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Discounting and efficiency in coalitional bargaining with random proposers

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Abstract

This paper analyzes a random−proposer coalitional bargaining game with different discount factors, which is a generalized version of Okada's (1996) model. We consider limit subgame efficiency which means that when the discount factors are sufficiently close to unity, the full coalition is formed in each subgame. In this paper, a negative result is shown: The limit subgame efficiency is attained if and only if values of the characteristic function are zero for all coalitions but the grand coalition. This result implies that under different discount factors, even under a naturally generalized condition of Okada's necessary and sufficient condition for the limit subgame efficiency, the limit subgame efficiency is not necessarily achieved. On the other hand, it is shown that under a condition on the region of players' discount factors, the generalized condition of Okada's condition is almost necessary and sufficient for the limit subgame efficiency.

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1. Introduction

Okada (1996) introduced a noncooperative coalitional bargaining game with random proposers, in which a player is randomly recognized as a proposer in each round. He showed that in equilibrium of the game, no delay occurs. Also, he provided the equivalent condition for the limit subgame efficiency, which means that when the discount factor is sufficiently close to unity, the full coalition is formed in each subgame. The condition is that values of coalitions *per capita* are monotonic with respect to sizes of coalitions.

This paper analyzes a generalized game of Okada's model. In the generalized game, players' discount factors and recognition probabilities are different. Especially, difference of discount factors is a key factor of this paper. Under different discount factors, even if the equivalent condition for the limit subgame efficiency of Okada (1996) holds, the limit subgame efficiency is not necessarily achieved (Example 1). This is intuitively explained as follows: A proposer may obtain a larger payoff by proposing subcoalitions with less patient players than the full coalition because by proposing such coalitions, she does not have to give larger allocations to more patient players, whose approval seems expensive. Thus, under different discount factors, the proposer may have an incentive to propose subcoalitions. Moreover, however close to 1 each player's discount factor is (however small the difference among players' discount factors is), the difference among players' discount factors matters.

In this paper, we show a negative result: In the generalized Okada model, the limit subgame efficiency is attained if and only if values of the characteristic function are 0 for all coalitions but the grand coalition. On the other hand, it is shown that under a condition on the region of players' discount factors, a naturally generalized condition of Okada's condition above is almost necessary and sufficient for the limit subgame efficiency.

This paper is related to Yan (2002), which generalized a variant of Okada model. In the generalized model, recognition probabilities are different but discount factors are common. According to Yan (2002), if the core of the underlying characteristic function form game is not empty, the limit efficiency is attained in a noncooperative game with some recognition probability tuple. The results of this paper, however, imply that under different discount factors, even if the core is not empty, the limit efficiency does not hold for any recognition probability tuple.

The paper is organized as follows: Section 2 defines a coalitional bargaining game; Section 3 investigates the efficiency.

2. Model

Take a characteristic function form game (N, v). Suppose that $N \equiv \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ such that $n \geq 2$. Let $\mathfrak{C} \equiv 2^N \setminus \{\emptyset\}$. Suppose that $v(\{i\}) = 0$ for all $i \in N$, $v(N) > 0$ and $\forall S, T \in \mathfrak{C} : S \cap T = \emptyset \Rightarrow v(S \cup T) \geq v(S) + v(T)$.

For $\delta \equiv (\delta_k)_{k \in N} \in (0,1)^n$, define a noncooperative bargaining game $G(\delta)$, which is a generalized model of Okada (1996) with different discount factors and different recognition probabilities.

The structure of the game is inductively defined. In the game, there are several states. The set of states is $\mathfrak{S} \equiv \left\{S \in 2^N \mid v(S) > 0\right\}$. At state $S \in \mathfrak{S}$, a player $i \in S$ is selected as a proposer with probability p_i^S, where $p^S \equiv \left(p_k^S\right)_{k \in S} \in \left\{(p_k)_{k \in S} \in \mathbb{R}_{++}^{|S|} \mid \sum_{k \in S} p_k = 1\right\}$. The proposer i proposes a pair of a coalition including i and a payoff distribution for the coalition,

$$(C, x) \in \left\{ \left(C', (x'_k)_{k \in C'}\right) \mid C' \in 2^S \wedge i \in C' \wedge (x'_k)_{k \in C'} \in \mathbb{R}_+^{|C'|} \wedge \sum_{k \in C'} x'_k \leq v(C') \right\}.$$

Then, each member in the proposed coalition C other than proposer i announces accepting or rejecting the proposal according to some predetermined order. If every responder accepts the proposal, the proposal is enforced and the state goes to state $S \setminus C$ (if $v(S \setminus C) = 0$, the game ends). Otherwise, the state remains to be S. The game begins with state N. We say that a proposal is *accepted* if every responder accepts it and *rejected* otherwise.

