E C O N O M I C S   B U L L E T I N

He said that he said that I am a J

Nicolas Houy
EUREQua

Abstract

We axiomatize the collective identity function selecting the agents that are indirectly designated by all the individuals in the society.

Citation: Houy, Nicolas, (2006) "He said that he said that I am a J." *Economics Bulletin*, Vol. 4, No. 4 pp. 1−6
Submitted: November 6, 2005. **Accepted:** January 17, 2006.
URL: http://www.economicsbulletin.com/2006/volume4/EB−05D70031A.pdf

1 Introduction

The first paper dealing with group identification was Kasher and Rubinstein (1997).[1] In a remark, the following method, axiomatized by Dimitrov et al. (2003), is suggested. In a first step, the persons who are designated by all the individuals are members of the group (the "J"s[2]). In the next step, those that are designated by a J become Js themselves and we reiterate this latter step. When the set of Js cannot grow anymore, we have the final identity of the Js. Said differently, those and only those that are indirectly designated (i.e. through a sequence of designations) by an unanimous J are Js. One of the interesting features of this method is that all the Js are indirectly designated by all the others (including non-Js) and themselves as deserving to be a J. However, it is not always the case that all those that are indirectly designated as Js by all the others and themselves are Js.

In Houy (2005), we have axiomatized the Collective Identification Function allowing the people with the greatest number of indirect designations to be identified as Js.[3] In this article we axiomatize the Collective Identification Function which allows the individuals that are indirectly designated by all the individuals (including themselves) to be Js. We will call this function the Indirect Consensus Function.

2 Framework and Theorem

Let $N = \{1, ..., n\}$ be the set of individuals. Let 2^N be the set of all subsets of N. A designation vector for i, $G_i \in 2^N$ is the subset of individuals that i designates as deserving to be accepted as group members (a group member will be called a J). A profile $G = (G_1, ..., G_n) \in (2^N)^n$ is a n-tuple of designation vectors, one for each individual in the society. $\forall G =$

[1]See the correction by Dimitrov and Sung (2005).

[2]We keep the letter J for members of the group to stick to the notations used in the literature but there is no judgement behind it.

[3]This axiomatization was more appropriate in a voting framework.

$(G_1, ..., G_n) \in (2^N)^n, \forall g \in 2^N$ and $\forall i \in N$ we define $G + (g, i) = (G'_1, ..., G'_n)$ as $\forall j \neq i, G'_j = G_j$ and $G'_i = g$. Then, $G + (g, i)$ is the same profile as G for all individuals but i who changes his designation for $g \in 2^N$.

A collective identity function (CIF) is a function that assigns a designation vector to any profile, i.e. f is a CIF if and only if $f : (2^N)^n \rightarrow 2^N$. We will say that a CIF f is more welcoming than a CIF g if and only if $\forall G \in (2^N)^n, g(G) \subseteq f(G)$. Let $\mathcal{F}$ be a set of CIFs and let $f \in \mathcal{F}$, we will say that f is the most welcoming CIF of $\mathcal{F}$ if and only if it is more welcoming than any other CIF in $\mathcal{F}$.

To designate what is the indirect consensus function (ICF), we need a few definitions. Let $G = (G_1, ..., G_n) \in (2^N)^n$. We will say that i_0 indirectly designates j, ($j \in ID(i_0; G)$) if and only if there exists $r \in \mathbb{N}$ and a sequence $(i_k)_{k=1}^r \in N^r$, such that $\forall k \in [\![0; r-1]\!], i_{k+1} \in G_{i_k}$ and $j = i_r$. The indirect consensus set ($IC(G)$) is defined as the set of those that are unanimously indirectly designated, i.e. $IC(G) = \{i / \{j / i \in ID(j; G)\} = N\}$. Now we can define ICF as the CIF always designating the indirect consensus set as Js.

Definition 1 (Indirect Consensus Function (ICF))
A CIF f is ICF if and only if $\forall G \in (2^N)^n$, $f(G) = IC(G)$.

We can now give our axioms.

