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Investment, irreversibility and financial imperfections: the rush to invest and the option to wait

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Abstract

The impact of combinations of frictions on investment activity is poorly understood. We develop a model of investment under financial frictions and irreversibility. We show that the possibility of encountering financial constraints in future raises irreversible investment today over that arising under irreversibility alone. By contrast, investment under both frictions is lower than under future financial constraints alone.

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1. Introduction

We contribute to the literature analysing the effects of irreversibility and financial constraints in conjunction on investment. Under irreversibility alone, and with the *option to wait* for uncertainty to be resolved, a firm reduces investment today, compared to the frictionless case, to avoid holding too much capital in future. We show that the possibility of encountering financial constraints in the future may accelerate current irreversible investment over that arising under irreversibility alone. This *rush to invest* effect arises as firms try to take advantage of benign conditions today to avoid having too little capital in future. Under this effect investment exceeds that under irreversibility alone but remains lower than in the frictionless case. We consider firms' investment decisions, but similar issues may be present in durable goods expenditures, human capital accumulation, labour demand and elsewhere.

The literature on irreversible investment is one response to the empirical failure of the (frictionless and convex adjustment cost versions of) the neoclassical investment model, Dixit and Pindyck (1994), Smit and Trigeorgis (2004). It may account for the history dependence of investment decisions, periods of inaction, and the predominance of quantity over price variables in investment equations. The presence of financial imperfections offers a competing explanation of observed investment dynamics, see Hubbard (1998). Only recently has work begun to examine the nature of the interaction between both frictions and the consequences for investment dynamics. Holt (2003) examines how firms' investment and dividend policies interact under irreversibility and financial constraints. He emphasises the life-cycle nature of the resultant optimal policies in which small (young) firms invest, and grow relaxing financial frictions before beginning to pay dividends. Bayer (2006) finds nonlinear patterns of short-run investment in UK firm level data which are consistent with a model in which firms face non-convex adjustment costs and financial frictions. Caggese (2007) studies investment under irreversibility and financial constraints when the firm uses irreversible, fixed capital and perfectly-flexible variable capital as complements in production. He shows that investing in fixed capital today exacerbates the impact of (future) financial constraints on variable capital. So the complementarity of fixed and variable capital in production leads to a *precautionary* effect of future financial constraints in his model which retards fixed capital investment beyond the effect of irreversibility alone.

The rush to invest effect which we highlight here has not been analysed in existing work. Our analysis corresponds to an environment where either the production complementarities studied by Caggese (2007) are absent, or where other factors of production are also subject to non-convex costs of adjustment rather than being freely adjustable. The effect we study arises because the possibility of being unable to expand capacity in future, due to financial constraints, raises the cost of waiting for the resolution of uncertainty, which partly offsets the value of the option to wait that arises under irreversibility. The combination of irreversibility and financial constraints raises current investment when compared with the effect of irreversibility alone. It tends to retard investment compared to a situation with financial constraints alone. We show that concerns over being financially constrained in future do not lead investment to exceed that in the frictionless case. Finally, increased uncertainty may raise investment in our model whereas it tends to reduce investment under irreversibility alone.

2. A Model of Optimal Capacity Choice

We extend the two period framework of Abel et al. (1996) to allow for present and future financial constraints. This enables us to characterise behaviour without relying on numerical analysis.[1] We first outline the basic framework and then characterise optimal investment in various environments: without frictions, under irreversibility alone, under financial constraints alone and under both constraints together. Unless otherwise indicated, proofs are in the appendix.