δ_i is i's discount factor. Player i obtains a payoff of $\delta_i^{t-1} x_i$ if a proposal $\left(C, (x_k)_{k \in C}\right)$ such that $C \ni i$ is enforced at the t-th round and nothing otherwise.

In this paper, consider pure strategies. The equilibrium concept employed in the paper is the stationary subgame perfect equilibrium (SSPE), which is the subgame perfect equilibrium such that each player takes the same actions at all rounds with the same state.

We introduce some notations. For $n \in \mathbb{N}$, $\epsilon > 0$ and $a \in \mathbb{R}^n$, let $B_\epsilon^n(a)$ be the ϵ-open ball of a in the n-dimensional Euclidean space. For any family $a \equiv (a_\lambda)_{\lambda \in \Lambda}$ and $\lambda \in \Lambda$, let $\operatorname{pr}_\lambda a \equiv a_\lambda$. Let $\mathbf{1}$ be a tuple whose elements are 1.

By the same logic as Okada (1996), there is no delay in $G(\delta)$ for any $\delta \in (0,1)^n$.

3. Efficiency

In this section, we consider the efficiency defined according to Okada (1996).

Definition 1. Take any $\delta \in (0,1)^n$. $G(\delta)$ is a *subgame efficient* if there exists an SSPE of $G(\delta)$ such that every player proposes full coalition S at a round with any state $S \in \mathfrak{S}$.

Let $\mathfrak{D} \equiv \left\{\Delta \in 2^{(0,1)^n} \mid \forall \epsilon > 0 : \Delta \cap B_\epsilon^n(\mathbf{1}) \neq \emptyset\right\}$.

Definition 2. Take any $\Delta \in \mathfrak{D}$. G is *limit subgame efficient on* Δ if there exists $\epsilon > 0$ such that for all $\delta \in B_\epsilon^n(\mathbf{1}) \cap \Delta$, $G(\delta)$ is subgame efficient.

Definition 3. G is a *limit subgame efficient* if G is a limit subgame efficient on $(0,1)^n$.

Note that the efficiency is defined for games in the above whereas it is defined for equilibria in Okada (1996).

In the case that $p_i^S = \frac{1}{|S|}$ for all $S \in \mathfrak{S}$ and all $i \in S$, according to Okada (1996), G is limit subgame efficient on Δ if and only if $\frac{v(S)}{|S|} \geq \frac{v(T)}{|T|}$ for all $S, T \in \mathfrak{C}$ such that $S \supset T$, where $\Delta \equiv \left\{(\delta'_k)_{k \in N} \in (0,1)^n \mid \forall k, l \in N : \delta'_k = \delta'_l\right\}$.

The following example implies that even if the equivalent condition above is satisfied, G is not limit subgame efficient on some $\Delta \in \mathfrak{D}$, which obviously implies that G is not limit subgame efficient.

Example 1. Suppose that $n = 3$, $p_i^S = \frac{1}{|S|}$ for all $S \in \mathfrak{S}$ and all $i \in S$, and $v(N) = 1$ and $v(\{i, j\}) = \frac{2}{3}$ for all $i, j \in N$ $i \neq j$. Obviously, characteristic function v satisfies Okada's equivalent condition. Let $\Delta \equiv \left\{(\delta_k)_{k \in N} \in (0,1)^n \mid \delta_1 = \delta_2 = \sqrt{\delta_3}\right\}$.

