E c o n o m i c s   B u l l e t i n

Equal probability for the best and the assignment of identical indivisible objects

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Abstract

We consider the problem of allocating several units of an indivisible object among agents with single-peaked utility functions. We introduce an axiom called equal probability for the best, and show that it is equivalent to both equal treatment of equals and symmetry in the presence of Pareto optimality. Moreover, we also show that the randomized uniform rule is the only randomized rule satisfying strategy-proofness, Pareto optimality, and equal probability for the best.

1 Introduction

We consider the situation where the manager of a firm allocates several identical recruited workers to some departments such as sales, production, and so on. Each department has an optimal number of workers, because if the department has fewer workers than the number, it cannot accomplish its task, and if it has more, it needs extra equipment and incurs additional training costs.

Since the optimal number of each department is unknown to the manager, he/she has to make each department report its optimal number in order to allocate the workers among them. On this occasion, the departments may try to misrepresent their own optimal numbers: each department would report the number which is more or less than his/her optimal number if such misrepresentation leads to the allocation favorable to the department. Hence, in order to attain a desirable allocation, the manager has to induce the departments to reveal their true optimal numbers.

The above mentioned problem faced by the manager is an example of the problem of allocating several units of an indivisible and identical object among a finite group of agents with single-peaked, risk-averse, and von-Neumann and Morgenstern utility functions. In this paper, we analyze the problem axiomatically, which was initiated by Sasaki (1997).

One of the difficulties we face when we allocate the indivisible objects is to obtain a fair allocation. For example, suppose that the manager has to allocate three recruited workers to two departments, A and B. Further, we assume that both departments have the same utility function and want more than three workers, that is, their peaks are three. In the deterministic environment the manager will allocate more workers to one department than the other due to indivisibility. However, such allocation is not fair for the departments. In order to deal with the difficulty due to indivisibility, not deterministic rules but randomized rules, which associate with each profile of the utility functions a probability distribution on feasible allocations, are considered. Randomization enables us to judge whether the allocation is fair in the ex-ante sense. Ehlers and Klaus (2003) used the randomized version of the axioms relating fariness such as symmetry, equal treatment of equals, and no-envy in evaluating allocation rules.

In this paper, we introduce a new axiom relating to fairness, called *equal probability for the best*. Given an agent and a probability distribution which the randomized rule outputs, let us call the number of the object which gives him/her the highest utility with a strictly positive probability the best for the agent. Then, equal probability for the best requires the randomized rules to output the probability distribution which has the following marginal dis-

tribution: if two agents have the same utility function, then they should get *the same best* with *the same probability*. In the same example in the previous paragraph, let us consider the case in which a randomized rule outputs a probability distribution which gives department A the marginal distribution that places probability zero on getting three workers, and probability 1/3 on getting two, one, and no worker(s) respectively. We can say that getting two workers is the best for department A in this case. Then, this axiom requires that the probability distribution should give department B the marginal distribution where getting two workers is the best for it and is placed probability 1/3 on. However, the probability placed on getting one worker and that on getting no worker do not need to be equal between the departments.

The contributions of this paper are as follows. First, our paper shows that in the presence of Pareto optimality, equal probability for the best, equal treatment of equals, and symmetry are equivalent. Equal treatment of equals says that if two departments' utility functions are same, their expected utilities should also be same. Symmetry says that if two departments' utility functions are same, their marginal distributions should be same. Note that equal probability for the best is logically independent to equal treatment of equals. Equal probability for the best is the first axiom that is indigenous to indivisibility or randomization. The axioms used in previous papers on probabilistic allocation of the indivisible objects, such as equal treatment of equals, symmetry, and so on, are just the randomized version of those used in the deterministic environment. However, equal probability for the best has no counterpart in the deterministic environment. In addition, equal probability for the best is a weak axiom in the sense that it requires that only a part of the marginal distribution should be equal between the departments with the same utility function, whereas some other axioms require the whole part should be.

