E C O N O M I C S   B U L L E T I N

Efficient equilibrium side contracts

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Abstract

We analyze two−stage games where players may make binding offers of schemes for side payment acceptance (or rejection) as well as those for side payments before choosing actions. We find that any set of efficient actions maximizing the total payoff is played on an equilibrium path of the two−stage game when such bilateral contracts on side payments are interdependent.

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1. Introduction

Coase (1960) put forth an idea that if property rights are well-defined, and bargaining is costless, then rational agents playing a game with externalities should contract to come to an efficient point. Coase (1960) was not explicit about the type of agreements between agents that are necessary as a form of bargaining to reach efficiency, but the idea has been widely accepted by economists.

Contrary to the widespread belief in the idea, Jackson and Wilkie (2000) pointed out that side contracting does not always lead to efficiency even when there are no transactions costs, complete information, and binding contracts. They studied games where agents may make binding offers of strategy-contingent side payments before choosing actions, and found that if there are only two agents, the agents are not always able to come to an agreement that supports an efficient strategy profile as an equilibrium point of the game. What kind of contracts will agents need to reach efficiency generally?

According to Jackson and Wilkie (2000), if there are three or more players, each efficient strategy profile is played on an equilibrium path in the game with side payments. However, Jackson and Wilkie (2000) only focused on voluntarily offered side payments and assumed that such side payments would always be accepted by transferees. This assumption might be thought of as arbitrary since voluntarily offered side payments could be invalidated by spontaneous rejection to receive them. Moreover, the results of Jackson and Wilkie (2000) depended upon another assumption as well that there is no budget constraint with players' transfer. Thus the question proposed above seems to remain unanswered. What kind of bilateral contracts would lead to efficiency even when agents face budget constraint with their transfer, no matter what number of players there are? This is the question we address in this paper.

We are to analyze two-stage games where players may make binding offers of schemes for side payment acceptance (or rejection) as well as those for side payments before choosing actions. A side payment from a player, say 1, to another, say 2, is implemented if and only if 1 offers the payment and 2 accepts it. If 2 rejects, then 1's offer is not in effect, and the payoff for the transfer remains with 1. We will see that every efficient strategy profile is played on an equilibrium path of the two-stage game, no matter what number of players there are, when the bilateral side contracts (transfer and receipt schemes) are somehow interdependent. Moreover, we will reach a similar result even when equilibrium contracts are required to meet agents' budget constraint with their transfer.

In what follows we present the model in Section 2 and our analysis in Section 3. Our concluding remarks appear in Section 4.

2. The Model

We consider two-stage games played as follows.

Stage 1: Each player announces a transfer function profile (transfer scheme) and a receipt function profile (transfer acceptance/rejection scheme), each of which is assumed to be binding.
Stage 2: Each player chooses an action.

2.1 The Underlying Game

The players are given by a set $N = \{1, \ldots, n\}$. A player i's finite pure strategy space in the second stage game is denoted by X_i, with $X = \times_i X_i$. Let $\Delta(X_i)$ denote the set of mixed strategies for i, and let $\Delta = \times_i \Delta(X_i)$. We denote by x_i, x, μ_i, and μ generic elements of X_i, X, $\Delta(X_i)$, and Δ respectively. For simplicity, we sometimes use x_i and x to denote μ_i and μ respectively that place probability one on x_i and x. A player i's payoffs in the second stage game are given by a von Neumann-Morgenstern utility function $v_i : X \to \mathbb{R}$.

2.2 The Contracts

We are interested in the contracts that are interdependent: each agent's transfer scheme (indirectly) depends on the others' receipt schemes and each agent's receipt scheme depends on the others' transfer schemes.

A transfer function profile announced by player i in the first stage is denoted by $t_i = \left(t_{i1}, \ldots, t_{i(i-1)}, t_{i(i+1)}, \ldots, t_{in}\right)$, where $t_{ij} : X \times Z \to \mathbb{R}_+$ with $Z = \{0, 1\}$ represents i's promises to j as a function of actions chosen in the second stage and indicators 0 and 1. Indicator 0 means that according to the transfer and receipt schemes announced in the first stage, a player rejects transfer from some other. Indicator 1 means that according to the transfer and receipt schemes announced in the first stage, every player accepts transfer from any other.

Note that if $t_{ij}(x,z) = z\tau_{ij}(x)$ for some $\tau_{ij} : X \to \mathbb{R}_+$, then the transfer scheme becomes degenerate, or $t_{ij}(x,z) = 0$ for all x, unless every player accepts transfer from the others. That is, when players are expected to promise acceptance to each other, such transfer function can be sensitive to a player's deviation on the receipt scheme.