We show that G is not limit subgame efficient on Δ. Take any $\delta \equiv (\delta_k)_{k\in N} \in \Delta$ such that $\delta_1 > \frac{-1+\sqrt{37}}{6} \in (0,1)$. Let $d \equiv \delta_1$. Suppose that $G(\delta)$ is subgame efficient. Then, there exists an SSPE of $G(\delta)$ such that every player proposes the full coalition at any round. For $i \in N$, let v_i be player i's payoff in the equilibrium. Then, by the efficiency and no delay, the equilibrium payoff profile must satisfy

$$(v_1, v_2, v_3) = \left(\frac{(1 - dv_2 - d^2 v_3) + 2dv_1}{3}, \frac{(1 - dv_1 - d^2 v_3) + 2dv_2}{3}, \frac{(1 - dv_1 - dv_2) + 2d^2 v_3}{3}\right).$$

Thus, $v_1 = v_2 = \frac{1+d}{3+2d}$ and $v_3 = \frac{1}{3+2d}$. At player 1's proposing node in the first round, player 1 obtains a payoff of $1 - d\frac{1+d}{3+2d} - d^2\frac{1}{3+2d} = \frac{3+d-2d^2}{3+2d}$ in the equilibrium. Consider player 1's deviation from proposing the grand coalition to proposing coalition $\{1,3\}$. By the deviation, player 1 obtains a payoff of $\frac{2}{3} - d^2\frac{1}{3+2d} = \frac{6+4d-3d^2}{3(3+2d)}$. Thus, Player 1's gain from the deviation is equal to $\frac{-3+d+3d^2}{3(3+2d)}$, which is greater than 0 since $d > \frac{-1+\sqrt{37}}{6}$. Thus, $G(\delta)$ is not subgame efficient for all $d > \frac{-1+\sqrt{37}}{6}$. Hence, G is not limit subgame efficient on Δ.

Note that there exists an equilibrium in mixed strategies. In the equilibrium, player $i \in \{1,2\}$ stochastically proposes grand coalition N and subcoalition $\{i,3\}$ at any round with state N.

The inefficiency is intuitively explained as follows: By proposing subcoalitions with less patient players, a proposer does not have to give larger payoffs to more patient players, whose approval is expensive. She may obtain a larger payoff by proposing such subcoalitions than the full coalition. Thus, under different discount factors, she may have an incentive to propose subcoalitions. Moreover, however close to 1 each player's discount factor is (however small the difference among players' discount factors is), the difference among players' discount factors matters.

In the following, we investigate conditions for the efficiency under different discount factors. To do so, introduce some notations. For any nonempty and finite set K, any tuple $p \equiv (p_k)_{k\in K} \in \mathbb{R}_+^{|K|}$ such that $\sum_{k\in K} p_k = 1$ and any tuple $a \equiv (a_k)_{k\in K} \in \mathbb{R}^{|K|}$, let $H_p(a)$ and $A(a)$ denote the harmonic mean of a weighted by p and the arithmetic mean of a, respectively: $H_p(a) \equiv \left(\sum_{k\in K} p_k a_k^{-1}\right)^{-1}$ and $A(a) \equiv \frac{1}{|K|}\sum_{k\in K} a_k$. For any $S \in \mathfrak{S}$, any $T \in \mathfrak{C}$ such that $T \subset S$ and any $i \in T$, let $p_i^{T|S} \equiv \frac{p_i^S}{\sum_{k\in T} p_k^S}$, $p^{T|S} \equiv \left(p_k^{T|S}\right)_{k\in T}$ and $p_T^S \equiv \left(p_k^S\right)_{k\in T}$.

The following lemma provides a condition for $G(\delta)$ to be subgame efficient given δ.

Lemma 1. *Take any $\delta \equiv (\delta_k)_{k\in N} \in (0,1)^n$. Then, $G(\delta)$ is subgame efficient if and only if*

$$\frac{H_{p^S}(\mathbf{1} - \delta_S)}{H_{p^{T|S}}(\mathbf{1} - \delta_T)}\frac{v(S)}{|S|} + \left(\frac{1}{\sum_{k\in T} p_k^S} - 1\right) H_{p^S}(\mathbf{1} - \delta_S)\frac{v(S)}{|S|} \geq \frac{v(T)}{\frac{A\left(p_T^S\right)}{A\left(p^S\right)}|T|} \quad (1)$$

for all $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$, where $\delta_C \equiv (\delta_k)_{k\in C}$ for $C \in \mathfrak{C}$.