Axiom 2 (Cooptation, Coop)
A CIF f satisfies Cooptation if and only if $\forall G = (G_1, ..., G_n) \in (2^N)^n$, $\forall i \in N$, $i \in f(G) \Leftrightarrow \exists j \in N$ such that $j \in f(G)$ and $i \in G_j$.

Axiom 3 (Robustness of the Js, RJ)
A CIF f satisfies Robustness of the Js if and only if $\forall G = (G_1, ..., G_n) \in (2^N)^n$, $\forall i \in N$, $\forall g \in 2^N \setminus \emptyset$, $i \in f(G) \Rightarrow i \in f(G + (g, i))$.

Axiom 4 (Non Triviality, NT)
A CIF f satisfies Non Triviality if and only if if there exists $G \in (2^N)^n$ such that $f(G) \neq \emptyset$.

The first axiom, Coop, requires that all Js must be themselves coopted by a J and conversely, the number of designations an individual gets is not big enough for him to be a J as long as he is not coopted.[4] The second axiom, RJ, requires that if a J changes his vote without abstaining, then he is still a J. The NT axioms requires that there can be Js. We can now show that the ICF is the most welcoming CIF respecting Coop and RJ and that the ICF is equivalent to the three axioms given above.[5]

Proposition 5

ICF is the most welcoming CIF that satisfies Coop and RJ.

Theorem 6

A CIF satisfies Coop, RJ, and NT if and only if it is ICF. Moreover, Coop, RJ and NT are independent.

3 Proofs

3.1 Proof of proposition 5

Lemma 7

Let f satisfy Coop and RJ. Then, $i \notin ID(i, G^0) \Rightarrow i \notin f(G^0)$.

Proof. Imagine G^0 such that $i \notin ID(i, G^0)$ and $i \in f(G^0)$.

Algorithmic proof: Set k=0. 1) $J(G^k) = \{j \in N \setminus \{i\} / i \in ID(j, G^k), j \in f(G^k), \{i\} \neq G_j^k\}$. 2) Consider $j_k \in J(G^k)$ if $J(G^k) \neq \emptyset$, else set K=k and stop. 3) $G^{k+1} = G^k + (\{i\}, j_k)$. 4) Set $k = k + 1$ and go to step 1.

$i \in f(G^0)$ by assumption. In step 3, by RJ, $j_k \in f(G^{k+1})$ and by Coop, $i \in f(G^k) \Rightarrow i \in f(G^{k+1})$. By definition of $J(G^k)$, we have $\forall k, \mid J(G^k) \mid \leq n - k$ since is is straightforward to see that $\forall k' > k, j_k \notin J(G^{k'})$, then this algorithm has an end. At the end of it, we have, for all j, $j \notin f(G^K)$ or

[4] The French Academy is an instance of institution whose members are decided through cooptation.

[5] Notice that as a corollary, we can claim that a CIF satisfying Coop and RJ is either always empty or the ICF.

$G_j^K = \{i\}$ or $i \notin ID(j, G^K)$ and we have $i \in f(G^K)$ which is a contradiction with Coop.□

Lemma 8

If a CIF f satisfies Coop and RJ then $\forall G \in (2^N)^n, f(G) \in \{\emptyset, IC(G)\}$.

Proof. Let us have $k \in IC(G)$, $k \notin f(G)$ and $k' \in f(G)$. By definition of $IC(G)$, $k \in ID(k', G)$. Then, by definition, $\exists$ a sequence $i_0, i_1, ..., i_l$ (possibly empty in which case $k \in G_{k'}$) such that $i_0 \in G_{k'}$, $i_1 \in G_{i_0}$, ..., $i_l \in G_{i_{l-1}}$, $k \in G_{i_l}$. Then, by Coop, $k' \in f(G)$ implies that $i_0 \in f(G)$. Again, by Coop, $i_0 \in f(G)$ implies $i_1 \in f(G)$ and implementing, $i_l \in f(G)$ implies $k \in f(G)$ which is a contradiction.