Suppose that a firm exists for two periods, possibly of different lengths. The firm initially holds no capital. It makes two investment decisions: how much capital to acquire today, K_1, and how much additional capital to acquire next period, $K_2 - K_1$. Investment becomes productive immediately. Capital does not depreciate between period 1 and 2, but there is 100% depreciation at the end of period 2. There are no other factors of production. The firm obtains revenues $r(K_1)$ in period 1. We assume that $r(0) = 0$, $r'(K_1) > 0$ and $r''(K_1) < 0$ for $K_1 > 0$, due to either diminishing marginal product of capital or a downward sloping product demand function. There is uncertainty over the profitability of capital in period 2. We write revenues as $R(K_2, e)$, where $K_2 \geq 0$ and $e, -\infty < e < \infty$, is a random variable with distribution function $G(e)$, which captures all uncertainty affecting the profitability of the firm. Future capital, K_2, is chosen after the realisation of e. Assume $R(K_2, e)$ is twice continuously differentiable in both arguments. Period 2 marginal revenue product of capital is positive, $R_K(K_2, e) > 0$, strictly decreasing in K, $R_{KK}(K_2, e) < 0$ and strictly increasing in e, $R_{Ke}(K_2, e) > 0$. The unit cost, P, of acquiring new capital remains constant over time. The firm chooses K_1 and K_2 so as to maximise expected discounted profits, where $\beta > 0$ is its discount factor.

As our primary aim is to characterise the consequences of irreversibility and financial constraints, rather than to model how these constraints originate, we take the financial constraint and the irreversibility constraint as exogenously given.[2] Irreversibility of investment requires $K_1 \geq 0$ and $K_2(e) - K_1 \geq 0$. To capture financial constraints let the firm hold wealth, $W_1 > 0$ at the start of period 1, but suppose it is unable to raise external finance. This constraint requires $W_1 - P \cdot K_1 \geq 0$ in period 1 and $W_2 - P[K_2 - K_1] \geq 0$ in period 2, where $W_2 = [W_1 - PK_1 + r(K_1)]$, and, for simplicity, the firm earns no interest on internal wealth.

2.1 Optimal Capacity without Frictions.

To establish a benchmark consider the frictionless case. In the absence frictions, the firm adjusts date 2 capital until $R_K(K_2(e), e) = P$. This determines K_2 as

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[1] Our 2-period set up offers a simple treatment of the presence and resolution of uncertainty, current and future financial constraints and the outcomes across different combinations of constraints.

[2] This approach is standard in the literature on irreversibility in investment, Dixit and Pindyck (1994). We recognise that informational asymmetries between lenders and borrowers in financial relationships, or between seller and purchaser in the secondary market for capital equipment, or limited commitment may underpin financial and irreversibility constraints. Our approach not only has the virtue of simplicity but also seems well suited to the objectives of the paper.

a function of e. Let $V^N(K_1) = r(K_1) + \beta\int_{-\infty}^{\infty}\{R(K_2(e),e) - P[K_2(e) - K_1]\}dG(e)$ be the value of capital installed at date 1. The firm solves: $\max_{K_1} V^N(K_1) - P \cdot K_1$. Let K_1^N be the optimal capital stock in the frictionless case. The first order condition is:

$$V^{N'}(K_1^N) = r'(K_1^N) + \beta P = P\,. \tag{1}$$

2.2 Optimal Capacity under Irreversible Investment.

Due to the irreversibility constraint, the firm must hold date 2 capital stock $K_2 \geq K_1$. For any initial capital K_1, there exists a threshold value of $e = e_I$, $R_K(K_1, e_I) = R_K(K_2(e_I), e_I) = P$, where for $e \leq e_I$, $K_2 = K_1$ and for $e > e_I$, $K_2(e)$ is determined by the first order condition $R_K(K_2(e), e) = P$. Let the date 1 value of installed capital under irreversibility be: $V^I(K_1) = r(K_1) + \beta\int_{-\infty}^{e_I} R(K_1, e)dG(e) + \beta\int_{e_I}^{\infty}\{R(K_2(e),e) - P[K_2(e) - K_1]\}dG(e)$. The firm solves the problem, $\max_{K_1} V^I(K_1) - P \cdot K_1$, to determine the optimal period 1 capital, K_1^I (with associated threshold e_I^I).[3] The first order condition is:

$$V^{I'}(K_1^I) = r'(K_1^I) + \beta\int_{-\infty}^{e_I^I}\left[R_K(K_1^I, e) - P\right]dG(e) + \beta P = P\,. \tag{2}$$

Lemma 1: $K_1^I < K_1^N$ *: date 1 capital is lower under an irreversibility constraint than in the frictionless case.*[4]

Here, the loss in marginal revenue associated with holding too much capital in the future, due to irreversibility, leads the firm reduce investment today.