Remark. Needless to say, $A\left(p^S\right) = \frac{1}{|S|}$.

Proof. See Appendix A. **Q.E.D.**

Using Lemma 1, we present a equivalent condition for G to be limit subgame efficient. For $\delta \equiv (\delta_k)_{k\in N} \in (0,1)^n$, $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$, let $L(\delta, S, T)$ be the left hand side of (1).

Theorem 1. *G is limit subgame efficient if and only if $v(S) = 0$ for all $S \in \mathfrak{C} \setminus \{N\}$.*

Remark. If $v(S) = 0$ for all $S \in \mathfrak{C} \setminus \{N\}$, $G(\delta)$ is essentially the same game as Baron and Ferejohn's (1989) model with the unanimity rule.

Proof. See Appendix B. **Q.E.D.**

According to this theorem, G is limit subgame efficient only under a very strong condition. The key underlying the theorem is the first fraction of the first term of the left hand side of (1). If $S \neq T$, for any $r > 0$, for any $\epsilon > 0$, there exists $\delta \in B_\epsilon^n(\mathbf{1}) \cap (0,1)^n$ such that the fraction is less than r. This implies that, if there exists $S \in \mathfrak{C} \setminus \{N\}$ such that $v(S) > 0$, for any $\epsilon > 0$, there exists $\delta \in B_\epsilon^n(\mathbf{1}) \cap (0,1)^n$ such that $L(\delta, N, S) < \frac{v(S)}{\frac{A\left(p_S^N\right)}{A(p^N)}|S|}$, which means that G is not limit subgame efficient.

On the other hand, in the following, we show that under a condition on $\Delta \in \mathfrak{D}$, a generalized version of Okada's condition is "almost equivalent" to the limit subgame efficiency *on* Δ of G. For $\Delta \in \mathfrak{D}$, let $\Delta_1 \equiv \{\mathrm{pr}_1 \delta \mid \delta \in \Delta\}$. For $\Delta \in \mathfrak{D}$, $i \in N$ and $\delta_1 \in \Delta_1$, let

$$f_{\Delta,i}^{+}(\delta_1) \equiv \sup \{\mathrm{pr}_i \delta \mid \delta \in \Delta \wedge \mathrm{pr}_1 \delta = \delta_1\}$$

and

$$f_{\Delta,i}^{-}(\delta_1) \equiv \inf \{\mathrm{pr}_i \delta \mid \delta \in \Delta \wedge \mathrm{pr}_1 \delta = \delta_1\}.$$

For $S \in \mathfrak{C}$, let $f_{\Delta,S}^{+}(\delta_1) \equiv \left(f_{\Delta,k}^{+}(\delta_1)\right)_{k \in S}$ and $f_{\Delta,S}^{-}(\delta_1) \equiv \left(f_{\Delta,k}^{-}(\delta_1)\right)_{k \in S}$.

Using these notations, we introduce the following concept:

Definition 4. $\Delta \in \mathfrak{D}$ is *limit-equivalent* if for all $i \in N \setminus \{1\}$, $\lim_{\delta_1 \to 1} \frac{1 - f_{\Delta,i}^{+}(\delta_1)}{1-\delta_1} = \lim_{\delta_1 \to 1} \frac{1 - f_{\Delta,i}^{-}(\delta_1)}{1-\delta_1} = 1$, i.e., $\forall r > 0 \exists \bar{\delta}_1 \in \Delta_1 : \delta_1 \in \left(\bar{\delta}_1, 1\right) \cap \Delta_1 \Rightarrow \left|\frac{1 - f_{\Delta,i}^{+}(\delta_1)}{1-\delta_1} - 1\right| < r$ and $\forall r > 0 \exists \bar{\delta}_1 \in \Delta_1 : \delta_1 \in \left(\bar{\delta}_1, 1\right) \cap \Delta_1 \Rightarrow \left|\frac{1 - f_{\Delta,i}^{-}(\delta_1)}{1-\delta_1} - 1\right| < r$.

For example, $\left\{(\delta_k)_{k \in N} \in (0,1)^n \mid \forall i \in N \setminus \{1\} : \log \delta_1 + 1 \leq \delta_i \leq e^{\delta_1 - 1}\right\}$ is limit-equivalent.

