ECONOMICS BULLETIN

Multifractality: Theory and Evidence an Application to the French Stock Market

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Abstract

This article presents the basics of multifractal modelling and shows the multifractal properties of the French Stock Market (CAC40). Monte Carlo simulations prove that the Multifractal Model of Asset Returns (MMAR) is a better model to replicate the scaling properties observed in the CAC40 series than alternative specifications like GARCH or FIGARCH.

I would like to thank Julien Matheron and Tristan−Pierre Maury for their helpful comments. All remaining errors are mine.

Citation: Fillol, Jérôme, (2003) "Multifractality: Theory and Evidence an Application to the French Stock Market." *Economics Bulletin*, Vol. 3, No. 31 pp. 1−12

Submitted: September 10, 2003. **Accepted:** November 29, 2003.

URL: http://www.economicsbulletin.com/2003/volume3/EB−03C50005A.pdf

1 Introduction

The Multifractal Model of Asset Returns (MMAR) was introduced by Calvet *et al.* (1997abc) and Calvet and Fisher (2002) as a development of Mandelbrot's findings (1974) in the field of multifractal measures. MMAR processes pay tribute to the fractional Brownian motion theory (Mandelbrot and Ness, 1968) by incorporating the trading time approach[1] (Mandelbrot and Taylor, 1967). MMAR processes were built to exhibit the same properties as financial series: long-range dependence and long-tails. Authors have developed a method - based on scaling function and multifractal spectrum - to detect multifractality of a process. These tools have allowed Calvet and Fisher (2002) to detect the multifractal properties of the Deutsche Mark / US Dollar exchange rate. We apply these tools to the CAC40 series[2].

Section 2 presents MMAR modelisation. Section 3 introduces multifractal formalism. In Section 4, we detect the multifractality of the CAC40 and we model it with MMAR. Using Monte Carlo simulations we compare the performances of MMAR and alternative specifications (GARCH, FIGARCH) to replicate the scaling properties exhibited in CAC40. Section 5 concludes.

2 Multifractal Model

The MMAR's principle is to consider a stochastic process $\{P(t); 0 \leq t \leq T\}$

$$X(t) = \ln P(t) - \ln P(0) \tag{1}$$

where $\{X(t)\}$ is a multifractal process which verifies the following three definitions[3]

Definition 1 $X(t)$ *is a compound process*

$$X(t) \equiv B_H[\theta(t)] \tag{2}$$

where $B_H(t)$ *is a fractional Brownian motion,* H *being the Hurst exponent* $(0 < H < 1)$*, and* $\theta(t)$ *is a stochastic trading time defined in Definition* 2*.*

Definition 2 *Trading time* $\theta(t)$ *is the cumulative distribution function of a multifractal measure* μ *defined on* $[0, T]$*.*

[1] This approach is based on a random time-deformation process. The trading time has been used to build financial models, see for example, Dacorogna *et al.* (1993) or Ghysels *et al.* (1996).

[2] CAC40 (quotation assisted uninterrrupted) is the main index of the French Stock Market. It is composed of the 40 most important companies quoted in the French Stock Market.

[3] Calvet and Fisher (2002) and Calvet *et al.* (1997a).

Definition 3 *$\{B_H(t)\}$ and $\{\theta(t)\}$ are independent.*

Calvet and Fisher (2002) propose an innovation in financial modelisation through Definition 2. This definition rests on the multifractal measure[4] which is paramount to the MMAR. To build a MMAR it is necessary to estimate the parameters of the trading time $\theta(t)$ and the Hurst exponent of the fractional Brownian motion (B_H). The next section presents the multifractal formalism whose tools allow us to detect the multifractality of the CAC40 and to estimate the MMAR parameters.

3 Multifractal Formalism

3.1 Scaling Function and Multifractal Spectrum

The scaling function's notion is extracted from multifractal formalism. We present Calvet and Fisher's (2002) definition.