2.3 Optimal Capacity Choice under Irreversibility and Financial Constraints

With financial constraints the maximum capital stock that can be held on date 2 is $K_2(K_1) = K_1 + [W_1 - PK_1 + r(K_1)]/P$. Lemma 2 shows results about $K_2(K_1)$.

Lemma 2 a) $K_2(K_1) > K_1$, $\forall K_1 \in [0, W_1/P]$; **b)** $dK_2(K_1)/dK_1 > (=,<) 0$ as $K_1 < (=,>) K_1^N$; **c)** $d^2K_2(K_1)/(dK_1)^2 < 0$.

For arbitrary capital stock, K_1, and wealth level, W_1, there are two threshold values of e: e_I and e_F, which are defined by the conditions

$$R_K(K_1, e_I) = R_K(K_2(e_I), e_I) = P\;, \tag{3}$$

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[3] The associated threshold value of e is e_I^I, where the subscript denotes a threshold associated with irreversibility and the superscript denotes that threshold associated with *optimal* capital stock under irreversibility, K_1^I. This approach to notation for thresholds is used throughout.

[4] Proofs of statements are contained in the appendix unless otherwise indicated.

$$R_K\left(K_2\left(K_1\right), e_F\right) = R_K\left(K_2\left(e_F\right), e_F\right) = P. \tag{4}$$

Now $e_F > e_I$, since $K_2(K_1) - K_1 > 0$ by Lemma 2, while $R_{KK}(\cdot,\cdot) < 0$ and $R_{Ke}(\cdot,\cdot) > 0$ by assumption. So period 2 is characterised by 3 regimes. For $e < e_I$ the firm's best response is inaction, so the irreversibility constraint binds and the marginal revenue product of irreversibly held period 1 capital lies below that of the optimally chosen frictionless period 2 capital, $R_K\left(K_1, e\right) < R_K\left(K_2(e), e\right) = P$. For $e > e_F$ the firm's best response is constrained investment, as the financial constraint binds and the marginal revenue product of capital under financial constraints exceeds that in the absence of financial frictions: $R_K\left(K_2(K_1), e\right) > R_K\left(K_2(e), e\right) = P$. For intermediate values $e \in \left[e_I, e_F\right]$ neither constraint binds and capital stock attains the same value as in the frictionless case: $R_K\left(K_2(e), e\right) = P$. These regimes exist regardless of whether the financial constraint also binds in period 1, but the threshold locations depend on the capital stock and wealth inherited from period 1. Hence, the date 1 value of the firm's capital is:

$$V^{FI}\left(K_1\right) = r\left(K_1\right) + \beta \begin{bmatrix} \int_{-\infty}^{e_I} R\left(K_1, e\right) dG(e) + \int_{e_I}^{e_F} \left\{R\left(K_2(e), e\right) - P\left[K_2(e) - K_1\right]\right\} dG(e) \\ + \int_{e_F}^{\infty} \left\{R\left(K_2\left(K_1\right), e\right) - P\left[K_2\left(K_1\right) - K_1\right]\right\} dG(e) \end{bmatrix}$$