Second, using the result of Ehlers and Klaus' (2003), we can characterize the randomized uniform rule, which is a well-known randomized rule, by strategy-proofness, Pareto optimality, and equal probability for the best. Our result is an alternative characterization of the randomized uniform rule. An advantage of the result is using the axiom that is indigenous to randomization in order to characterize the randomized version of the uniform rule.

This paper is organized as follows: Section 2 describes the model. Several axioms are presented in Section 3. Our theorems are presented in Section 4.

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2   The Model

We consider the problem of allocating several units $X \in \mathbb{Z}_{++}$ of *an indivisible object* among a finite group of agents, $N = \{1, 2, \ldots, n\}$. The objects are identical. We denote the set of the indivisible objects by $C = \{0, 1, 2, \ldots, X\}$.

Each agent $i \in N$ has a von-Neumann and Morgenstern utility function $u_i : C \to \mathbb{R}$ which is *single-peaked* and *risk-averse*. u_i is *single-peaked* if there is *a unique peak* $p(u_i) \in C$ such that for all $k, k' \in C$ with $p(u_i) \leq k < k'$ or $k' < k \leq p(u_i)$, $u_i(p(u_i)) \geq u_i(k) > u_i(k')$. u_i is *risk-averse* if for any $k \in C \backslash \{0, X\}$, $u_i(k) - u_i(k-1) > u_i(k+1) - u_i(k)$. We assume that the utility functions are only privately known. Let U be the set of all single-peaked, risk-averse, and von-Neumann and Morgenstern utility functions. $u = (u_1, u_2, \ldots, u_n) \in U^n$ is called a profile of the utility functions. u_{-i} stands for $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n) \in U^{n-1}$.

(N, u, X) is called a problem. We say that the problem is in excess demand if $\sum_{i \in N} p(u_i) \geq X$. If $\sum_{i \in N} p(u_i) < X$, the problem is in excess supply. A feasible allocation is a vector $x = (x_1, x_2, \ldots, x_n) \in C^n$ with $\sum_{i \in N} x_i = X$. Free-disposability is not allowed. Let $A(X)$ be the set of all feasible allocations. f is a probability distribution on $A(X)$ such that $f : A(X) \to [0, 1]$ and $\sum_{x \in A(X)} f(x) = 1$. Let $S(X)$ be the set of all probability distributions on $A(X)$. A *randomized rule* is a function $\phi : U^n \to S(X)$. We define the probability with which a probability distribution f gives k units of the indivisible object to agent i by $f_i(k) \equiv \sum_{x \in \{x \in A(X) | x_i = k\}} f(x)$. This means that f_i is *the marginal distribution* of f with respect to the number of the object that the agent i obtains. As demonstrated in Example 3 by Ehlers and Klaus (2003), two distinct probability distributions may have the same marginal distribution.

Given a probability distribution f, we call $\sum_{k \in C} f_i(k) u_i(k)$ the expected utility of agent i. We abuse notation, and write $\phi(u)(x)$ as $\phi(x; u)$ and $\phi_i(u)(k)$ as $\phi_i(k; u)$. $\phi(x; u)$ is the probability which the randomized rule ϕ places on the feasible allocation $x \in A(X)$ when the reported utility profile is $u \in U^n$. $\phi_i(k; u)$ is the probability with which agent i obtains k units of the object under the randomized rule ϕ when the reported utility profile is u.

3   Axioms

We are interested in the following axioms. The first is related to fairness. We call it *equal probability for the best*. Given a probability distribution and an agent, there exists a number of the object which gives him/her the

highest utility with a strictly positive probability. The number is called the best. It should be noted that since every utility function is single-peaked the agent has at most two bests: one is on the left side of his/her peak and the other is on the right side. Also note that each agent has only one best if the randomized rule satisfies same-sidedness.

Then, equal probability for the best requires the randomized rule to output the probability distribution which has the following marginal distribution: if two agents have the same utility function, then at least one of the bests and the corresponding probabilities should be same between the agents. Note that the marginal distribution of each agent does not need to be same between the agents.

In order to define equal probability for the best formally, for any $i \in N$ and any $f \in S(X)$, we need B_f^i: the set of all pairs consisting of both the number of the object which gives agent i the highest utility with a strictly positive probability and the corresponding probability of the number under the probability distribution f. Note that B_f^i is a singleton or a doubleton.

