ECONOMICS BULLETIN

Rationalizability of one-to-one matchings with externalities

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Abstract

In this paper, we show that the one-to-one matching model of Mumcu and Saglam (2008) studying stability under interdependent preferences is refutable. We also give a sufficient characterization of the set of matchings that are rationalizable inside the core.

An early draft of this paper was written while the second author was affiliated with Bogazici University and the authors were respectively visiting the University of Pennsylvania and Massachusetts Institute of Technology, to which they are grateful for their hospitality. The second author also acknowledges the support of the Turkish Academy of Sciences, in the framework of Distinguished Young Scientist Award Program (TUBA-GEBIP). The usual disclaimer applies.

Citation: Mumcu, Ayþe and Ismail Saglam, (2008) "Rationalizability of one-to-one matchings with externalities." *Economics Bulletin*, Vol. 4, No. 32 pp. 1-8

Submitted: September 1, 2008. **Accepted:** November 18, 2008.

URL: http://economicsbulletin.vanderbilt.edu/2008/volume4/EB-08D70031A.pdf

1 Introduction

The last 45 years following the appearance of the seminal work by Gale and Shapley (1962) have witnessed a rapidly growing literature in matching theory studying the microfoundations of equilibrium in marriage and labor markets, in college admissions and school choice problems, and recently in organ exchange. Undoubtedly, stability, as the relevant notion of economic efficiency, has invariably been one of the main concerns of researchers and market designers in evaluating possible matching rules and procedures. While many efforts have in this literature been spent on characterizing the set of stable matchings in a given market or for a given problem, an existential question as to the validity of matching models with regard to the used stability concepts was delayed until it was very recently posed by Echenique (2008): Can there be any set of matchings for a given society or a market that is incompatible with the predictions of the matching model at hand with respect to the employed stability notions? As Echenique (2008) points out, the answer to this question is important when the preferences of individuals are unknown as it allows one to know whether a matching theory at hand has testable implications.

In this paper, we extend the inquiry of Echenique (2008) that he answers in a marriage model under independent preferences to the marriage model of Mumcu and Saglam (2008) that characterizes stable one-to-one matchings under *interdependent* preferences.[1]

Following Echenique (2008), we say that a set of matchings $\mathcal{H}$ in a given marriage market is rationalizable inside the stable set if there exists a preference profile such that the corresponding stable set contains $\mathcal{H}$. Similarly, we say that the set $\mathcal{H}$ is rationalizable inside the core if there exists a preference profile such that the corresponding core contains $\mathcal{H}$.

We show that Mumcu and Saglam's (2008) marriage model with external-

[1]Matching under externalities was also studied for marriage markets by Sasaki and Toda (1996), Roy Chowdhury (2004), and Hafalir (2008f), and for a housing market by Mumcu and Saglam (2007).

ities is refutable, since for any society facing at least two different matchings there exists at least one collection of matchings, e.g. the set of all conceivable matchings, that is not rationalizable inside the stable set or inside the core. We also give a sufficient characterization of the set of matchings that are rationalizable inside the core.

The organization of the rest of the paper is as follows: In Section 2, we introduce our model that borrows from Mumcu and Saglam (2008). We present our results in Section 3. Finally, Section 4 concludes.

2 The Model

We consider a marriage market involving a set of men, M and a set of women, W. We assume that M and W are nonempty, finite and disjoint, and satisfy $|M||W| \geq 2$, i.e. there exist at least three agents in the society and at least one member from each gender. We denote a generic agent by i, a generic man by m, and a generic woman by w. We denote the society by $N = M \cup W$.

A matching is a one-to-one function, μ, from N to itself, such that for each $m \in M$ and for each $w \in W$ we have $\mu(m) = w$ if and only if $\mu(w) = m$. Moreover, either $\mu(m) \in W$ or $\mu(m) = m$, and similarly either $\mu(w) \in M$ or $\mu(w) = w$. If $\mu(m) = w$, then m and w are matched to each other. If $\mu(i) = i$, then i is single. Let $\mathcal{M}^N$ denote the set of all matchings in society N.

