E C O N O M I C S   B U L L E T I N

Quasi−linear peferences with Auspitz−Lieben−Pareto complementarity

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Abstract

I show that if preferences are quasi−linear (non−linear in goods x1, …, xn but linear in xn+1) and the sub−utility function defined over [x1, …, xn] is strongly concave and exhibits Auspitz−Lieben−Pareto complementarity, then goods x1−xn must be gross and compensated complements for each other and xn+1 must be a compensated substitute for all other goods. Also, an increase in its price of xn+1 must reduce the demand for goods x1−xn. The effects of uncompensated changes in the prices of goods x1−xn on the demand for good xn+1 vary predictably with income.

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1. Introduction

The general quasi-linear utility function,

$$v(x) = u(x_1, x_2, ..., x_n) + x_{n+1}, \tag{1}$$

has found wide application in economic theory, especially in cases where analytical tractability requires eliminating income effects from the analysis at hand. While the best known applications of quasi-linear preferences no doubt include consumer surplus analysis and analysis of economies with public goods or externalities (see, e.g., the textbook treatment in Varian, 1992), quasi-linear preferences have also proven useful in analysis of optimal taxation (Weymark, 1987), rational expectations equilibria (Shi, 1988a), equilibria in insurance markets (Shi, 1988b), Nash equilibrium (Boylan, 1998), imperfect competition (Vives, 1999), and the construction of optimal sales mechanisms (Che and Gale, 2000), among others.

Despite the widespread application of quasi-linear preferences, however, relatively little seems to be known regarding the comparative statics properties of demand functions derived from quasi-linear preferences, aside from the implications that the matrix of compensated cross price effects must be symmetric and negative semidefinite (which of course applies to demand functions derived from any maximizing any quasiconcave utility function subject to a linear budget constraint), that x_{n+1} is zero below a certain threshold income level, but is consumed in positive quantities and is necessarily normal above that level, and that the demand functions for x_1 ... x_n must be completely income inelastic above the threshold level of income at which x_{n+1} is demanded in a positive quantity. For a discussion of our ignorance beyond these propositions, see, e.g., the apparent uncertainty concerning cross price effects for quasi-linear utility functions in Vives (1999, p. 145). This note fills in part of this gap in the existing literature by deriving the cross price effects among all n+1 goods for quasi-linear utility functions for the special case where the sub-utility function, u( ), is twice continuously differentiable and strongly concave with all goods x_1 ... x_n exhibiting weak Auspitz-Lieben (1889)-Pareto (1909) complementarity, so that $u_{ij}(x) \geq 0$ for $i \neq j$.

Previous analyses of general (i.e., non quasi-linear) strongly concave utility functions with Auspitz-Lieben-Pareto complementarity include Chipman (1977),[1] who showed that if the utility function is strongly concave (which implies, among other restrictions, that all goods must have strictly diminishing marginal utility), so that its Hessian matrix is negative definite, then all goods will be normal and thus have downward sloping demand curves if all goods are weak Auspitz-Lieben-Pareto complements. More recently, Weber (2000, 2004) has shown that if u(x) is strongly concave and all goods are Auspitz-Lieben-Pareto complements, then: 1) all goods must be compensated (utility constant) substitutes; and 2) if an increase in p_j increases the marginal utility of income, then all goods must be gross complements for good j in the sense that an uncompensated increase in the price of good j increases the demands for all other goods.[2]

Compared to the utility function considered by Chipman (1977) and Weber (2000, 2004), the functional form in (1) drops the assumption of strong concavity of v(x) since $v_{n+1,n+1} = 0$, replacing it with weak concavity. The functional form in (1) also replaces $v_{i,n+1}(x) \geq 0$ with the stronger condition $v_{i,n+1}(x) = 0$. It is well known that if the demand for x_{n+1} is positive, then the demands for goods x_1 ... x_n are completely income inelastic, which shows, not surprisingly that Chipman's (1977) result that for strongly concave utility with Auspitz-Lieben-Pareto complementarity, all goods are normal does not extend to all non-strictly concave utility functions. The purpose of this note is to determine how sensitive the empirical implications derived in Weber (2000, 2004) are to the same changes in the structure of the household's preferences. We will see below that these apparently minor changes in the household's preference structure also lead to results for cross price effects which are quite different from those developed in Weber (2000, 2004).