The following theorem states that if $\Delta \in \mathfrak{D}$ is limit-equivalent, a generalized version of Okada's condition is "almost equivalent" to the limit subgame efficiency on Δ of G.

Theorem 2. *Take any $\Delta \in \mathfrak{D}$. Suppose that Δ is limit-equivalent. Then, (i) G is limit subgame efficient on Δ only if*

$$\frac{v(S)}{|S|} \geq \frac{v(T)}{\frac{A\left(p_T^S\right)}{A(p^S)}|T|}$$

for all $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$; and (ii) G is limit subgame efficient on Δ if

$$\frac{v(S)}{|S|} > \frac{v(T)}{\frac{A\left(p_T^S\right)}{A(p^S)}|T|}$$

for all $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$.

Proof. See Appendix C. **Q.E.D.**

If Δ is limit-equivalent, for any $r > 0$, there exists $\epsilon > 0$ such that for all $\delta \in B_{\epsilon}^{n}(\mathbf{1}) \cap \Delta$, the first fraction of the first term of the left hand side of (1) is in the r-open ball of 1 for all S and T. From this, the theorem is obtained.

The results of this paper imply importance of similarity of time preferences. Theorems 1 and 2 mean that in general, the efficiency of coalitional bargaining is scarcely attained, but if players' time preferences are similar, the efficiency is achieved under a moderate condition.

Appendix

A. Proof of Lemma 1

Proof. (Necessity) Suppose that there exists an SSPE σ of $G(\delta)$ such that every player proposes the full coalition at any round. Take any $S \in \mathfrak{S}$ and any $T \in \mathfrak{C}$ such that $T \subset S$. For $i \in N$, let v_i be player i's payoff by σ in any subgame with state S. By the efficiency and no delay,

$$v_i = p_i^S \left(v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k \right) + \sum_{k \in S \setminus \{i\}} p_k^S \delta_i v_i = p_i^S \left(v(S) - \sum_{k \in S} \delta_k v_k \right) + \delta_i v_i. \tag{2}$$

This implies

$$v_i = \frac{p_i^S}{1-\delta_i} \left(v(S) - \sum_{k \in S} \delta_k v_k \right). \tag{3}$$

(2) and (3) yield $\sum_{i \in S} v_i = v(S) = \left(\sum_{i \in S} \frac{p_i^S}{1-\delta_i} \right) \left(v(S) - \sum_{k \in S} \delta_k v_k \right)$. Thus,

$$v(S) - \sum_{k \in S} \delta_k v_k = H_{p^S}(\mathbf{1} - \delta_S)\, v(S). \tag{4}$$

From (3) and (4), we have

$$v_i = \frac{H_{p^S}(\mathbf{1} - \delta_S)}{1-\delta_i} p_i^S v(S). \tag{5}$$

By the efficiency, $v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k \geq v(T) - \sum_{k \in T \setminus \{i\}} \delta_k v_k$ must hold for $i \in T$. Add $\delta_i v_i$ to both sides of the above inequality. Then, $v(S) - \sum_{k \in S} \delta_k v_k \geq v(T) - \sum_{k \in T} \delta_k v_k$. Substitute (4) and (5) into the left hand side and the right hand side of the above inequality, respectively. Then,

$$H_{p^S}(\mathbf{1} - \delta_S)\, v(S) \geq v(T) - \left(\sum_{k \in T} \frac{\delta_k p_k^S}{1-\delta_k} \right) H_{p^S}(\mathbf{1} - \delta_S)\, v(S).$$

Add and subtract $\left(\sum_{k \in T} \frac{p_k^S}{1-\delta_k} \right) H_{p^S}(\mathbf{1} - \delta_S)\, v(S)$ to and from the right hand side of the above inequality. Then,

$$H_{p^S}(\mathbf{1} - \delta_S)\, v(S) \geq v(T) - \left(\sum_{k \in T} \frac{p_k^S}{1-\delta_k} \right) H_{p^S}(\mathbf{1} - \delta_S)\, v(S) + \left(\sum_{k \in T} p_k^S \right) H_{p^S}(\mathbf{1} - \delta_S)\, v(S).$$