For any probability distribution $f \in S(X)$, any $i \in N$, and any $u_i \in U$, let $B_f^i \equiv \{(b, f_i(b)) \in C \times (0, 1] \mid u_i(b) \geq u_i(k) \text{ for all } k \in C \text{ such that } f_i(k) > 0\}$.

Using B_f^i, equal probability for the best can be represented as follows.

Equal probability for the best. For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $B_{\phi(u)}^i \cap B_{\phi(u)}^j \neq \emptyset$.

Equal treatment of equals says that if two agents' utility functions are same, their expected utilities should also be same.

Equal treatment of equals. For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $\sum_{k \in C} \phi_i(k; u) u_i(k) = \sum_{k \in C} \phi_j(k; u) u_j(k)$.

Other axiom relating to fairness is symmetry.

Symmetry. For any $u \in U^n$ and any $i, j \in N$, if $u_i = u_j$, then $\phi_i(k; u) = \phi_j(k; u)$ for all $k \in C$.

Remark 1. Both equal probability for the best and equal treatment of equals are implied by symmetry. It is easy to check that equal probability for the best and equal treatment of equals are logically independent.

A standard axiom that needs no further explanation is Pareto optimality.

Pareto optimality. For any $u \in U^n$, there is no $f \in S(X)$ such that for all $i \in N$, $\sum_{k \in C} f_i(k) u_i(k) \geq \sum_{k \in C} \phi_i(k; u) u_i(k)$ and for some $j \in N$, $\sum_{k \in C} f_j(k) u_j(k) > \sum_{k \in C} \phi_j(k; u) u_j(k)$.

Pareto optimality implies same-sidedness, which requires that for any agent every number of the object which he/she obtains with a strictly positive

probability should be equal to or less (more) than his/her own peak when the problem is in excess demand (supply).

Same-sidedness. For any $u \in U^n$ and any $i \in N$,
(i) when $\sum_{j \in N} p(u_j) \geq X$, for all $k \in C$, $\phi_i(k; u) > 0$ implies $k \leq p(u_i)$, and
(ii) when $\sum_{j \in N} p(u_j) < X$, for all $k \in C$, $\phi_i(k; u) > 0$ implies $k \geq p(u_i)$.

Lemma 1 (Sasaki (1997)). *If ϕ is Pareto optimal, then it is same-sided.*

Proof of Lemma 1. See Sasaki (1997). □

For the converse statement of Lemma 1 to be true we need property B, which is shown as Lemma 3 in Appendix.

The final is the requirement that no agent ever benefits from misrepresenting his/her utility function.

Strategy-proofness. For any $i \in N$, any $u_i, \hat{u}_i \in U$, and any $u_{-i} \in U^{n-1}$,
$\sum_{k \in C} \phi_i(k; u) u_i(k) \geq \sum_{k \in C} \phi_i(k; \hat{u}_i, u_{-i}) u_i(k)$.

4 The Results

We have the following results on equal probability for the best.

Theorem 1. *In the presence of Pareto optimality, the following three axioms, equal probability for the best, equal treatment of equals, and symmetry are equivalent.*

Before proceeding to prove Theorem 1, we need the following lemma. Lemma 2 says that Pareto optimality implies property B, which requires that randomized rules assign the probability distribution where each agent receives at most two numbers of the object which are adjacent with each other.

Property B (Sasaki (1997)). For any $u \in U^n$ and any $i \in N$, there exists $\alpha_i \in C$ with $\phi_i(\alpha_i + 1; u) > 0$ such that $\phi_i(\alpha_i; u) + \phi_i(\alpha_i + 1; u) = 1$.

Lemma 2 (Sasaki (1997)). *If ϕ satisfies Pareto optimality, then it satisfies property B.*

Proof of Lemma 2 is in Appendix.