On date 1, the firm solves the problem: $\max_{K_1} V^{FI}\left(K_1\right) - P \cdot K_1 + \lambda\left[W - P \cdot K_1\right]$. The condition $\lambda\left[W - P \cdot K_1\right] \geq 0$ holds with complementary slackness. When the financial constraint does not bind in period 1, $W_1 > P \cdot K_1$, $\lambda = 0$. Then:

$$V^{FI'}\left(K_1^{FI}\right) = r'\left(K_1^{FI}\right) + \beta \begin{bmatrix} \int_{-\infty}^{e_I^{FI}} \left[R_K\left(K_1^{FI}, e\right) - P\right] dG(e) + \\ \dfrac{dK_2}{dK_1^{FI}} \int_{e_F^{FI}}^{\infty} \left[R_K\left(K_2\left(K_1^{FI}\right), e\right) - P\right] dG(e) \end{bmatrix} + \beta P = P \tag{5}$$

where K_1^{FI} is the optimal date 1 capital stock and e_I^{FI}, e_F^{FI} are the associated thresholds (for irreversibility and financial constraints respectively). If the financial constraint does bind in period 1, then $\lambda > 0$. Let $\overline{K}_1^{FI}$ be the optimal date 1 capital given the binding constraint, then $W = P \cdot \overline{K}_1^{FI}$. The first order condition in that case is a modified version of equation (5) in which $K_1^{FI} = \overline{K}_1^{FI}$ and $V^{FI'}\left(\overline{K}_1^{FI}\right) = P[1 + \lambda]$.

Next, define the derivative of V^{FI} evaluated for arbitrary $K_1 = K_1^*$ as

$$J\left(K_1^*\right) = r'\left(K_1^*\right) + \beta \int_{-\infty}^{e_I^*\left(K_1^*\right)} \left[R_K\left(K_1^*, e\right) - P\right] dG(e) + \beta \frac{dK_2}{dK_1^*} \int_{e_F^*\left(K_2\left(K_1^*\right)\right)}^{\infty} \left[R_K\left(K_2\left(K_1^*\right), e\right) - P\right] dG(e) + \beta P$$

where e_I^*, e_F^* are the threshold values for equations (3) and (4) associated with K_1^*. Lemma 3 develops a useful intermediate result on the monotonicity of $J\left(K_1^*\right)$ in K_1^*.

Lemma 3. *For* $K_1^* \in (0, K_1^N]$, $\partial J\left(K_1^*\right) / \partial K_1^* < 0$.

Our main result summarises the effect of financial constraints, present and future, on irreversible investment.

Proposition 1: a) For $\lambda = 0$, $K_1^I < K_1^{FI} < K_1^N$**. b)** For $\lambda > 0$ **i)** $\overline{K}_1^{FI} < K_1^{FI}$, **ii)** $\overline{K}_1^{FI} > (=,<) K_1^I$ as $J(K_1^I) - V^{I'}(K_1^I) < (=,>) \lambda P$.

Proof: a) First we show that $K_1^I < K_1^{FI}$, by contradiction. Suppose that $K_1^I = K_1^{FI}$**,** then the irreversibility thresholds are identical $e_I^I = e_I^{FI}$. Yet, at $K_1^* = K_1^I$, we find $J(K_1^*) - V^{I'}(K_1^I) = J(K_1^I) - V^{I'}(K_1^I) = [dK_2/dK_1]\int_{e_F}^{\infty}[R_K(K_2(K_1), e) - P]dG(e)$. The integral is positive. Lemma 1 guarantees that $K_1^I < K_1^N$, so from Lemma 2 $dK_2(K_1)/dK_1 > 0$. Thus $J(K_1^I) - V^{I'}(K_1^I) > 0$ and we must have $K_1^I \neq K_1^{FI}$. From Lemma 3, $\partial\left[J(K_1^*) - V^{I'}(K_1^I)\right]\Big/\partial K_1^* = \partial J/\partial K_1^* < 0$ so $K_1^I < K_1^{FI}$**.** Next we show that $K_1^{FI} < K_1^N$ using a similar approach. Suppose that $K_1^{FI} = K_1^N$, then, evaluated at $K_1^* = K_1^N$, $J(K_1^*) - V^{N'}(K_1^N)$ is $J(K_1^N) - V^{I'}(K_1^N) = \beta\int_{-\infty}^{e_I^N(K_1^N)}[R_K(K_1^N, e) - P]dG(e) < 0$. So we must have $K_1^{FI} \neq K_1^N$**.** Since $\partial\left[J(K_1^*) - V^{N'}(K_1^N)\right]\Big/\partial K_1^* = \partial J/\partial K_1^* < 0$, from Lemma 3, it follows that $K_1^{FI} < K_1^N$**.**