Given any matching μ, let $\mu_{m,w}$ denote the matching at which (i) m and w are a couple, i.e., $\mu_{m,w}(m) = w$, (ii) their mates under μ, if they exist, become single, i.e., $\mu_{m,w}(\mu(m)) = \mu(m)$ if $\mu(m) \notin \{m, w\}$ and $\mu_{m,w}(\mu(w)) = \mu(w)$ if $\mu(w) \notin \{w, m\}$, and (iii) the marital status and the mates of all other agents are preserved, i.e., $\mu_{m,w}(i) = \mu(i)$ for all $i \notin \{m, w, \mu(m), \mu(w)\}$.

Each agent has a complete, transitive, and strict preference relation over the matchings in $\mathcal{M}^N$. P^i represents the preference relation of agent i, while $P = (P^i)_{i \in N}$ denotes the preference profile of the society. We respectively write $\mu >_i \mu'$ and $\mu \geq_i \mu'$ to mean i strictly and weakly prefers μ to μ'. A marriage market is a triple (M, W, P).

For any profile P and any $l \in \{1, 2, \dots, |\mathcal{M}^N|\}$, let $P^i[l]$ denote the lth-ranked matching from top in the ordering P^i of agent i.

We say that agent i *individually blocks* matching μ (via $\mu_{i,i}$) if $\mu_{i,i} >_i \mu$. A matching is *individually rational* if it is not individually blocked by any agent. For a given matching μ, (m, w) is a *blocking pair* if $\mu(m) \neq w$, $\mu_{m,w} >_m \mu$ and $\mu_{m,w} >_w \mu$. A matching is *stable* if it is individually rational and if there are no blocking pairs. We denote the set of stable matchings (the stable set) for the marriage market (M, W, P) by $S(M, W, P)$.

A matching $\hat{\mu}$ *dominates* another matching μ via a blocking coalition $\hat{M} \cup \hat{W}$ of men and women such that $\hat{\mu}(\hat{M} \cup \hat{W}) = \hat{M} \cup \hat{W}$, $\hat{\mu}(\mu(\hat{m})) = \mu(\hat{m})$ for any $\hat{m} \in \hat{M}$ if $\mu(\hat{m}) \notin \hat{W} \cup \{\hat{m}\}$, $\hat{\mu}(\mu(\hat{w})) = \mu(\hat{w})$ for any $\hat{w} \in \hat{W}$ if $\mu(\hat{w}) \notin \hat{M} \cup \{\hat{w}\}$, $\hat{\mu}(i) = \mu(i)$ for any $i \notin \hat{M} \cup \hat{W} \cup \mu(\hat{M} \cup \hat{W})$, and $\hat{\mu} >_i \mu$ for all $i \in \hat{M} \cup \hat{W}$. In the above definition, members of the blocking coalition can only be matched within the coalition. In addition, the previous mate, if exists, of any agent in the blocking coalition becomes single under the new matching unless he or she is inside the blocking coalition, too. Moreover, the mates and marital status of all other agents are unchanged.

The set of all matchings dominated by no other matching is called the *core* and denoted by $C(M, W, P)$.

For a given society N, let $\mathcal{H} \subset \mathcal{M}^N$ be a subset of available matchings. We say that $\mathcal{H}$ is *rationalizable inside the stable set* if there exists a preference profile P such that $\mathcal{H} \subset S(M, W, P)$. Similarly, we say that $\mathcal{H}$ is *rationalizable inside the core* if there exists a preference profile P such that $\mathcal{H} \subset C(M, W, P)$.

We simply note that a set $\mathcal{H} \subset \mathcal{M}^N$ is rationalizable inside the core only if it is rationalizable inside the stable set. Echenique (2008) shows that under independent preferences $\mathcal{M}^N$ is not rationalizable inside the stable set (equalling the core) if the number of men and the number of women are the same and at least three. We extend this result in our first proposition.