Proof of Theorem 1. Let ϕ be a randomized rule satisfying Pareto optimality. Fix $u \in U^n$ and consider the case when $\sum_{i \in N} p(u_i) \geq X$. The other case is similar. From Lemmas 1 and 2 ϕ satisfies same-sidedness and property B. By property B, for any agent $i \in N$, there exists $\alpha_i \in C$ with $\phi_i(\alpha_i + 1; u) > 0$ such that $\phi_i(\alpha_i; u) + \phi_i(\alpha_i + 1; u) = 1$. From same-sidedness, we have $p(u_i) \geq \alpha_i + 1 > \alpha_i$ for all $i \in N$. Hence, for any agent i the best for him/her is $\alpha_i + 1$.

Suppose that ϕ satisfies equal probability for the best. Pick any agent $i, j \in N$ such that $u_i = u_j$. Then, from equal probability for the best, we

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have $B^i_{\phi(u)} \cap B^j_{\phi(u)} \neq \emptyset$. From the previous paragraph, we have $B^i_{\phi(u)} = \{(\alpha_i + 1, \phi_i(\alpha_i + 1; u))\}$ for any $i \in N$. Therefore, $B^i_{\phi(u)} \cap B^j_{\phi(u)} \neq \emptyset$ gives us that for any $i, j \in N$, $\alpha_i = \alpha_j$, $\phi_i(\alpha_i; u) = \phi_j(\alpha_j; u)$, and $\phi_i(\alpha_i + 1; u) = \phi_j(\alpha_j + 1; u)$. Hence, from property B it is clear that $\phi_i(k; u) = \phi_j(k; u)$ for all $k \in C$, which means that ϕ satisfies symmetry. It is obvious that symmetry implies equal treatment of equals.

Next, suppose that ϕ satisfies equal treatment of equals. Pick any agent $i, j \in N$ such that $u_i = u_j$. Then, we have $\sum_{k \in C} \phi_i(k; u) u_i(k) = \sum_{k \in C} \phi_j(k; u) u_j(k)$. From the first paragraph, we have $\sum_{k \in C} \phi_i(k; u) k = \phi_i(\alpha_i; u)\alpha_i + \phi_i(\alpha_i + 1; u)(\alpha_i + 1) = \phi_j(\alpha_j; u)\alpha_j + \phi_j(\alpha_j + 1; u)(\alpha_j + 1) = \sum_{k \in C} \phi_j(k; u) k$. Same-sidedness implies that $p(u_i) \geq \alpha_i + 1 > \alpha_i$ for all $i \in N$. Therefore, we have $\alpha_i = \alpha_j$ and $\phi_i(\alpha_i + 1; u) = \phi_j(\alpha_j + 1; u)$. Since $\alpha_i + 1$ and $\alpha_j + 1$ are the best for agent i and j respectively, we have $B^i_{\phi(u)} = \{(\alpha_i + 1, \phi_i(\alpha_i + 1; u))\}$ and $B^j_{\phi(u)} = \{(\alpha_j + 1, \phi_j(\alpha_j + 1; u))\}$, which implies that ϕ satisfies equal probability for the best. $\square$

Remark 2. Only in the presence of property B, symmetry is not equivalent to equal probability for the best.

The main feature of Theorem 1 is that we clarify the relation of our axiom, equal probability for the best to other axioms relating to fairness. Our axiom is indigenous to indivisibility or randomization so that it has no counterpart in the deterministic environment. On the other hand, in previous papers on randomization of the uniform rule, all axioms on fairness, such as equal treatment of equals, symmetry, and so on, are just the randomized version of those used in the deterministic environment.

Ehlers and Klaus (2003) showed that the class of the randomized uniform rules is characterized by equal treatment of equals, Pareto optimality, and strategy-proofness. Hence, using Theorem 1, the randomized uniform rule can be characterized by strategy-proofness, Pareto optimality, and equal probability for the best. *The randomized uniform rule* is a randomized extension of the uniform rule.[1]

Definition 1. *The randomized uniform rule* $\Phi : U^n \to S(X)$ *is the function defined as follows: for any* $u \in U^n$,
(i) when $\sum_{i \in N} p(u_i) \geq X$,

$$
\begin{cases}
\Phi_j(p(u_j); u) = 1 & \text{if } p(u_j) \leq \lambda, \\
\Phi_j(\mu + 1; u) = \lambda - \mu \\
\Phi_j(\mu; u) = 1 - (\lambda - \mu) & \text{if } p(u_j) > \lambda,
\end{cases}
$$

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[1] See Benassy (1982) for the uniform rule.