2. Comparative Statics for Quasi-Linear Utility with Auspitz-Lieben Complementarity

I begin by examining the signs of the cross price effects for all goods other than good n+1. Let $p = [p_1, p_2, ..., p_{n+1}]$ and I denote, respectively, the vector of prices which the household faces and household income. Let Ω_I denote the set of price-income pairs with $p_i > 0$, $i \in \{1, ..., n+1\}$ and $I > 0$. The household's Marshallian or uncompensated demand function will be denoted h: $\Omega_I \rightarrow \Re_+^{n+1}$. The range of h will be denoted $X = \{x: x = h(p, I) \text{ for some } (p, I) \in \Omega_I\}$. To begin, we have:

THEOREM: Assume that the household's preferences generate a single-valued, differentiable demand function, h(p, I), defined on Ω_I and satisfying the budget identity:

$$p \cdot h(p, I) = I \quad \text{for all } (p, I) \in \Omega_I. \tag{2}$$

with $h_{n+1} > 0$. Let Ω_I^0 be a neighborhood whose image $X^0 = h(\Omega_I^0)$ is in the interior of $\Re_+^{n+1}$. Assume that the household's preferences over X^0 can be represented by a real valued, twice continuously differentiable function v: $X^0 \rightarrow \Re^1$, which satisfies:

(i) *quasi-linearity*: $v(x) = u(x_1, x_2, ..., x_n) + x_{n+1}$;

(ii) *strong concavity of u*: for all i and $j \in \{1, \ldots, n\}$, the matrix $U(x) \equiv [u_{ij}(x)] \equiv [\partial^2 u(x)/\partial x_i \partial x_j]$ is negative definite;

(iii) *weak Auspitz-Lieben-Pareto complementarity*: for all i and $j \in \{1, \ldots, n\}$ and $i \neq j$, $u_{ij}(x) \geq 0$.

Then $\partial h_i(p, I)/\partial p_j \leq 0$ for all i and $j \in \{1, \ldots, n\}$, $i \neq j$ and $(p, I) \in \Omega_I^0$.

PROOF: For $h_{n+1} > 0$, it is well established that assumption (i) implies that the marginal utility of income depends only on p_{n+1} and that changes in I do not affect x_i for $i \in \{1, \ldots, n\}$. Together, these two facts imply that for $i \in \{1, \ldots, n\}$, the h_i's jointly solve:[3]

$$\max_{\{x_i\}_{i=1}^n} \quad u(x_1, \ x_2, \ \ldots, \ x_n) - \sum_{i=1}^n p_i x_i. \tag{3}$$

Under the conditions stated in the theorem, the first order conditions for solving (3) are given by:

$$u_k(h(p, \ I)) = p_k, \qquad (p, \ I) \in \Omega_I^0, \qquad k \in \{1, \ 2, \ \ldots \ n\}. \tag{4}$$

Differentiating each of the n different versions of (4) and using a mathematical result due to McKenzie (1960),[4] Rader (1968) has shown that conditions (i)-(iii) imply $\partial h_i(p, I)/\partial p_j \leq 0$,[5] which was the result to be shown.

This theorem immediately implies the following corollaries:

COROLLARY 1: Since assumption (i) of theorem 1 implies that income effects are zero for goods 1-n, the Slutsky equation implies that goods 1-n must be compensated complements for each other.

COROLLARY 2: Since each good must have at least one compensated substitute (Hicks, 1946), corollary 1 implies that good n+1 must be a compensated substitute for each of goods 1-n.

Corollaries 1 and 2 imply that the sign pattern for the symmetric $(n+1)^2$ matrix of compensated own and cross price effects is:

$$\begin{bmatrix} - & - & \cdots & - & + \\ - & - & \cdots & - & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & - & \cdots & - & + \\ + & + & \cdots & + & - \end{bmatrix}.$$

It bears emphasizing that the functional form in (1) is not the only one which results in this sign pattern. It has been known since Slutsky (1915) that when v(x) is additively separable, at most one good can have increasing marginal utility if the second order conditions for a constrained maximum are to be satisfied. Hicks and Allen (1934) showed that in the case of three goods the two goods with diminishing marginal utility must be compensated complements for each other and compensated substitutes for the good with increasing marginal utility. Silberberg (1972) generalized this result to the case of an arbitrary number of goods: the one good with increasing marginal utility will be a compensated substitute for all of the other goods, while the remaining goods will be compensated complements for each other.