Divide both sides of the above inequality by $|S| \sum_{k \in T} p_k^S$. Then,

$$\frac{1}{\sum_{k \in T} p_k^S} H_{p^S}(\mathbf{1} - \delta_S) \frac{v(S)}{|S|} \geq \frac{v(T)}{\frac{A(p_T^S)}{A(p^S)} |T|} - \frac{H_{p^S}(\mathbf{1} - \delta_S)}{H_{p^{T|S}}(\mathbf{1} - \delta_T)} \frac{v(S)}{|S|} + H_{p^S}(\mathbf{1} - \delta_S) \frac{v(S)}{|S|}.$$

Obviously, this inequality is equivalent to (1).

(Sufficiency) Suppose that (1) holds for all $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$. Consider strategy profile σ such that at a round with state $S \in \mathfrak{S}$, (i) every proposer $i \in S$ offers $\left(S, (x_k^i)_{k \in S}\right)$ with $x_i^i \equiv v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k^S$ and $x_j^i \equiv \delta_j v_j^S$ for $j \neq i$, and (ii) every responder i accepts proposal $\left(T, (y_k)_{k \in T}\right)$ with $T \ni i$ if and only if $y_i \geq \delta_i v_i^S$, where $v_i^S \equiv \frac{H_{p^S}(\mathbf{1} - \delta_S)}{1 - \delta_i} p_i^S v(S)$ for $i \in S$. In σ, every proposer offers the full coalition at any round. We want to show that σ is an SSPE. The stationarity obviously holds. Take any $S \in \mathfrak{S}$. Player i's payoff at a round with state S by σ is $p_i^S \left(v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k^S\right) + \sum_{k \in S \setminus \{i\}} p_k^S \delta_i v_i^S$, which is equal to v_i^S by the definition of v_i^S. Thus, each player's responding actions of σ are unimprovable. At player i's proposing node at a round with state S, player i's gain by one deviation from σ to offering proposal to be accepted with coalition $T \subset S$ is at most

$$\begin{aligned}
&\left(v(T) - \sum_{k \in T \setminus \{i\}} \delta_k v_k^S\right) - \left(v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k^S\right) \\
&\quad = v(T) - A\left(p_T^S\right)|T| \frac{H_{p^S}(\mathbf{1} - \delta_S)}{H_{p^{T|S}}(\mathbf{1} - \delta_T)} v(S) - \left(1 - A\left(p_T^S\right)|T|\right) H_{p^S}(\mathbf{1} - \delta_S)\, v(S),
\end{aligned}$$

which is less than or equal to 0 by (1). At the node, player i's gain by one deviation from σ to offering a proposal to be rejected is $\delta_i v_i^S - \left(v(S) - \sum_{k \in S \setminus \{i\}} \delta_k v_k^S\right) = -H_{p^S}(\mathbf{1} - \delta_S)\, v(S) < 0$. Thus, each player's proposing actions of σ are unimprovable. From the argument above, the One Deviation Principle implies that σ is an SPE. **Q.E.D.**

B. Proof of Theorem 1

Proof. (Necessity) Prove the contraposition. Suppose that there exists $S \in \mathfrak{C} \setminus \{N\}$ such that $v(S) > 0$. For $i \in N$ and $x \in (0,1)$, let

$$\delta_i(x) \equiv \begin{cases} 1 - \sqrt{1 - x^2} & \text{if } i \in S \\ x & \text{if } i \in N \setminus S. \end{cases}$$

Let $\delta(x) \equiv (\delta_k(x))_{k \in N}$ and $\delta_S(x) \equiv (\delta_k(x))_{k \in S}$. Then,

$$\begin{aligned}
\frac{H_{p^N}(\mathbf{1} - \delta(x))}{H_{p^{S|N}}(\mathbf{1} - \delta_S(x))} &= \frac{\left\{\sum_{k \in S} p_k^N \sqrt{1 - x^2}^{-1} + \sum_{k \in N \setminus S} p_k^N (1 - x)^{-1}\right\}^{-1}}{\sqrt{1 - x^2}} \\
&= \frac{1}{\sum_{k \in S} p_k^N + \left(\sum_{k \in N \setminus S} p_k^N\right) \frac{\sqrt{1 - x^2}}{1 - x}}.
\end{aligned}$$