b) i) For $K_1^* = K_1^{FI}$, $J(K_1^{FI}) - V^{FI'}(K_1^{FI}) = 0 < \lambda P = V^{FI'}(\overline{K}_1^{FI}) - V^{FI'}(K_1^{FI})$ and $\partial\left[J(K_1^*) - V^{FI'}(K_1^{FI})\right]\Big/\partial K_1^* = \partial J/\partial K_1^* < 0$. So it follows that $\overline{K}_1^{FI} < K_1^{FI}$.

ii) For $K_1^* = K_1^I$, $J(K_1^I) - V^I(K_1^I) = (dK_2/dK_1^I)\int_{e_F(K_1^I)}^{\infty}[R_K(K_2(K_1^I), e) - P]dG(e)$. Since $K_1^I < K_1^N$, Lemma 2 ensures that $dK_2(K_1)/dK_1 > 0$. Also, the integral term is positive so it follows that $J(K_1^I) - V^{I'}(K_1^I) > 0$. Now $\partial\left[J(K_1^*) - V^{I'}(K_1^I)\right]\Big/\partial K_1^* = \partial J/\partial K_1^* < 0$ and $V^{FI}(\overline{K}_1^{FI}) - V^{FI}(K_1^{FI}) = \lambda P > 0$. So $\overline{K}_1^{FI} > (=,<) K_1^I$ as $J(K_1^I) - V^{I'}(K_1^I) < (=,>) \lambda P$.■

Result **a)** shows that if a firm does not face financial constraints today and its investment decisions are irreversible, the possibility of encountering financial constraints in the future raises current investment over that arising under irreversibility alone.[5] The firm takes advantage of benign conditions to guard against the possibility of having too little capital in future. The possibility of being financially constrained in future raises the cost of waiting which partly offsets the value of the option to wait that arises under irreversibility. Yet while this rush to invest may attenuate the effect of irreversibility it does not override it. This follows because investment does not exceed that in the frictionless case, even without the

[5] Under irreversibility alone, $K_1^I < K_1^N$. At this point $dK_2(K_1)/dK_1 > 0$ so the possibility of future financial constraints financial constraints raises optimal period 1 investment to $K_1^{FI} > K_1^I$.

complementarities in production assumed in Caggese (2007). Result **b)** shows that when a firm currently faces financial constraints and its investment decisions are irreversible, the rush to invest effect is still present but may be outweighed by the other constraints affecting the firm. Results **b) i)** and **b) ii)** show that currently binding financial constraints reduce investment compared with result **a)**.

Finally, consider the effects of uncertainty over the future as captured by a mean-preserving spread in the distribution function: $G(e)$ to $\tilde{G}(e)$ in equation (5)**.** This raises the mass in the tails of the profitability shock distribution which increases $\beta\left[dK_2/dK_1^{FI}\right]\int_{e_F^{FI}}^{\infty}\left[R_K\left(K_2\left(K_1^{FI}\right),e\right)-P\right]dG(e)$ and makes $\beta\int_{-\infty}^{e_I^{FI}}\left[R_K\left(K_1^{FI},e\right)-P\right]dG(e)$ more negative. This second integral represents the loss in marginal revenue associated with holding too much capital in the future, due to irreversibility. This leads the firm to reduce investment today, exploiting the option to wait. However, a firm must balance this consideration against the possibility that by acquiring too little capital today, it fails to generate sufficient internal funds to take advantage of high future profitability levels. This is represented by $\beta\left[dK_2/dK_1^{FI}\right]\int_{e_F^{FI}}^{\infty}\left[R_K\left(K_2\left(K_1^{FI}\right),e\right)-P\right]dG(e)$. Depending on the locations of the thresholds and the shape of the distribution function for profitability, a rise in uncertainty may lead to a rise in investment today by increasing the latter rush to invest effect more than the former option to wait. However the model would also be consistent with empirical evidence of a negative effect of uncertainty on investment, Carruth et al. (2000).