Definition 4 *A stochastic process $\{X(t)\}$ is called multifractal if it has stationary increments and satisfies*

$$E\left(|X(t)|^q\right) = c(q)\, t^{\tau(q)+1}, \quad \textit{for all } t \in B, q \in Q \tag{3}$$

where, E(.) is the expectation operator and c(q) is called the prefactor, B and Q are intervals on the real line. Moreover, B and Q have positive lengths, and $0 \in B$, $[0,1] \subseteq Q$.

The scaling function is denoted $\tau(q)$ and takes into account the influence of the time t on the moments q. To explain the scaling function's notion, we briefly present the scaling function in the particular case of an unifractal process like the fractional Brownian motion. A fractional Brownian motion, with a Hurst exponent H, satisfies[5]

$$X(t) = t^H X(1) \tag{4}$$

which implies that

$$E\left(|X(t)|^q\right) = t^{Hq} E\left(|X(1)|^q\right) \tag{5}$$

In this special case, we obtain the prefactor $c(q) = E\left(|X(1)|^q\right)$ and the scaling function $\tau(q) = Hq - 1$. So, the scaling function is linear if the process is unifractal. On the contrary, multifractal processes present multiscaling properties that imply the nonlinearity of the scaling function.

[4] We present the notion of multifractal measure in Appendix 1.

[5] See for example Baillie (1996).

The scaling function exhibit interesting properties. Indeed, Calvet and Fisher (2002) show that the scaling function $\tau(q)$ presents the following properties

$$\tau\left(\frac{1}{H}\right) = 0 \quad (6)$$
$$\tau_X(q) = \tau_\theta(Hq) \quad (7)$$

which allow us to estimate the MMAR parameters. The relation (6) gives the particular value (inverse of the Hurst exponent) for which the scaling function is equal to zero. So (6) allows us to estimate the Hurst exponent $\widehat{H}$ of the process $X(t)$[6] (i.e $B_{\widehat{H}}$ in the MMAR). The relation (7) connects the scaling function of the price series τ_X to the scaling function of the trading time τ_θ. So, the estimation of the scaling function of the price series allows us to know the properties of the trading time (i.e $\theta(t)$ in the MMAR).

It is possible to detect a multifractal process by studying the multifractal spectrum, which is defined by the following theorem[7].

Theorem 1 *The multifractal spectrum $f(\alpha)$ is the Legendre transform of the scaling function $\tau(q)$*

$$f(\alpha) = \inf_q [q\alpha - \tau(q)] \quad (8)$$

A unifractal process presents a linear scaling function and a multifractal spectrum that equals to a single point $H : f(\alpha) = H$. A multifractal process presents a nonlinear scaling function and a concave multifractal spectrum[8].

In the next section, we present estimations of the scaling function and of the multifractal spectrum with the partition function.

3.2 Partition Function

To estimate the scaling function, Calvet and Fisher (2002) propose a method based on the partition function. The partition function of X_t is denoted $\pi_\delta(X, q)$, defined for each moment q, and obtained by partitioning the series into n subintervals of length δ

$$\pi_\delta(X, q) = \sum_{i=1}^{n} \left| X_{\lceil i \cdot \delta \rceil} - X_{\lceil (i-1) \cdot \delta \rceil} \right|^q \quad (9)$$

where $\lceil \cdot \rceil$ is the integer part operator.

[6] The scaling function methodology to estimate the long memory parameter is presented in Fillol and Tripier (2003).
[7] See Calvet *et al.* (1997b).
[8] Notice that the particular value of α_0 satisfaying $f(\alpha_0) = 1$ is the maximum of the multifractal spectrum.

Using (3) gives us

$$\log\left(\pi_\delta\left(X,q\right)\right) = \tau\left(q\right)\log\left(\delta\right) + \log\left(c_q\right) + \log\left(T\right) \tag{10}$$

For a given series X_t, computing its partition function according to (9) for various moments q allows us to deduce its scaling function according to (10). Thus, the partition function gives an estimate of the scaling function $\widehat{\tau}\left(q\right)$. Using relation (8), we obtain an estimation of the multifractal spectrum $\widehat{f}(\alpha)$

$$\widehat{f}(\alpha) = \min_q\left[q\alpha - \widehat{\tau}(q)\right] \tag{11}$$

We estimate the multifractal spectrum with the Legendre transform of the scaling function. In the next section we apply the above methodology to detect the multifractality of the CAC40 and to estimate the MMAR parameters.