Let us have $k \notin IC(G)$ and $k \in f(G)$. Then, necessarily we have $\exists k' \in N \setminus \{k\}$ s.t. $k \notin ID(k', G)$ (by lemma 7, $k \in f(G) \Rightarrow k \in ID(k, G)$). Define $G' = G + (\{k'\}, k)$. We have $k \notin ID(k, G')$. By lemma 7, $k \notin f(G')$ which contradicts RJ. □

Following lemma 8, to prove proposition 5, it is enough to check that ICF satisfies Coop and RJ.

3.2 Proof of theorem 6

Lemma 9

Let f satisfy Coop and RJ. $IC(G) = IC(G') \neq \emptyset$, $f(G) = IC(G)$, $k \notin IC(G)$ and $G' = G + (A, k)$ with $A \neq \emptyset$ imply that $f(G') = f(G)$.

Proof. Let us have $i \in IC(G)$. Let $G'' = G + (\{k\}, i)$. By RJ, $i \in f(G'')$ and by Coop, $k \in f(G'')$. Let $G^3 = G'' + (A, k)$. By RJ, $k \in f(G^3)$ and by definition $i \in IC(G^3)$, hence by Coop, $i \in f(G^3)$. Let $G^4 = G^3 + (G_i, i)$. By definition, $G^4 = G'$. By RJ, $i \in f(G')$ and by lemma 8, $f(G) = f(G')$. □

Imagine G such that $f(G) \neq \emptyset$. By NT, this exists. Let $i \in f(G)$ and let $G' = G + (N, i) + (N, 1) + ... + (N, n)$. Then, by definition $\forall j \in N, G'_j = N$. By Coop and RJ implemented at each stage of the construction of G', $f(G') = N$.

Now, let us consider $G'' = \{G''_1, ..., G''_n\}$ such that $IC(G'') = \{i_1, ..., i_k\} \neq \emptyset$. Then $\forall j \in \{1, ..., k-1\}$, $\bigcup_{r \in \{1,...j\}} G''_r \nsubseteq \{i_1, ..., i_r\}$ (or else, i_k would not be in $IC(G'')$). Let us set $G^3 = G + (G''_{i_1}, i_1) + ... + (G''_{i_k}, i_k)$. By Coop and RJ implemented at each stage of the construction of G^3, $f(G^3) = IC(G'')$. Then, by lemma 9, $f(G'') = IC(G'') \neq \emptyset$. This with lemma 8 proves the theorem.

Independence: Let f_{-NT} be defined as $\forall G \in (2^N)^n$, $f_{-NT}(G) = \emptyset$. f_{-NT} satisfies Coop and RJ but not NT. Let $f_{-Coop\Rightarrow}$ be defined as $\forall G \in (2^N)^n$, $f_{-Coop\Rightarrow}(G) = N$. $f_{-Coop\Rightarrow}$ satisfies NT, RJ and the "$\Leftarrow$" part of Coop but not the "$\Rightarrow$" part of Coop. Let $f_{-Coop\Leftarrow}$ be defined as $f_{-Coop\Leftarrow}(G) = \{1\}$ if $1 \in IC(G)$ and $1 \in G_1$, $f_{-Coop\Leftarrow}(G) = \emptyset$ if $1 \notin IC(G)$ and $1 \in G_1$, $f_{-Coop\Leftarrow}(G) = IC(G)$ otherwise. $f_{-Coop\Rightarrow}$ satisfies NT, RJ and the "$\Rightarrow$" part of Coop but not the "$\Leftarrow$" part of Coop. Let f_{-RJ} be defined as $\forall G \in (2^N)^n, f_{-RJ} = \{i \in /i \in ID(i, G)\}$. f_{-RJ} satisfies Coop and NT but not RJ.

References

Dimitrov, D., S.-C. Sung, and Y. Xu (2003) "Procedural Group Identification" CentER Discussion Paper, 2003-10, Tilburg University.

Dimitrov D., and S.-C. Sung (2005) "On the axiomatic characterization of "Who is a J?"" *Logique Analyse*, forthcoming.

Houy N. (2005) "De la Stabilité des Procédures et Constitutions en Théorie du Choix Social" Ph.D. Thesis, University Paris I.

Kasher A., and A. Rubinstein (1997) "On the Question "Who is a J?" A Social Choice Approach" *Logique Analyse* **160**, 385-395.