E C O N O M I C S   B U L L E T I N

Pure strategy dominance with quasiconcave utility functions

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Abstract

By a result of Pearce (1984), in a finite strategic form game, the set of a player's serially undominated strategies coincides with her set of rationalizable strategies. In this note we consider an extension of this result that applies to games with continuous utility functions that are quasiconcave in own action. We prove that in such games, when the players are endowed with compact, metrizable, and convex action spaces, a strategy of some player is dominated by some other pure strategy if and only if it is not a best reply to any belief over the strategies adopted by her opponents. For own-quasiconcave games, this can be used to give a characterization of the set of rationalizable strategies, different from the one given by Pearce. Moreover, expected utility functions defined on the mixed extension of a game are always own-quasiconcave, and therefore the result in this note generalizes Pearce's characterization to infinite games, by a simple shift of perspective.

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1. Introduction

The sets of serially undominated actions and of rationalizable actions of a game give rise to two well known set-valued solution concepts on that game. Each of these notions singles out a subset of actions for each player in a way that is justified by a certain reasoning process, but the reasoning process behind either concept is different. Nevertheless, by a celebrated result of Pearce (1984), in any finite strategic form game, the set of a player's serially undominated actions coincides with her set of rationalizable actions, provided the players are expected utility maximizers. A recent paper by Zimper (2005) considered the generalization of Pearce's result to games with infinite strategy sets; Zimper elegantly proved the equivalence of the sets of serially undominated actions and rationalizable actions, under the conditions that each player's action set is a compact subset of a metrizable space and the best reply correspondences are upper-hemicontinuous.

This note further extends the results by Pearce, and essentially also those by Zimper. To this end, we consider games with continuous utility functions that are quasiconcave in own action. We prove that in such games, when players are endowed with compact, metrizable, and convex action spaces, an action a of some player i is (strictly) dominated by some other action if and only if it is not a best reply to any belief over the strategies adopted by her opponents. Note that in this equivalence we ask whether there is a *pure* strategy available to player i that dominates a. The mixed strategies of player i play no role. The notion of dominance considered in this note is therefore subtly different from the one that is usually considered in the literature, as an action is usually said to be dominated when there is a *mixed* strategy available to the player which gives unambiguously higher expected utility. As a consequence, our result can be used to prove that the set of each player's serially "purely"-undominated actions coincides with her set of rationalizable actions in games with quasiconcave utility functions—a characterization of the set of rationalizable actions which differs from the one given by Pearce.

There is, however, an intimate connection between the above characterization for own-quasiconcave games and the usual one for games that are not necessarily quasiconcave in own action, and in fact the latter can be obtained as a corollary. This is similar in spirit to a result by Glicksberg (1952) on the existence of pure strategy Nash equilibria in games with utility functions that are quasiconcave in own action. Glicksberg, and more recently, Aliprantis et al. (2006), proved the existence of pure strategy Nash equilibria for such games, by generalizing Nash's arguments that establish the existence of an equilibrium in mixed strategies (Nash Jr, 1950) to games with such utility functions. For any compact metrizable joint action set A, the expected utility function of a Von-Neumann expected utility maximizing agent over the mixed extension of A is quasiconcave (indeed, linear). Nash's original result can be obtained by identifying the joint strategy space with the mixed extension of A. In a similar vein, the conditions that utility functions are quasiconcave in own action and that action sets are convex are stronger than the assumptions at the basis of Zimper's results. However, for games that do not satisfy these strong conditions, the usual equivalence of serially undominated actions and rationalizable actions still follows, using the linearity of the expected utility functions of Von-Neumann maximizing agents over the mixed extension of the joint action set.

When Glicksberg used this type of argument to show his result implied the original theorem of Nash, he implicitly assumed that the expected utility function is jointly continuous on the mixed strategy extension of a game. Unfortunately, this is not at all obvious. Aliprantis et al. explicitly proved this; we make use of this fact, and Lemma 2 in the appendix is an immediate corollary of it.

The equivalence proof in Zimper (2005) revolves around an argument that makes use of a generalized separating hyperplane theorem. In contrast, the proof below follows Pearce's original line of reasoning, by constructing a zero-sum two player game and proving it has a saddle point using a minimax theorem. Specifically, we use the minimax theorem for quasiconcave-quasiconvex functions due to Sion (1958). Interestingly, Sion's minimax theorem is not proved using the separating hyperplane theorem, but using the well known Knaster-Kuratowski-Mazurkiewicz fixed point theorem, which is in turn proved using Sperner's lemma, a combinatorial result.

2. Main Result

We define a game in the standard way: N is a countable set of players, N_{-i} denotes the set $N-\{i\}$; for each $i \in N$, A_i is an action set and π_i is a utility function $\pi_i : A_i \times A_{-i} \to \mathbb{R}$ where, as usual, A_{-i} is the set $\prod_{N_{-i}} A_j$. For any topological space S we write $\triangle(S)$ for the set of all probability measures on the Borel sets of S.