4 Empirical Application

Let us consider the CAC40 series (ranging from 1990/03/01 to 2003/05/20 in daily frequencies). Our objectives are to identify multifractality in CAC40, to estimate the associated MMAR and to compare this model to GARCH and FIGARCH models[9].

4.1 Estimating Multifractal Spectrum

Figures 1-2 present estimations of the partition and scaling functions. The particular value of the scaling function : $\widehat{\tau}\left(\frac{1}{H}\right) = 0$ allows us to estimate the Hurst exponent $\widehat{H} = 0.452$. We use $\widehat{H}$ to build the MMAR. Figure 3 presents the estimation of the multifractal spectrum. The spectrum is characterized by its maximum α_0 (notice that it is also the most probable Hölder exponent). Our estimation is reported in Table 1. We obtain a concave spectrum for $\alpha < \alpha_0$, similar to a quadratic function. The concavity of the spectrum implies the multifractality of the CAC40.

4.2 Multifractal Model of CAC40

Calvet *et al.* (1997b) show that the trading time function specified in Definition 2 should be the cumulative distribution function of a multifractal random measure with lognormal masses[10]. Authors give the following formulation for the spectrum of trading time $\left(\theta\left(t\right)\right)$

$$f_\theta(\alpha) = 1 - \frac{1}{2\ln b}\left(\frac{\alpha-\lambda}{\vartheta}\right)^2 \tag{12}$$

[9] These models are usually used in finance (see for example Bollerslev (1986) and Baillie et *al.* (1996)).
[10] This construction is explained in Appendix 1.

where f is hump-shaped and symmetric around its maximum at $\alpha_0 = \lambda$.

To build the MMAR associated with the CAC40, it is necessary to estimate the lognormal parameters (λ, ϑ^2) of the trading time. We estimate $\widehat{\lambda}$ by the following theorem 2[11]

Theorem 2 *Let $f_\theta(\alpha)$ denote the multifractal process of the trading time $\theta(t)$. Under definitions* [1] − [3], *processes $X(t)$ and $P(t)$ have the same multifractal spectrum $f_X(\alpha) = f_\theta(\frac{\alpha}{H})$.*

Therefore, for the log price spectrum we have the relation $\widehat{\alpha}_0 = \widehat{\lambda}\widehat{H}$, which give us $\widehat{\lambda}$ (Table 1). To obtain $\widehat{\vartheta}^2$, we can consider a further restriction by requiring average conservation of mass, and it is easy to show that this relation implies[12]

$$\log b = 2\frac{\lambda - 1}{\vartheta^2} \tag{13}$$

Estimation[13] of ϑ^2 is reported in Table 1. Our estimations of $\widehat{\lambda}$ the $\widehat{\vartheta}^2$ allow us to construct the trading time of CAC40 and to build the MMAR. In the next section we compare the MMAR to the GARCH and FIGARCH models.

4.3 Monte Carlo Simulations[14]

For each model (MMAR, GARCH and FIGARCH) we simulate 10000 paths with the same sample size $T = 4000$. We focus our analysis on the moments $q \in \{1, 2, 3, 4, 5\}$. We compare mean values of the scaling function for each moment q. Table 2 summarizes the results. Estimations show that for each moment q the MMAR model is very close to the empirical data. MMAR is a better model to replicate the main scaling features of the data than the GARCH and FIGARCH models. Moreover, the small size of confidence intervals show the robustness of the MMAR simulations.