(ii) when $\sum_{i \in N} p(u_i) < X$,

$$
\begin{cases}
\Phi_j(p(u_j); u) = 1 & \text{if } p(u_j) \geq \lambda, \\
\Phi_j(\mu + 1; u) = \lambda - \mu & \\
\Phi_j(\mu; u) = 1 - (\lambda - \mu) & \text{if } p(u_j) < \lambda,
\end{cases}
$$

for all $j \in N$, where $\lambda \in \mathbb{R}_+$ solves $\sum_{i \in N} \sum_{k \in C} \Phi_i(k)k = X$ and $\mu \equiv \lfloor \lambda \rfloor$.[23]

The randomized uniform rule associates with each profile of the utility functions a probability distribution that induces the marginal distribution of the following form. When the problem is in excess demand (supply), the agent whose peak is equal to or less (more) than a common bound λ gets his/her own peak with probability 1. The rest of the agents get $\mu+1$ with probability $\lambda - \mu$ and μ with probability $1 - \lambda + \mu$ respectively. μ is obtained through rounding down the common bound λ.

Theorem 2. The randomized uniform rule Φ is the only randomized rule satisfying strategy-proofness, Pareto optimality, and equal probability for the best.

An advantage of Theorem 2 is using the axiom that is indigenous to randomization when we characterize the randomized version of the uniform rule.

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[2]For any $a \in \mathbb{R}$, let $\lfloor a \rfloor = n$ be the integer such that $n \leq a < n + 1$.

[3]The randomized uniform rule outputs "a" probability distribution which induces each agent's marginal distribution defined as (i) and (ii) of Definition 1. Hence, the randomized uniform rule uniquely determines *a marginal distribution* for each agent, but it does not uniquely determine *a probability distribution* (although it uniquely chooses a probability distribution) because more than one probability distribution can have an identical marginal distribution.

Appendix

Lemma 3. If ϕ satisfies same-sidedness and property B, then it satisfies Pareto optimality.

Proof of Lemma 3. Suppose the contrary. There exists $f \in S(X)$ such that for all $i \in N$, $\sum_{k \in C} f_i(k)u_i(k) \geq \sum_{k \in C} \phi_i(k;u)u_i(k)$ and for some $j \in N$, $\sum_{k \in C} f_j(k)u_j(k) > \sum_{k \in C} \phi_j(k;u)u_j(k)$. Without loss of generality, we assume that $\sum_{i \in N} p(u_i) \geq X$. Let $\lambda_i \equiv \sum_{k \in C} f_i(k)k$ and $\mu_i \equiv \lfloor \lambda_i \rfloor$ for all $i \in N$. From risk-averseness, we have for all $i \in N$, $(1 - \lambda_i + \mu_i)u_i(\mu_i) + (\lambda_i - \mu_i)u_i(\mu_i + 1) \geq \sum_{k \in C} f_i(k)u_i(k)$. From property B, for all $i \in N$, there exists $\alpha_i \in C$ such that $\phi_i(\alpha_i;u) + \phi_i(\alpha_i + 1;u) = 1$. Therefore, we have $\sum_{k \in C} \phi_i(k;u)u_i(k) = \phi_i(\alpha_i;u)u_i(\alpha_i) + \phi_i(\alpha_i + 1;u)u_i(\alpha_i + 1)$ for all $i \in N$. Hence, we have $(1 - \lambda_i + \mu_i)u_i(\mu_i) + (\lambda_i - \mu_i)u_i(\mu_i + 1) \geq \phi_i(\alpha_i;u)u_i(\alpha_i) + \phi_i(\alpha_i + 1;u)u_i(\alpha_i + 1)$ for all $i \neq j$ and $(1 - \lambda_j + \mu_j)u_j(\mu_j) + (\lambda_j - \mu_j)u_j(\mu_j + 1) > \phi_j(\alpha_j;u)u_j(\alpha_j) + \phi_j(\alpha_j + 1;u)u_j(\alpha_j + 1)$.