Take $i \in N$ and let $B_{-i} \subseteq A_{-i}$. For our purposes, an action $a^* \in A_i$ will be called *purely-dominated given* B_{-i} (and for the given player i) if and only if there exists some $a_i \in A_i$ such that for each $a_{-i} \in B_{-i}$, the inequality $\pi_i(a^*, a_{-i}) < \pi(a_i, a_{-i})$ holds. The interpretation of this inequality is that the action a_i is, in a quite unambiguous sense, better than a^* when opponents choose actions from B_{-i}. As explained in the introduction, this notion of dominance differs from the usual one: usually one allows the dominating strategy a_i to be a "mixture"—that is, a point a_i in $\triangle(A_i)$ instead of A_i.

A point $\mu \in \triangle(A_{-i})$ is a probability measure over A_{-i} and can be interpreted as a belief of player i about the actions chosen by his opponents. For each fixed $a_i \in A_i$, we can regard $\pi_i(a_i, a_{-i})$ as a continuous function $\pi_i^{a_i}$ on A_{-i}. Given a point $\mu \in \triangle(A_{-i})$, we obtain the expected value of π when choosing a_i as:

$$\tilde{\pi}_i(a_i, \mu) \mapsto \int \pi_i^{a_i} \, d\mu. \tag{1}$$

In this case, $\tilde{\pi}_i(a_i, \mu)$ can be regarded as the utility of playing a_i given the belief μ of a Von-Neumann Morgenstern expected utility maximizing agent. For $B_{-i} \subseteq A_{-i}$, action $a^* \in A_i$ is called a *never-best-reply given* B_{-i} (for a given player i) if there is no $\mu \in \triangle(B_{-i})$ such that the inequality $\tilde{\pi}_i(a^*, \mu) \geq \tilde{\pi}_i(a_i, \mu)$ holds for all $a_i \in A_i$. The interpretation is that there is no belief over i's opponents' joint action profiles in B_{-i} that justifies choosing a^*.

Now consider any game satisfying the following additional assumptions (1)–(3) for each $i \in N$:

(1) A_i is a compact, convex and metrizable topological space;
(2) $\pi_i : A_i \times A_{-i} \to \mathbb{R}$ is a bounded real valued function that is jointly continuous with respect to the product topology;
(3) π_i is quasiconcave on A_i for fixed $a_{-i} \in A_{-i}$.[1]

[1] Recall that a function $f : S \to \mathbb{R}$ is called quasiconvex if for each real number c, the set $\{s \in S \mid f(s) \leq c\}$ is convex; f is quasiconcave if $-f$ is quasiconvex.

In this case, we have the following lemma that connects the never-best-replies to the purely-dominated actions.

Lemma 1. *Take $i \in N$ and for each $j \in N_{-i}$, let $B_j \subseteq A_j$ be compact. For player i, $a^* \in A_i$ is purely-dominated given $\prod_{N_{-i}} B_j$ if and only if it is a never-best-reply given $\prod_{N_{-i}} B_j$.*

The proof of the statement appears in the appendix.

3. Serially Undominated Strategies and Rationalizability

Using the above notions of dominated actions and never-best-replies, the iterative solution concepts of serially purely-undominated actions and rationalizable actions can be developed in the standard way, as follows.[2] Consider again any game satisfying the assumptions (1)–(3). For each $i \in N$, suppose $B_i \subseteq A_i$ and let $U_i(\prod_{N_{-i}} B_j)$ be the set defined as:

$$A_i - \{a \in A_i \mid a \text{ is purely-dominated for player } i \text{ given } \prod_{N_{-i}} B_j\}. \tag{2}$$

By a similar token let $R_i(\prod_{N_{-i}} B_j)$ be the set:

$$A_i - \{a \in A_i \mid a \text{ is a never-best-reply for player } i \text{ given } \prod_{N_{-i}} B_j\}. \tag{3}$$

Now recursively define $U_i^0 = U_i(\prod_{N_{-i}} A_j)$ and $U_i^n = U_i(\prod_{N_{-i}} U_j^{n-1})$ for all $n \in \mathbb{N}$ and each $i \in N$. The set of i's *serially purely-undominated* actions is the set $\bigcap_{n\in\mathbb{N}} U_i^n$—this is the set that survives the elimination due to applying U_i at each step $n \in \mathbb{N}$. Similarly, recursively define $R_i^0 = R_i(\prod_{N_{-i}} A_j)$ and $R_i^n = R_i(\prod_{N_{-i}} R_j^{n-1})$ for all $n \in \mathbb{N}$ and each $i \in N$. The set of i's *rationalizable actions* is the set $\bigcap_{n\in\mathbb{N}} R_i^n$. The two sets are identical.