To shed further light on the uncompensated cross price effects, we use the Slutsky equation explicitly. Thus, assume that the household chooses x to minimize its money expenditure, $E = p \cdot x$, subject to the constant utility constraint, $v(x) = v^0$. Let Ω_v denote the set of price-utility pairs such that $p_i > 0$, $i \in \{1, 2, \ldots n+1\}$ and v^0 is finite. The household's Hicksian or compensated demand function is denoted s: $\Omega_v \rightarrow \Re_+^{n+1}$. The range of s is $X = \{x\colon\ x = s(p, v^0) \text{ for some } (p, v^0) \in \Omega_v\}$. Thus, the Slutsky equation for the effect of a change in p_{n+1} is:

$$\frac{\partial h_i}{\partial p_{n+1}} = \frac{\partial s_i}{\partial p_{n+1}} - h_{n+1}\frac{\partial h_i}{\partial I}. \qquad (5)$$

Since, $\partial h_i/\partial I = 0$ for $i \in \{1, 2, \ldots n\}$, equation (5) and corollary 2 jointly yield:

COROLLARY 3: For $i \in \{1, 2, \ldots n\}$, goods i and n+1 are gross substitutes in the sense that

$$\frac{\partial h_i}{\partial p_{n+1}} = \frac{\partial s_i}{\partial p_{n+1}} \geq 0. \qquad (6)$$

Finally, the Slutsky equation for the effect of a change in p_j on the demand for x_{n+1} is:

$$\frac{\partial h_{n+1}}{\partial p_j} = \frac{\partial s_{n+1}}{\partial p_j} - h_j \frac{\partial h_{n+1}}{\partial I}. \qquad (7)$$

Since it is easily shown that with p_{n+1} held constant $\partial h_{n+1}/\partial I = 1/p_{n+1}$, multiplying (7) through by p_j/h_{n+1} to obtain the Slutsky equation in elasticity form yields:

$$\varepsilon^h_{n+1,j} = \varepsilon^s_{n+1,j} - \frac{p_j h_j}{p_{n+1} h_{n+1}}. \qquad (8)$$

From corollary 2, $\varepsilon^s_{n+1,j}$ is positive. Thus, the sign of $\varepsilon^h_{n+1,j}$ depends on the relative magnitudes of $\varepsilon^s_{n+1,j}$ and $p_j h_j/p_{n+1}h_{n+1}$. It is well known that quasi-linear preferences imply a threshold level of income below which $h_{n+1} = 0$ and above which any changes in income affect only h_{n+1}. Thus, if $\varepsilon^s_{n+1,j}$ is bounded above as income changes, equation (8) implies that good n+1 will be a gross complement for good n+1 only for sufficiently low income levels where $p_j h_j/p_{n+1}h_{n+1}$ is relatively large. Above the threshold, as income and hence h_{n+1} rises while p_j, h_j, and p_{n+1} all remain constant, $p_j h_j/p_{n+1}h_{n+1}$ falls monotonically, so that with $\varepsilon^s_{n+1,j}$ bounded above, the expression on the righthand side of (8) turns positive and goods j and n+1 become gross substitutes. This establishes:

LEMMA: Assume that conditions (i)-(iii) of the theorem hold and in addition that $\varepsilon^s_{n+1,j}$ is bounded above for all $(p, I) \in \Omega_I^0$. Then there exists I* such that $I < I^*$ implies $\varepsilon^h_{n+1,j} < 0$, while $I > I^*$ implies $\varepsilon^h_{n+1,j} > 0$.