Let T be the set of all possible t_{ij}. Let $t = (t_1, \ldots, t_n)$. A transfer function profile $t_i = \left(t_{i1}, \ldots, t_{i(i-1)}, t_{i(i+1)}, \ldots, t_{in}\right)$ announced by player i meets his budget constraint if $\sum_{j \neq i} t_{ij}(x,z) \leq \max\{0, v_i(x)\}$ for all x and all z. A profile $t = (t_1, \ldots, t_n)$ of transfer function profiles is called feasible if every t_i meets i's budget constraint.

A receipt function profile announced by player i in the first stage is denoted by $r_i = \left(r_{i1}, \ldots, r_{i(i-1)}, r_{i(i+1)}, \ldots, r_{in}\right)$, where $r_{ij} : \left(T^{n-1}\right)^n \to \{0,1\}$ represents i's acceptance (1) or rejection (0) of transfer from j as a function of profiles of transfer function profiles announced in the first stage. Let $r = (r_1, \ldots, r_n)$.

Given a profile t of transfer function profiles and a profile r of receipt function profiles in the first stage, and a play x in the second stage game, the payoff U_i to player i becomes

$$U_i(x,t,r) = v_i(x) + \sum_{j \neq i} \left(r_{ij}(t)\, t_{ji}(x, a(t,r)) - r_{ji}(t)\, t_{ij}(x, a(t,r))\right)$$

where $a(t,r) = \times_{i,j,i \neq j} r_{ij}(t)$.

Given a profile t of transfer function profiles and a profile r of receipt function profiles in the first stage, and a play μ in the second stage game, the expected payoff EU_i to player i becomes

$$EU_i(\mu,t,r) =$$
$$\sum_x \times_k \mu_k(x_k) \left(v_i(x) + \sum_{j \neq i} \left(r_{ij}(t)\, t_{ji}(x, a(t,r)) - r_{ji}(t)\, t_{ij}(x, a(t,r))\right)\right)$$

where $a(t,r) = \times_{i,j,i \neq j} r_{ij}(t)$. Let $EU_i(\mu) = \sum_x \times_k \mu_k(x_k)\, v_i(x)$.

Let $NE(t,r)$ denote the set of (mixed) Nash equilibria of the second stage game given (t,r) in the first stage. Let NE represent the set of (mixed) Nash equilibria of the underlying game (the second stage game without side contracts).

A pure strategy profile $x \in X$ of the second stage game together with a vector $u \in \mathbb{R}^n$ of payoffs such that $\sum_i u_i = \sum_i v_i(x)$ is *supportable* if there exists a subgame perfect equilibrium of the two stage game where some t and some r are announced in the first stage and x is played in the second stage on the equilibrium path, and $U_i(x,t,r) = u_i$.

A pure strategy profile $x \in X$ of the second stage game together with a vector $u \in \mathbb{R}^n$ of payoffs such that $\sum_i u_i = \sum_i v_i(x)$ is *feasibly supportable* if there exists a subgame perfect equilibrium of the two stage game where some feasible t and some r are announced in the first stage and x is played in the second stage on the equilibrium path, and $U_i(x,t,r) = u_i$.

3. Analysis

The following proposition holds in the model, which implies that any set of efficient actions maximizing the total payoff is supportable with some payoff distribution.

Proposition 1. $(\overline{x}, \overline{u})$ *such that* $\sum_i \overline{u}_i = \sum_i v_i(\overline{x})$ *is supportable if there exists* ${}_i\mu$ *for all* i *such that* ${}_i\mu \in NE$ *and* $EU_i({}_i\mu) \leq \overline{u}_i$.

Proof of Proposition 1. Suppose for $(\overline{x}, \overline{u})$ with $\sum_i \overline{u}_i = \sum_i v_i(\overline{x})$, there exists ${}_i\mu$ for all i such that ${}_i\mu \in NE$ and $EU_i({}_i\mu) \leq \overline{u}_i$.

Consider $\tau_i = \left(\tau_{i1}, \ldots, \tau_{i(i-1)}, \tau_{i(i+1)}, \ldots, \tau_{in}\right)$ where $\tau_{ij} : X \to \mathbb{R}_+$. Let $\tau = (\tau_1, \ldots, \tau_n)$ be such that $\tau_{ij}(x) = 0$ for all $x \neq \overline{x}$, $\overline{u}_i = v_i(\overline{x}) + \sum_{j \neq i} (\tau_{ji}(\overline{x}) - \tau_{ij}(\overline{x}))$, and $\tau_{ij}(\overline{x}) > 0$ for some j implies $\tau_{ji}(\overline{x}) = 0$ for all j. Let $\overline{t}$ and $\overline{r}$ be as follows.

$$\overline{t}_{ij}(x,z) = \begin{cases} z\left(\tau_{ij}(x) + \dfrac{\max\left\{0, v_i(x) + \sum_{j \neq i}(\tau_{ji}(x) - \tau_{ij}(x)) - \overline{u}_i\right\}}{n-1}\right) & \text{if } x = (\overline{x}_{-i}, x_i) \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{r}_{ij}(t) = \begin{cases} 1 & \text{if } t = \overline{t} \\ 0 & \text{otherwise} \end{cases}$$

Consider the following strategy profile (μ, t, r).