2.4 Optimal Capacity Choice Under Financial Constraints

To shed further light on the interaction of irreversibility with financial constraints, we contrast the results of Proposition 1 with those arising in an environment where financial constraints may bind at present or in the future but investment is *reversible*. For arbitrary capital stock, K_1, and wealth level, W_1, equation (4) defines a threshold value of e, e_F, such that for $e > e_F$ the financial constraint binds and the marginal revenue product of capital is $R_K\left(K_2\left(K_1\right),e\right) > R_K\left(K_2(e),e\right) = P$. The date 1 value of capital installed is:

$$V^F\left(K_1\right) = r\left(K_1\right) + \beta\begin{bmatrix}\int_{-\infty}^{e_F}\left\{R\left(K_2(e),e\right)-P\left[K_2(e)-K_1\right]\right\}dG(e)\\ +\int_{e_F}^{\infty}\left\{R\left(K_2\left(K_1\right),e\right)-P\left[K_2\left(K_1\right)-K_1\right]\right\}dG(e)\end{bmatrix}.$$

On date 1, the firm solves the problem: $\max_{K_1} V^F\left(K_1\right) - P\cdot K_1 + \lambda\left[W - P\cdot K_1\right]$ and the condition $\lambda\left[W - P\cdot K_1\right] \geq 0$ holds with complementary slackness. When $\lambda = 0$ the financial constraint does not bind in period 1 and the first order condition is $V^{F'}\left(K_1^F\right) = r'\left(K_1^F\right) + \beta\frac{dK_2}{dK_1^F}\int_{e_F^F}^{\infty}\left[R_K\left(K_2\left(K_1^F\right),e\right)-P\right]dG(e) + \beta P = P$, where K_1^F is the optimal capital stock. Let $\overline{K}_1^F$ denote the optimal capital stock when $\lambda > 0$ so $V^{F'}\left(\overline{K}_1^F\right) = P(1+\lambda)$. Implications for capital choice are captured in Proposition 2.

Proposition 2: a) For $\lambda = 0$, $K_1^F = K_1^N$. **b)** For $\lambda > 0$, $\overline{K}_1^F < K_1^N$.

Part **a.** indicates that when financial constraints may bind in future, but do not bind today, it is optimal invest until capital stock is at its frictionless level. Notice that with this level of capital stock, there remains some possibility that investment will be financially constrained at date 2. This is optimal because holding too much capital, i.e. $K_1^F > K_1^N$, reduces both the marginal revenue product of capital today and the value of wealth next period. This contrasts with the situation described in Proposition 1. There the possibility of future financial constraints raises capital stock above that observed under pure irreversibility but leaves it below the frictionless level. Although wealth is an increasing function of capital at that point, the firm does not hold as much capital as in the frictionless case. It balances the impact on marginal revenue of being financially constrained when $e > e_F$ (through having too little capital ex post) against the impact on marginal revenue of being irreversibility constrained when $e < e_I$ (through holding too much capital ex post).

3 Implications for Empirical Work

Our results highlight contrasting implications of the introduction of irreversibility and financial constraints. Comparison of Propositions 1 & 2 suggests that the introduction of irreversibility slows investment under financial constraints, because with both constraints, investment remains lower than in the frictionless case. Meanwhile, Proposition 1 says that the possibility of future financial constraints may raise investment under both frictions above that observed under irreversibility alone. This also contrasts the impact of future financial constraints, with that of current financial constraints, which lower investment.