5 Conclusion

The method developed by Calvet *et al.* (1997abc) and Calvet and Fisher (2002) which use concepts such as scaling function and multifractal spectrum, allows us to detect multifractality of the CAC40 series. This result implies the use of the MMAR to model the CAC40 series. Monte Carlo simulations show that MMAR seems to be a better model than GARCH or FIGARCH to replicate the main scaling features observed in the financial time series.

[11] See Calvet *et al.* (1997c) for a proof.

[12] See Appendix 2.

[13] We use $b = 2$, as Calvet and Fisher (2002).

[14] We present the construction of the MMAR in Appendix 3.

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Appendix 1 - Multifractal Measure

Multifractal measures can be built by iterating a simple procedure. We present one of the simplest examples: the binomial measure on $[0,1]$[15].

Consider the uniform probability measure μ_0 on the unit interval, and two positive numbers m_0 and m_1 adding up to 1. b denotes the number of masses, here, $b=2$. In the first step of the cascade, we define a measure μ_1 by uniformly spreading the mass m_0 on the left subinterval $[0,0.5]$, and the mass m_1 on the right subinterval $[0.5,1]$. The density of μ_1 is the step function. In the second stage, we split the interval into two subintervals of equal length. The left subinterval $[0,0.25]$ is allocated a fraction m_0 of $\mu_1[0,0.5]$, whereas the right subinterval $[0.25,0.5]$ receives a fraction m_1. Applying a similar procedure to $[0.5,1]$. Iteration of this procedure generates an infinite sequence of measure (μ_k) that weakly converges to the binomial measure μ.

To build a multifractal random measure with lognormal masses, we consider a random variable M such as $-\log_b M \sim N[\lambda,\theta^2]$. At each stage of the construction we draw the masses M.

Appendix 2 - Relation (13)

Conservation of mass imposes that $E(M)=\frac{1}{b}$ where M such as $-\log_b M \sim N[\lambda,\theta^2]$.

Let u be a random variable satisifies $E(u)=1/b$ and the following relation

$$v = -\log_b(u) \sim N(\lambda,\vartheta^2) \tag{14}$$

thus,

$$log(u) \sim N\left(-[\log(b)]\lambda, [\log(b)^2]\vartheta^2\right) \tag{15}$$

Let $y \sim N(m,s^2)$ and $x = exp(y)$. x $\sim$lognormal and verifies

$$E(x) = e^{m+s^2/2} \tag{16}$$

$$V(x) = e^{2m+s^2}(e^{s^2}-1) \tag{17}$$

we then obtain

$$E(u) = \left(\frac{1}{b}\right)^{\lambda} \exp\left\{\frac{[\log(b)\vartheta]^2}{2}\right\} = \frac{1}{b} \tag{18}$$

$$\log(b) = 2\frac{\lambda-1}{\vartheta^2} \tag{19}$$

[15] Calvet and Fisher (2002).

Appendix 3 - Data simulations of the MMAR

To build a MMAR, it is necessary to build a multifractal random measure and a fractional Brownian motion.

- The Multifractal Random Measure

- The scaling function of the CAC40 allows us to estimate parameters $(\widehat{\lambda}, \widehat{\vartheta}^2)$ of the multifractal random measure with lognormal masses : $-\log_b M \sim N[\widehat{\lambda}, \widehat{\theta}^2]$.
- If a simulation of length T is desired, we choose the minimum integer number of stages k such that $2^k \geq T$. Here we consider $k = 12$.

- The fractional Brownian motion

- We simulate a fractional Brownian motion with parameter $\widehat{H} = 0.452$. This estimation is given by the particular value of the scaling function : $\widehat{\tau}\left(\frac{1}{H}\right) = 0$.

- Interpolation provides value of the path $B_H[\theta(t)]$ at the simulated values from the path $\theta(t)$.