(1) $(t,r) = \left(\overline{t}, \overline{r}\right)$;

(2) if $(t,r) = \left(\overline{t}, (\overline{r}_{-i}, r_i)\right)$ for some i, where
$r_{ij}\left(\overline{t}\right) = 1$ for all $j \neq i$, then $\mu = \overline{x}$;

(2-1) if $(t,r) = \left(\left(\overline{t}_{-i}, t_i\right), (\overline{r}_{-i}, r_i)\right)$ for some i, where

$t_i \neq \bar{t}_i$ or $r_i \neq \bar{r}_i$ such that $r_{ij}\left(\bar{t}\right) = 0$ for some j, then $\mu =_i \mu$;
(2-2) otherwise $\mu \in NE\left(t, r\right)$.

Note first that for all i, $\bar{x} \in NE\left(\bar{t}, \left(\bar{r}_{-i}, r_i\right)\right)$ and $U_i\left(\bar{x}, \bar{t}, \left(\bar{r}_{-i}, r_i\right)\right) = \bar{u}_i$ if $r_{ij}\left(\bar{t}\right) = 1$ for all $j \neq i$.

Suppose $(t, r) = \left(\left(\bar{t}_{-i}, t_i\right), \left(\bar{r}_{-i}, r_i\right)\right)$ for some i, where $t_i \neq \bar{t}_i$. If $\mu = \left({}_i\mu_{-j}, \mu_j\right)$ for some j, then when $j \neq i$

$$
\begin{aligned}
&EU_j\left(\mu, t, r\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_j\left(x\right) + \sum_{k \neq j}\left(r_{jk}\left(t\right) t_{kj}\left(x, a\left(t, r\right)\right) - r_{kj}\left(t\right) t_{jk}\left(x, a\left(t, r\right)\right)\right)\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(\begin{array}{c} v_j\left(x\right) + \sum_{k \neq j, k \neq i}\left(r_{jk}\left(t\right) t_{kj}\left(x, a\left(t, r\right)\right) - r_{kj}\left(t\right) t_{jk}\left(x, a\left(t, r\right)\right)\right)\\ + \left(r_{ji}\left(t\right) t_{ij}\left(x, a\left(t, r\right)\right) - r_{ij}\left(t\right) t_{ji}\left(x, a\left(t, r\right)\right)\right)\end{array}\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_j\left(x\right) + \sum_{k \neq j, k \neq i}\left(0 \cdot 0 - 0 \cdot 0\right) + \left(0 \cdot t_{ij}\left(x, a\left(t, r\right)\right) - r_{ij}\left(t\right) \cdot 0\right)\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right) v_j\left(x\right) = EU_j\left({}_i\mu_{-j}, \mu_j\right) \leq EU_j\left({}_i\mu\right),
\end{aligned}
$$

and when $j = i$

$$
\begin{aligned}
&EU_i\left(\mu, t, r\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_i\left(x\right) + \sum_{k \neq i}\left(r_{ik}\left(t\right) t_{ki}\left(x, a\left(t, r\right)\right) - r_{ki}\left(t\right) t_{ik}\left(x, a\left(t, r\right)\right)\right)\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_i\left(x\right) + \sum_{k \neq i}\left(r_{ik}\left(t\right) \cdot 0 - 0 \cdot t_{ik}\left(x, a\left(t, r\right)\right)\right)\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right) v_i\left(x\right) = EU_i\left({}_i\mu_{-i}, \mu_i\right) \leq EU_i\left({}_i\mu\right) \leq \bar{u}_i.
\end{aligned}
$$