Notice that $\lim_{x \to 1} \frac{\sqrt{1 - x^2}}{1 - x} = \lim_{x \to 1} \frac{x}{\sqrt{1 - x^2}} = +\infty$ by L'Hôpital's Rule. Then,

$$\lim_{x \to 1} \frac{H_{p^N}(1 - \delta(x))}{H_{p^{S|N}}(\mathbf{1} - \delta_S(x))} = 0.$$

Thus, $\lim_{x\to1} L(\delta(x), N, S) = 0$. Notice that $\frac{v(S)}{\frac{A(p_S^N)}{A(p^N)}|S|} > 0$. Then, there exists $\bar{x} \in (0,1)$ such that for all $x > \bar{x}$, $L(\delta(x), N, S) < \frac{v(S)}{\frac{A(p_S^N)}{A(p^N)}|S|}$. Now, take any $\epsilon > 0$. Let $\hat{x} \equiv \max\left\{\bar{x}, \sqrt{\max\left\{1-\frac{\epsilon^2}{n}, 0\right\}}\right\}$. Obviously, $\hat{x} \in (0,1)$. Take an $x_* \in (\hat{x}, 1)$. Consider $\delta(x_*)$. Note that

$$\begin{aligned}||\delta(x_*) - \mathbf{1}|| &= \sqrt{|S|(1-x_*^2) + (n-|S|)(1-x_*)^2} < \sqrt{n(1-x_*^2)} \\ &< \sqrt{n(1-\hat{x}^2)} \leq \sqrt{n\left(1-\max\left\{1-\frac{\epsilon^2}{n}, 0\right\}\right)} \leq \epsilon.\end{aligned}$$

Since $x_* > \hat{x} \geq \bar{x}$, we have $L(\delta(x_*), N, S) < \frac{v(S)}{\frac{A(p_S^N)}{A(p^N)}|S|}$. Thus, from Lemma 1, $G(\delta(x_*))$ is not subgame efficient. Hence, G is not limit subgame efficient.

(Sufficiency) Suppose that $v(S) = 0$ for all $S \in \mathfrak{C} \setminus \{N\}$. Then, for any $(\delta_k)_{k\in N} \in (0,1)^n$, for any $S \in \mathfrak{S} = \{N\}$ and any $T \in \mathfrak{C}$ such that $T \subset S$, (1) holds. Thus, from Lemma 1, for any $\delta \in (0,1)^n$, $G(\delta)$ is subgame efficient. Hence, G is limit subgame efficient. **Q.E.D.**

C. Proof of Theorem 2

Proof. For any $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$, for any $\delta \equiv (\delta_k)_{k\in N} \in \Delta$,

$$\begin{aligned}&\frac{H_{p^S}\left(\mathbf{1} - f^-_{\Delta,S}(\delta_1)\right)}{H_{p^{T|S}}\left(\mathbf{1} - f^+_{\Delta,T}(\delta_1)\right)}\frac{v(S)}{|S|} + \left(\frac{1}{\sum_{k\in T} p_k^S} - 1\right) H_{p^S}(\mathbf{1}-\delta_S)\frac{v(S)}{|S|} \\ &\quad\leq L(\delta, S, T) \leq \frac{H_{p^S}\left(\mathbf{1} - f^+_{\Delta,S}(\delta_1)\right)}{H_{p^{T|S}}\left(\mathbf{1} - f^-_{\Delta,T}(\delta_1)\right)}\frac{v(S)}{|S|} + \left(\frac{1}{\sum_{k\in T} p_k^S} - 1\right) H_{p^S}(\mathbf{1}-\delta_S)\frac{v(S)}{|S|}\end{aligned}$$

holds, where $\delta_S \equiv (\delta_k)_{k\in S}$. Since Δ is limit-equivalent,

$$\lim_{\delta_1\to1} \frac{H_{p^S}\left(\mathbf{1} - f^-_{\Delta,S}(\delta_1)\right)}{H_{p^{T|S}}\left(\mathbf{1} - f^+_{\Delta,T}(\delta_1)\right)} = \lim_{\delta_1\to1} \frac{\frac{H_{p^S}\left(\mathbf{1} - f^-_{\Delta,S}(\delta_1)\right)}{1-\delta_1}}{\frac{H_{p^{T|S}}\left(\mathbf{1} - f^+_{\Delta,T}(\delta_1)\right)}{1-\delta_1}} = 1.$$