Recall that all of the uncompensated own price effects are non-positive since no good is inferior. Since the $(n+1)^2$ matrix of uncompensated cross price effects is generally asymmetric, we define this matrix such that the entry in row i column j gives the effect of a change in p_j on h_i. Assuming that the conditions stated in the theorem hold and that $\varepsilon^s_{n+1,j}$ is bounded above as income changes, we have established that for $I < I^*$, the sign pattern of the matrix is:

$$\begin{bmatrix} - & - & \cdots & - & + \\ - & - & \cdots & - & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & - & \cdots & - & + \\ - & - & \cdots & - & - \end{bmatrix},$$

but that for $I > I^*$, the sign pattern is the same as that of the matrix of compensated price effects, viz:

$$\begin{bmatrix} - & - & \cdots & - & + \\ - & - & \cdots & - & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & - & \cdots & - & + \\ + & + & \cdots & + & - \end{bmatrix}.$$

3. Conclusion

Chipman (1977) and Weber (2000, 2004) have shown that in the general (non quasi-linear) case, strong concavity and Auspitz-Lieben-Pareto complementarity (which includes additive separability and diminishing marginal utility for all goods as a special case) imply that:

1) all goods must be normal and hence have downward sloping demand curves;
2) all goods must be compensated substitutes;
3) if an increase in p_j increases the marginal utility of income, then all goods must be gross complements for good j.

It is well known that the first of these results no longer holds when the overall utility function, v(x) is quasi-linear. This note has shown specifically how the second and third results also change when the overall utility function is quasi-linear and the sub-utility function is strongly concave and exhibits Auspitz-Lieben-Pareto complementarity. Specifically, these restrictions imply that the goods in the sub-utility function, $u(x_1, x_2, ..., x_n)$, must be both gross and compensated complements for each other; also, for sufficiently small income, $\partial h_{n+1}(p, I)/\partial p_j$ has a different sign than $\partial h_i(p, I)/\partial p_j$, $i \in \{1, 2, ... n\}$.

One of the results developed here merits further attention. Specifically, since quasi-linear preferences imply zero income elasticities for goods $x_1, ..., x_n$, the fact that those goods must be gross and compensated complements suggests a possible connection between inferiority (in the weak sense of a non-positive income elasticity) and complementarity. Since this is not the first paper to imply such a connection (see, e.g., the related analyses of Liebhafsky (1969) and Silberberg (1972), who jointly demonstrate such a connection under additive separability of the utility function), a further analysis of the apparent connection between complementarity and inferiority might prove fruitful. In particular, it would be interesting to know at a deeper level why weak inferiority and complementarity seem to be linked together.

NOTES

1. Following standard practice, Chipman (1977) refers to Auspitz-Lieben-Pareto complementarity as Auspitz-Lieben-Edgeworth (1897)-Pareto complementarity. However, a closer examination of Edgeworth's discussion in his paper on monopoly (Edgeworth, 1897) shows that in fact, Edgeworth's approach to complementarity was much more modern than that of his contemporaries, in that it went beyond a classification based on the signs of the second order cross partial derivatives of the utility function and focused instead on the signs of the "cross quantity effects" in the inverse demand functions. See Weber (2003) for a more detailed discussion.

2. Defining gross substitutability and complementarity by the impact of a change in the price of good j on the demand for all other goods rather than by the impacts of changes in the prices of other goods on the demand for good j is important since, unlike compensated cross price effects, gross cross price effects are generally not symmetric and thus may have different signs, so that the definition of two goods as gross complements or gross substitutes may depend on which price is assumed to change.

3. Note that we could also derive (3) by letting x_{n+1} be the *numéraire*, solving the budget constraint for x_{n+1}, substituting the result into (1) to yield $v(x) = u(x_1, x_2, ..., x_n) + I - \Sigma^n_{i=1} p_i x_i$, and observing that with I fixed choosing $x_1, ..., x_n$ to maximize this version of v(x) is equivalent to choosing $x_1, ..., x_n$ to maximize the function in (3).

4. For other, related applications of McKenzie's result, see Chipman (1977) and Weber (2000, 2004).

5. Technically, Rader interpreted (3) as a profit maximization problem in which $u(x_1, ..., x_n)$ is a production function and the x_i's and p_i's are productive inputs and real factor prices, respectively. However, the mathematical structure of Rader's problem and of the problem considered here are identical so that they do not depend on the particular economic interpretation attached to u( ), x_i, or p_i. Hence, the comparative statics of the model do not depend on the economic meaning of these symbols. Thus, the result $\partial h_i(p, I)/\partial p_j \leq 0$ applies equally well to Rader's profit maximization problem and to the utility maximization problem with quasi-linear preferences considered here.

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