Suppose $(t, r) = \left(\bar{t}, \left(\bar{r}_{-i}, r_i\right)\right)$ for some i, where $r_i \neq \bar{r}_i$ such that $r_{ij}\left(\bar{t}\right) = 0$ for some j. If $\mu = \left({}_i\mu_{-j}, \mu_j\right)$ for some j, then when $j \neq i$

$$
\begin{aligned}
&EU_j\left(\mu, t, r\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_j\left(x\right) + \sum_{k \neq j}\left(r_{jk}\left(t\right) t_{kj}\left(x, a\left(t, r\right)\right) - r_{kj}\left(t\right) t_{jk}\left(x, a\left(t, r\right)\right)\right)\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(\begin{array}{c} v_j\left(x\right) + \sum_{k \neq j, k \neq i}\left(r_{jk}\left(t\right) t_{kj}\left(x, a\left(t, r\right)\right) - r_{kj}\left(t\right) t_{jk}\left(x, a\left(t, r\right)\right)\right)\\ + \left(r_{ji}\left(t\right) t_{ij}\left(x, a\left(t, r\right)\right) - r_{ij}\left(t\right) t_{ji}\left(x, a\left(t, r\right)\right)\right)\end{array}\right)\\
=&\sum_x \times_k \mu_k\left(x_k\right)\left(v_j\left(x\right) + \sum_{k \neq j, k \neq i}\left(1 \cdot 0 - 1 \cdot 0\right) + \left(1 \cdot 0 - r_{ij}\left(t\right) \cdot 0\right)\right)
\end{aligned}
$$

$$=\sum_x \times_k \mu_k (x_k)\, v_j(x) = EU_j\left({}_i\mu_{-j}, \mu_j\right) \leq EU_j\left({}_i\mu\right),$$

and when $j = i$

$$EU_i(\mu, t, r)$$
$$=\sum_x \times_k \mu_k(x_k)\left(v_i(x) + \sum_{k\neq i}\left(r_{ik}(t)\, t_{ki}(x, a(t,r)) - r_{ki}(t)\, t_{ik}(x, a(t,r))\right)\right)$$
$$=\sum_x \times_k \mu_k(x_k)\left(v_i(x) + \sum_{k\neq i}\left(r_{ik}(t)\cdot 0 - 1\cdot 0\right)\right)$$
$$=\sum_x \times_k \mu_k(x_k)\, v_i(x) = EU_i\left({}_i\mu_{-i}, \mu_i\right) \leq EU_i\left({}_i\mu\right) \leq \overline{u}_i.$$

Thus, (1)-(2-2) constitutes a subgame perfect equilibrium where $\left(\overline{t}, \overline{r}\right)$ is announced in the first stage and $\overline{x}$ is played in the second on the equilibrium path, and $U_i\left(\overline{x}, \overline{t}, \overline{r}\right) = \overline{u}_i$. ■

Remark 1. Proposition 1 corresponds to Theorem 6 in Jackson and Wilkie (2000). In contrast to that theorem, Proposition 1 holds even for two-player games.

Remark 2. Note that the side contracts $\left(\overline{t}, \overline{r}\right)$ proposed in the proof of Proposition 1 is equipped with a kind of invalidation mechanism which makes it possible to punish any individual deviation. If some i deviate to reject transfer from some other $(z = 0)$ when all the players j promise $\overline{t}_j$, then $\overline{t}(x, z) = \mathbf{0}$, or no transfer takes place, and ${}_i\mu$ becomes an equilibrium in the second stage game which punishes i. At the same time, if some i deviates from $\overline{t}_i$ $(t \neq \overline{t})$ while the others $j \neq i$ promise $\overline{t}_j$ and $\overline{r}_j$, then $\overline{r}_{-i}(t) = \mathbf{0}$ $(z = 0)$, which in turn results in $\overline{t}_{-i}(x, z) = \mathbf{0}$. Thus again no transfer takes place, and ${}_i\mu$ becomes an equilibrium strategy in the second stage game which punishes i.

Remark 3. It is a corollary of Proposition 1 that equilibrium strategies and outcomes of the underlying game are supportable.

Corollary 1. *If $x \in NE$, then $(x, v(x))$ is supportable.*

Note that $\overline{t}$ in the proof of Proposition 1 is sure to be feasible when $\overline{u}_i \geq 0$ for all i. That is, even the following proposition holds in the model, which implies that any set of efficient actions maximizing the total payoff is feasibly supportable with some payoff distribution if there exists an equilibrium of the underlying game in which each player enjoys nonnegative payoff without side payments.

Proposition 2. $(\overline{x}, \overline{u})$ *such that* $\sum_i \overline{u}_i = \sum_i v_i(\overline{x})$ *and* $\overline{u}_i \geq 0$ *is feasibly supportable if there exists* ${}_i\mu$ *for all* i *such that* ${}_i\mu \in NE$ *and* $EU_i({}_i\mu) \leq \overline{u}_i$.

4. Concluding Remarks

We found that there is a class of (feasible) side contracts which may induce play of efficient actions in equilibria not only in three-or-more-player games but also in two-player games. What to do next is to see whether the contracts proposed here are the simplest ones in the class. In fact we already know that there exist simpler (feasible) side contracts which may lead to efficiency for two-player games (Yamada 2002). We will find out whether three-or-more-player games also have such alternatives.

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