These implications may be helpful in assessing the relative contribution of these frictions to investment behaviour in greater detail than heretofore. However, studies of irreversibility in investment which fail to control for financial constraints may be compromised by incorrectly ascribing to irreversibility, or to the combination of both frictions, an effect that is really due to financial constraints alone. In particular, when financial constraints do not bind today but may do so in future, the effect of irreversibility may be attenuated (Proposition 1) and may thus be difficult to detect, while when financial constraints do bind today, they alone produce an effect with the same sign as that of irreversibility alone (Lemma 1, Propositions 1 and 2). Finally, since irreversibility exacerbates the effect of (current) financial constraints, it may be important to control for the effect of the former friction when assessing the impact of financial constraints on investment.

4. Summary

We developed a model of investment under financial frictions and irreversibility. The possibility of encountering financial constraints in the future raises current investment over that arising under irreversibility alone. Investment remains lower than in the frictionless case. These results cast doubt on empirical work on the effects of irreversibility that does not control for financial frictions, and suggest how one might distinguish between the effects of both frictions.

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Appendix

Proof of Lemma 1 Notice that in equation (2) $\beta\int_{-\infty}^{e_I^I}\left[R_K\left(K_1^I,e\right)-P\right]dG(e)<0$, since $R_K\left(K_1^I,e_I^I\right)=R_K\left(K_2\left(e_I^I\right),e_I^I\right)=P$ and $R_{Ke}(\cdot,\cdot)>0$. Comparison of equations (1) and (2) shows that we must have $r'\left(K_1^I\right)>r'\left(K_1^N\right)$, so $K_1^I<K_1^N$ as required.■

Proof of Lemma 2 a) From the date 1 financial constraint $W_1-P\cdot K_1\geq 0$. Now $W_1>0$ by assumption and $r(K_1)>0$ for $K_1>0$. So $W_1-PK_1+r(K_1)>0$, $\forall K_1\in\left[0,W_1/P\right]$ and $K_2(K_1)>K_1$.
b) $dK_2(K_1)/dK_1=\left[r'(K_1)+\beta P-P\right]/P$. Now $r'(K_1)>0$ is decreasing in K_1 and from equation (1) $r'\left(K_1^N\right)+\beta P-P=0$ so the desired result follows.
c) $d^2K_2(K_1)/(dK_1)^2=r''(K_1)/P<0$.■

Proof of Lemma 3:

$$\frac{\partial J}{\partial K_1^*}=\left\{\begin{aligned}&r''\left(K_1^*\right)+\beta\int_{-\infty}^{e_I^*\left(K_1^*\right)}R_{KK}\left(K_1^*,e\right)dG(e)+\beta\left[R_K\left(K_1^*,e_I^*\left(K_1^*\right)\right)-P\right]g\left(e_I^*\left(K_1^*\right)\right)\frac{\partial e_I^*}{\partial K_1^*}\\&+\frac{d^2K_2}{\left(dK_1^*\right)^2}\beta\int_{e_I^*\left(K_1^*\right)}^{\infty}\left[R_K\left(K_1^*,e\right)-P\right]dG(e)+\left(\frac{dK_2}{dK_1^*}\right)^2\beta\int_{e_F^*\left(K_1^*\right)}^{\infty}R_{KK}\left(K_1^*,e\right)dG(e)\\&-\beta\left[R_K\left(K_2\left(K_1^*\right),e_F^*\left(K_2\left(K_1^*\right)\right)\right)-P\right]g\left(e_F^*\left(K_2\left(K_1^*\right)\right)\right)\frac{\partial e_F^*}{\partial K_2}\frac{dK_2}{dK_1^*}\end{aligned}\right\}$$