Similarly, $\lim_{\delta_1\to1} \frac{H_{p^S}\left(\mathbf{1} - f^+_{\Delta,S}(\delta_1)\right)}{H_{p^{T|S}}\left(\mathbf{1} - f^-_{\Delta,T}(\delta_1)\right)} = 1$. Thus, for any $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$, for any $r > 0$, there exists $\epsilon > 0$ such that for all $\delta \in B_\epsilon^n(\mathbf{1}) \cap \Delta$,

$$\frac{v(S)}{|S|} - r \leq L(\delta, S, T) \leq \frac{v(S)}{|S|} + r. \tag{6}$$

(On (i)) Prove the contraposition. Suppose that $\frac{v(S)}{|S|} < \frac{v(T)}{\frac{A(p_T^S)}{A(p^S)}|T|}$ for some $S \in \mathfrak{S}$ and $T \in \mathfrak{C}$ such that $T \subset S$. Then, there exists $\rho \in \mathbb{R}$ such that $0 < \rho < \frac{v(T)}{\frac{A(p_T^S)}{A(p^S)}|T|} - \frac{v(S)}{|S|}$. Take $\frac{v(T)}{\frac{A(p_T^S)}{A(p^S)}|T|} - \frac{v(S)}{|S|} - \rho > 0$

as r of (6). Then, there exists $\epsilon > 0$ such that for any $\delta \in B^n_\epsilon(\mathbf{1}) \cap \Delta$, $L(\delta, S, T) \leq \frac{v(T)}{\frac{A\left(p^S_T\right)}{A(p^S)}|T|} - \rho$. Thus, there exists $\epsilon > 0$ such that for any $\delta \in B^n_\epsilon(\mathbf{1}) \cap \Delta$, $L(\delta, S, T) < \frac{v(T)}{\frac{A\left(p^S_T\right)}{A(p^S)}|T|}$ by $\rho > 0$. Therefore, there exists $\epsilon > 0$ such that for any $\delta \in B^n_\epsilon(\mathbf{1}) \cap \Delta$, $G(\delta)$ is not subgame efficient. This implies that G is not limit subgame efficient on Δ.

(On (ii)) Take any $S \in \mathfrak{S}$ and any $T \in \mathfrak{C}$ such that $T \subset S$. Take $\frac{v(S)}{|S|} - \frac{v(T)}{\frac{A\left(p^S_T\right)}{A(p^S)}|T|} > 0$ as r of (6). Then, there exists $\epsilon(S,T) > 0$ such that for any $\delta \in B^n_{\epsilon(S,T)}(\mathbf{1}) \cap \Delta$, $L(\delta, S, T) \geq \frac{v(T)}{\frac{A\left(p^S_T\right)}{A(p^S)}|T|}$. Since $\{(S,T) \in \mathfrak{S} \times \mathfrak{C} \mid S \supset T\}$ is finite, function ϵ has a minimizer. Let ϵ_* be the minimum of ϵ. By definition, for all $\delta \in B^n_{\epsilon_*}(\mathbf{1})$, for all $S \in \mathfrak{S}$ and all $T \in \mathfrak{C}$ such that $T \subset S$, $L(\delta, S, T) \geq \frac{v(T)}{\frac{A\left(p^S_T\right)}{A(p^S)}|T|}$. Lemma 1 implies that for all $\delta \in B^n_{\epsilon_*}(\mathbf{1}) \cap \Delta$, $G(\delta)$ is subgame efficient. Therefore, G is limit subgame efficient on Δ. **Q.E.D.**

References

Baron, D.P. and J. Ferejohn (1989) "Bargaining in legislatures" *American Political Science Review* **83**, 1181-1206.

Okada, A. (1996) "A noncooperative coalitional bargaining game with random proposers" *Games and Economic Behavior* **16**, 97-108.

Yan, H. (2002) "Noncooperative selection of the core" *International Journal of Game Theory* **31**, 527-540.