3 Results

Proposition 1. *For any society N satisfying $|M||W| \geq 2$ and having strict and interdependent preferences, $\mathcal{M}^N$ is not rationalizable inside the stable set (hence not rationalizable inside the core).*

Proof. Suppose, $\mathcal{M}^N$ is rationalizable inside the stable set by some preference profile P; i.e., $\mathcal{M}^N \subset S(M, W, P)$. Let μ^s denote the matching at which every agent is single. Pick any $(m, w) \in M \times W$. Denote by $\mu^s_{m,w}$ the matching at which (m, w) is the unique married couple. Then, $\mu^s_{m,w} >_m \mu^s$ and $\mu^s_{m,w} >_w \mu^s$ by the assumed stability of $\mu^s_{m,w}$. This implies that μ^s cannot be in $S(M, W, P)$, a contradiction. ∎

Proposition 1 establishes that the whole set of matchings cannot be rationalizable, hence our matching model is testable. As the proof of the proposition clearly shows, what drives this impossibility result is the presence of the matching, called μ^s, where every individual is single inside the set of all matchings, $\mathcal{M}^N$. In fact, the following example shows that one cannot argue that $\mathcal{M}^N \backslash \{\mu^s\}$ is not rationalizable.

Example 1. Consider $M = \{m_1, m_2\}$ and $W = \{w_1\}$. The three possible matchings are denoted by μ_1, μ_2, and μ_3. At μ_1 and μ_2, w_1 is matched to m_1 and m_2 respectively, while at μ_3 every agent is single. Let the preferences be $P^{m_1} = \mu_2\mu_1\mu_3$, $P^{m_2} = \mu_1\mu_2\mu_3$, and $P^{w_1} = \mu_1\mu_2\mu_3$. It is easy to check that $S(M, W, P) = C(M, W, P) = \{\mu_1, \mu_2\}$. So, $\mathcal{H} = \{\mu_1, \mu_2\}$ is rationalizable inside the core (hence inside the stable set).

It is then natural to ask here which proper subsets of $\mathcal{M}^N$ can be rationalizable. When the preferences are independent, Echenique (2008) is able to show that any set of matchings in which no agent is matched with the same partner under different matchings is rationalizable. Below, we will establish a similar result under interdependent preferences. But, we have to first introduce the following definitions.

For any society N with the preference profile P, we call a proper subset $\mathcal{V}$ of matchings $\mathcal{M}^N$ *top-matching collection* if $\mathcal{V}$ is nonempty, and for all $i \in N$ we have $P^i[k] \in \mathcal{V}$ for all $k \in \{1, 2, \ldots, |\mathcal{V}|\}$. Given a society N and an agent $i \in N$, two matchings $\mu, \mu' \in \mathcal{M}^N$ are called *connected by agent* i if $\mu(i) = \mu'(i)$ and *unconnected by agent* i otherwise. Given a society N and a coalition T of agents in N, a matching μ' is *reachable by* T from another matching μ if the set of all individuals that connect μ to μ' is $N \backslash T$. Let $R(\mu, \mu')$ denote the unique coalition by which μ' is reachable from μ.

Our main result, Proposition 2 below, shows that any collection of matchings, any two members of which are unconnected by each member of the society, is rationalizable inside the core. To give the intuition underlying this result, we will simply sketch the proof here before its formal introduction. When the elements of a collection of matchings are pairwise unconnected, the unique blocking coalition that is capable to reach from one matching to another one always involves the smallest of the set of men and the set of women, say without loss of genarality the set of men. Then it suffices for us to find a preference profile under which (i) a given collection of matchings will be a top-matching collection and (ii) for any two matchings inside the given collection, there will always exist a man as a member of the corresponding blocking coalition who will be in conflict with the rest of the coalition as to the comparison of the two matchings. In the proof of Proposition 2, we use the well-known 'Condorcet cycle' over the collection of matchings commonly top-ranked by all agents in the society so as to construct a preference profile that rationalizes a given collection of pairwise-unconnected matchings inside the core.