For any thresholds $e_I^*\left(K_1^*\right)$, $e_F^*\left(K_2\left(K_1^*\right)\right)$, we have $R_K\left(K_1^*,e_I^*\left(K_1^*\right)\right)=P$ and $R_K\left(K_2\left(K_1^*\right),e_F^*\left(K_2\left(K_1^*\right)\right)\right)=P$, by definition. So we simplify to give:

$$\frac{\partial J}{\partial K_1^*}=\left\{\begin{aligned}&r''\left(K_1^*\right)+\beta\int_{-\infty}^{e_I^*\left(K_1^*\right)}R_{KK}\left(K_1^*,e\right)dG(e)+\frac{d^2K_2}{\left(dK_1^*\right)^2}\beta\int_{e_F^*\left(K_2\left(K_1^*\right)\right)}^{\infty}\left[R_K\left(K_2\left(K_1^*\right),e\right)-P\right]dG(e)\\&+\left(\frac{dK_2}{dK_1^*}\right)^2\beta\int_{e_F^*\left(K_2\left(K_1^*\right)\right)}^{\infty}R_{KK}\left(K_2\left(K_1^*\right),e\right)dG(e)\end{aligned}\right\}$$

Now $r''(\cdot)<0$ and $R_{KK}(\cdot,\cdot)<0$ so the first and second terms are negative for $K_1^*>0$. For the third term $R_K\left(K_2\left(K_1^*\right),e\right)>P$ for $e>e_F^*\left(K_2\left(K_1^*\right)\right)$, so the integral is positive. However, $d^2K_2/\left(dK_1^*\right)^2=r''\left(K_1^*\right)/P<0$ from Lemma 2. So the third term is negative. For the final term, the integral is negative, while $\left(dK_2/dK_1^*\right)^2$ is zero for $K_1^*=K_1^N$ and positive otherwise. Therefore the final term is non-positive. As a consequence $\partial J/\partial K_1^*<0$. ■

Proof of Proposition 2

a) $V^{F'}\left(K_1^F\right)-V^{N'}\left(K_1^N\right)=r'\left(K_1^F\right)-r'\left(K_1^N\right)+\frac{dK_2}{dK_1^F}\int_{e_F^F}^{\infty}\left[R_K\left(K_2\left(K_1^F\right),e\right)-P\right]dG(e)$.

Notice that this expression equals 0 when $K_1^F=K_1^N$, because $\frac{dK_2}{dK_1^F}=0$ by Lemma 2. By contrast if $K_1^F<K_1^N$ then $r'\left(K_1^F\right)>r'\left(K_1^N\right)$ and $\frac{dK_2}{dK_1^F}>0$ by Lemma 2, so $V^{F'}\left(K_1^F\right)-V^{N'}\left(K_1^N\right)>0$. While if $K_1^F>K_1^N$ then $r'\left(K_1^F\right)<r'\left(K_1^N\right)$ and $\frac{dK_2}{dK_1^F}<0$ by Lemma 2, so $V^{F'}\left(K_1^F\right)<V^{N'}\left(K_1^N\right)>0$. Therefore we have $K_1^F=K_1^N$.

b) $V^{F'}\left(\overline{K}_1^F\right)-V^{N'}\left(K_1^N\right)=r'\left(\overline{K}_1^F\right)-r'\left(K_1^N\right)+\frac{dK_2}{d\overline{K}_1^F}\int_{e_F^F}^{\infty}\left[R_K\left(K_2\left(\overline{K}_1^F\right),e\right)-P\right]dG(e)=\lambda P>0$

Now if $\overline{K}_1^F=K_1^F=K_1^N$, then $V^{F'}\left(\overline{K}_1^F\right)-V^{N'}\left(K_1^N\right)=0<\lambda P$, so $\overline{K}_1^F\neq K_1^F$. Also $V^{F'}\left(K_1^*\right)$ is decreasing in K_1^* so we must have $\overline{K}_1^F>K_1^F=K_1^N$ ■