From same-sidedness, we have for all $i \in N$, $p(u_i) \geq \sum_{k \in C} \phi_i(k;u)k$. Hence, single-peakedness implies that for all $i \neq j$, $(1 - \lambda_i + \mu_i)\mu_i + (\lambda_i - \mu_i)(\mu_i + 1) \geq \phi_i(\alpha_i;u)\alpha_i + \phi_i(\alpha_i + 1;u)(\alpha_i + 1)$ and $(1 - \lambda_j + \mu_j)\mu_j + (\lambda_j - \mu_j)(\mu_j + 1) > \phi_j(\alpha_j;u)\alpha_j + \phi_j(\alpha_j + 1;u)(\alpha_j + 1)$.

Summing up gives us $\sum_{i \in N}(1 - \lambda_i + \mu_i)\mu_i + \sum_{i \in N}(\lambda_i - \mu_i)(\mu_i + 1) > \sum_{i \in N} \phi_i(\alpha_i;u)\alpha_i + \sum_{i \in N} \phi_i(\alpha_i + 1;u)(\alpha_i + 1)$. The left hand side becomes $\sum_{i \in N} \lambda_i = \sum_{i \in N} \sum_{k \in C} f_i(k)k = X$. The last equality follows from feasibility. Moreover, the right hand side also becomes $\sum_{i \in N} \sum_{k \in C} \phi_i(k;u)k = X$ by feasibility. This is a contradiction. $\square$

Proof of Lemma 2 (Sasaki (1997)). Suppose the contrary. Then, there exist $a, b, c \in C$ such that $\phi_i(a;u) > 0$, $\phi_i(c;u) > 0$ and $a < b < c$. Then, there exist x' and $x'' \in A(X)$ such that $x'_i = a, x''_i = c, \phi(x';u) > 0$ and $\phi(x'';u) > 0$. Without loss of generality, we assume that $\phi(x';u) \geq \phi(x'';u)$. $\sum_{l \in N} x'_l = \sum_{l \in N} x''_l = X$ and $a < c$ imply that there exists $j \neq i$ such that $x'_j > x''_j$. Let us define $\tilde{x}$ and $\hat{x} \in A(X)$ as

$$
\begin{cases}
\tilde{x}_i &=& a + 1, \\
\tilde{x}_j &=& x'_j - 1, \\
\tilde{x}_l &=& x'_l \quad \text{for all } l \neq i, j,
\end{cases}
\quad \text{and} \quad
\begin{cases}
\hat{x}_i &=& c - 1, \\
\hat{x}_j &=& x''_j + 1, \\
\hat{x}_l &=& x''_l \quad \text{for all } l \neq i, j.
\end{cases}
$$

Let us define $f \in S(X)$ as

$$
\begin{cases}
f(x') &=& \phi(x';u) - \phi(x'';u), \\
f(x'') &=& 0, \\
f(\tilde{x}) &=& \phi(\tilde{x};u) + \phi(x'';u), \\
f(\hat{x}) &=& \phi(\hat{x};u) + \phi(x'';u), \\
f(y) &=& \phi(y;u) \quad \text{for all } y \neq x', x'', \tilde{x}, \hat{x}.
\end{cases}
$$

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Note that if $\tilde{x} = \hat{x}$, then let us define $f(\tilde{x}) = f(\hat{x}) \equiv \phi(\hat{x}; u) + 2\phi(x''; u)$. Then, by risk-averseness, we find that $\sum_{k \in C} f_i(k) u_i(k) > \sum_{k \in C} \phi_i(k; u) u_i(k)$, $\sum_{k \in C} f_j(k) u_j(k) \geq \sum_{k \in C} \phi_j(k; u) u_j(k)$ and $\sum_{k \in C} f_l(k) u_l(k) = \sum_{k \in C} \phi_l(k; u) u_l(k)$ for all $l \neq i, j$. This contradicts Pareto optimality of ϕ. $\qquad\square$

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