Proposition 2. *For any society N satisfying $|M||W| \geq 2$ and having strict and interdependent preferences, consider $\mathcal{H} \subset \mathcal{M}^N$ such that $|\mathcal{H}| \leq \min\{|M|, |W|\}$ and no pair of matchings $\mu_k, \mu_l \in \mathcal{H}$ are connected by any agent in N. Then $\mathcal{H}$ is rationalizable inside the core (hence inside the stable* *set).*

Proof. Consider any society N satisfying $|M||W| \geq 2$ and having strict and interdependent preferences. Assume without loss of generality that $|M| \leq |W|$. Pick any $\mathcal{H} = \{\mu_1, \mu_2, \ldots, \mu_H\} \subset \mathcal{M}^N$ such that no pair of matchings $\mu_k, \mu_l \in \mathcal{H}$ are connected by any agent in N.

Now, enumerate agents from 1 to $|N|$, and let $M = \{1, 2, \ldots, |N| - |W|\}$, i.e. individuals enumerated with the smallest $|M|$ numbers are all men. Let P be a preference profile such that $P^i[k] = \mu_l$ with $l = (k + i - 2) \bmod_H$ for all $i \in \{1, \ldots, H\}$ and $k \in \{1, 2, \ldots, H\}$. Explicitly writing the first H components of P as

$$\begin{array}{cccccccc} P^1 & = & \mu_1 & \mu_2 & \cdots & \mu_{H-1} & \mu_H & \cdots \\ P^2 & = & \mu_2 & \mu_3 & \cdots & \mu_H & \mu_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P^H & = & \mu_H & \mu_1 & \cdots & \mu_{H-2} & \mu_{H-1} & \cdots \end{array}$$

we can notice the (well-known Condorcet) cycle over the top-H ranked matchings. Also, let $P^i = P^1$ for all $i \in \{H + 1, \ldots, |N|\}$.

Obviously, $\mathcal{H}$ is a top-matching collection under the constructed preference profile. So, no coalition of individuals can block any matching in $\mathcal{H}$ via any other matching in $\mathcal{M}^N \backslash \mathcal{H}$. It is obvious that $C(M, W, P) = \mathcal{H}$ if $|\mathcal{H}| = 1$. Now, suppose $|\mathcal{H}| \geq 2$. Pick any two matchings $\mu, \mu' \in \mathcal{H}$ such that $\mu \neq \mu'$. We have $R(\mu, \mu') \supset M \supset \{1, 2, \ldots, H\}$, since $H \leq |M| = \min\{|M|, |W|\}$ by assumption. Then, by the construction of $\langle P^1, \ldots, P^H \rangle$, there always exists an agent (a man) in $R(\mu, \mu')$ who prefers μ to μ' and withstands the coalition $R(\mu, \mu')$. Since μ and μ' were arbitrary, we have $\mathcal{H} \subset C(M, W, P)$. ∎

Example 2. Let $M = \{m_1, m_2\}$ and $W = \{w_1\}$, and consider the three possible matchings μ_1, μ_2, and μ_3 as defined in Example 1. The sets $\mathcal{H}_1 = \{\mu_1\}$ and $\mathcal{H}_2 = \{\mu_2\}$ satisfy the connectedness hypothesis in Proposition 2. Moreover, $|\mathcal{H}_1| = |\mathcal{H}_2| = 1 = \min\{|M|, |W|\}$. Therefore $\mathcal{H}_1$ and $\mathcal{H}_2$ are both rationalizable by Proposition 2. Indeed, the preference profiles that rationalize $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively place μ_1 and μ_2 at the top position for each agent.

We should finally remark that the studied refutable matching model is not exactly identifiable, as similarly to Echenique (2008) there may exist many different preference profiles that rationalize some sets of matchings.

4 Concluding Remarks

In this paper, we have showed that Mumcu and Saglam's (2008) marriage model with externalities is refutable, and hence it has testable implications (Proposition 1). We have also established that if a collection of matchings is not rationalizable inside the core, then some agents must have the same mate under more than one matching (Proposition 2). We should here emphasize that our second result simply characterizes collections of matchings that are not rationalizable. However, a sufficiency result such as Proposition 2 is still valuable, as already remarked by Echenique (2008) in his framework of independent preferences, since it has an important implication for empirical tests of matching theory at hand, requiring some pairs of agents to be identified under more than one matching in the available data.

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