Series	$\widehat{H}$	$\widehat{\alpha_0}$	$\widehat{\lambda}$	$\widehat{\vartheta}^2$
CAC	0.452	0.491	1.09	0.26

Table 1: MMAR parameters

	CAC	MMAR	FIGARCH	GARCH
q/τ	τ_{emp}	$\overline{\tau}$	$\overline{\tau}$	$\overline{\tau}$
1	$\mathbf{-0.52}$	$\mathbf{-0.52}$	-0.48	-0.5
		$[-0.53,-0.49]$	$[-0.51,-0.46]$	$[-0.52,-0.47]$
2	$\mathbf{-0.08}$	$\mathbf{-0.08}$	0.02	-0.0
		$[-0.13,-0.01]$	$[-0.016,0.08]$	$[-0.05,0.05]$
3	**0.31**	**0.31**	0.53	0.48
		[0.20,043]	[0.45,0.61]	[0.37,0.59]
4	**0.65**	**0.66**	1.01	0.94
		[0.45,0.86]	[0.87,1.15]	[0.75,1.14]
5	**0.94**	**0.96**	1.47	1.37
		[0.62,1.28]	[1.26,1.68]	[1.09,1.67]

τ_{emp} : is the empirical values of the CAC40 series

$\overline{\tau}$: is the estimated mean values

Table 2: Means and confidence intervals of each model

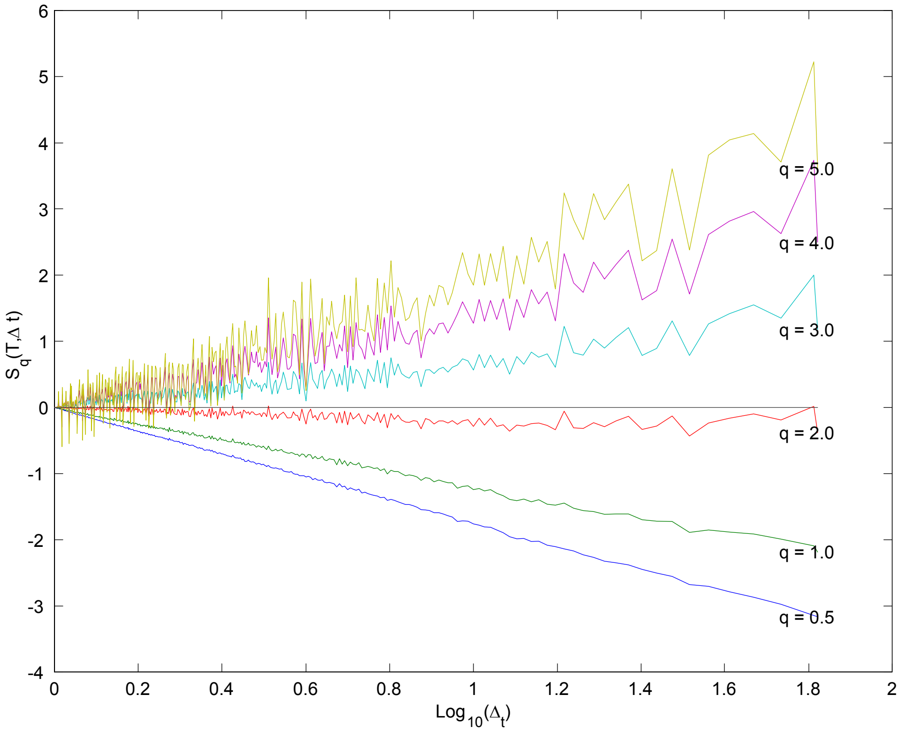

Figure 1: Partition function of the CAC40 series

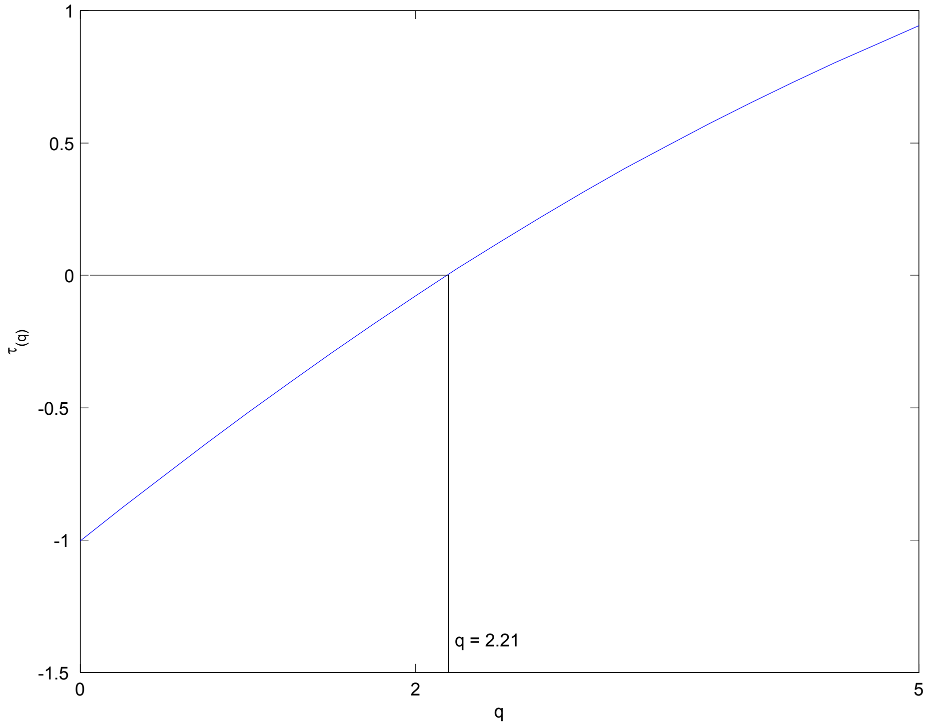

Figure 2: Scaling function of the CAC40 series ($\tau(2.21) = 0$ then $\widehat{H} = 0.452$)

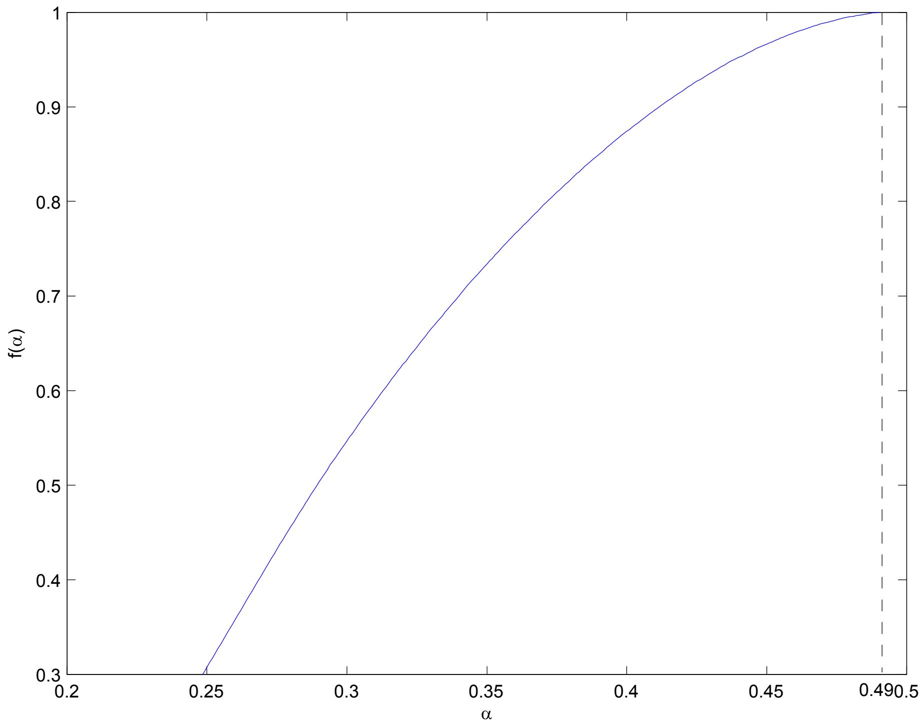

Figure 3: Multifractal spectrum of the CAC40 series