{h}{k}| = |\frac{1}{2} - \frac{k-h}{k}| = \frac{1}{2k} + (\frac{k-1}{2} - h)\frac{1}{k}$$

[4] We can prove Proposition 4 without using local maximization conditions (the first order and the second order conditions) by showing that given other locations of other firms the profit of firm 1 is non-increasing in x_1 for $x_1 \in [0, 1/2]$ and strictly decreasing in x_1 for $x_1 \in (0, 1/2)$. We avoid this alternative proof because it requires tedious calculations and takes much space. This proof is available from the authors upon request.

and $|1/2 - 0| = 1/2$, we have

$$\sum_{j \neq i} |\frac{1}{2} - x_j| = \frac{2m}{k}\Big(\frac{1}{2} + \frac{3}{2} +, ..., + \frac{k-2}{2}\Big) + \frac{m-1}{2}.$$

Substituting these equations into (9), we have that the second order condition is satisfied if and only if

$$-\frac{2m}{k}\frac{k-1}{4} + \frac{m-1}{2} \leq 0 \Longleftrightarrow \frac{m}{k} \geq \frac{m-1}{k-1},$$

and it is satisfied if and only if $k \geq m$. ∎

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