Theorem. *For each player $i \in N$, the set of i's serially purely-undominated actions and the set of i's rationalizable actions coincide.*

4. Remarks

Lemma 1 holds for games with infinite, compact, convex, metrizable action sets and quasi-concave utility functions. The conditions that utility functions are continuous, quasiconcave in own action and that action sets are convex are a strengthening of the conditions in Zimper (2005). However, for games with continuous utility functions, the results obtained above can be seen to imply results similar to those of Zimper—which essentially mirrors the way Glicksberg (1952) extended Nash's theorem.[3] For each $i \in N$, let A_i be any compact, metrizable set of actions, not necessarily convex. Furthermore, suppose each π_i is jointly continuous on $A_i \times A_{-i}$ but not necessarily quasiconcave in own action. Then, by Lemma 2 in the appendix, each π_i is continuous on $\triangle(A_i) \times A_{-i}$. The set $\triangle(A_i)$ is also metrizable and compact, and moreover, convex; and expected utility is quasiconcave (indeed, linear) on $\triangle(A_i)$ for fixed a_{-i}. Thus by identifying each player i's strategy set with the set of mixed strategies $\triangle(A_i)$ over A_i, we obtain strategy spaces and utility functions that satisfy all the stated assumptions (1)–(3). The actions of the original game are embeddable in the strategy set

[2] As noted by Apt (2007), several other ways to define these solution concepts are found in the literature. See the remarks in section 4.

[3] The assumption that the utility functions are continuous implies Zimper's assumption that the best reply correspondences are upper-hemicontinuous.

$\triangle(A_i)$ and correspond to the pure strategies. Using this embedding, we may apply Lemma 1 to obtain the following result. An action $a_i \in A_i$ of the original game is dominated by some $\alpha \in \triangle(A_i)$, that is, by some other (possibly mixed) strategy, if and only if it is a never-best-reply among all the strategies in $\triangle(A_i)$. And indeed, if a_i is a never-best-reply among *all* the strategies in $\triangle(A_i)$, then—using the linearity of the expected utility function—it is a never-best-reply among all the *pure* strategies in $\triangle(A_i)$, *viz.*, among the actions of the original game, which lie embedded in $\triangle(A_i)$. This result is just what is needed to develop the usual equivalence of the set of rationalizable actions and serially undominated actions (as in e.g. Osborne and Rubinstein (1994), proposition 62.1, and the proposition in Zimper (2005)).

By the definition of the operator R_i in expression (3), an action survives the elimination procedure at stage n if it is a best reply in the original game (i.e. among A_i) to some belief over the surviving strategies of the other players. In fact, this how the set of rationalizable actions was originally defined by Bernheim (1984). In an alternative approach, considered by Pearce (1984), an action survives elimination if it is a best reply among the *remaining* actions of player i (i.e. among R_i^{n-1}) to some belief over the surviving strategies of the other players. The latter is a weaker condition, since the criterion for survival is that an action is a best reply among a smaller set of competing actions. By a similar token, there is a strong and a weak way to define the operator U_i. The various ways to define the operators U_i and R_i, and the conditions under which the sets of serially undominated actions and rationalizable actions are invariant under the definition adopted is investigated in Apt (2007). Indeed, his results entail that the strong and weak approaches are equivalent for the kind of games considered in this note.

Dominance by pure strategies seems to have first been considered by Börgers (1993), apparently out of a mild dissatisfaction with some facets of the expected utility approach in game theory. Börgers assumed preference relations on a (finite) set of possible outcomes of a game and derived the utility functions π_i from these relations; then he proceeded to prove that an action a maximises the expected utility of player i under some belief μ for some monotonic transformation of π_i, if and only if a is not dominated by another action. Thus, in this approach the utility function π_i does not uniquely represent the utility of player i. The implications of this are obviously quite different from those of Lemma 1 above. In contrast, in this note we take the utility functions as primitives, under the assumption that players maximize expected utility.

Appendix

Lemma 2. *Let A and B be compact, metrizable topological spaces and $f : A \times B \to \mathbb{R}$ be jointly continuous and bounded. For each fixed $a \in A$ we may regard f as a continuous function $f_a : B \to \mathbb{R}$. Let $\tilde{f} : A \times \triangle(B) \to \mathbb{R}$ be defined by:*

$$\tilde{f}(a, \mu) \mapsto \int f_a \, d\mu.$$

Endow $\triangle(B)$ with the weak topology.[4] *Then $\tilde{f}$ is jointly continuous on $A \times \triangle(B)$.*

[4] $\mu_n \to \mu$ in the weak topology if and only if $\int f \, d\mu_n \to \int f \, d\mu$ for any bounded continuous function f (Aliprantis and Border (2006), p. 507).

Proof. Endow both $\triangle(A)$ and $\triangle(B)$ with the weak topology. The product topology on $\triangle(A) \times \triangle(B)$ is metrizable (see Aliprantis and Border (2006), Theorem 15.11) and A can be embedded into $\triangle(A)$ through the mapping $a \mapsto \delta_a$, where δ_a is the point-mass on a (Aliprantis and Border (2006), theorem 15.8). It suffices to show that if (a_n, μ_n) is a sequence that converges to (a, μ), then

$$\tilde{f}(a_n, \mu_n) := \iint f \, d\delta_{a_n} d\mu_n \to \iint f \, d\delta_a d\mu = \int f_a \, d\mu =: \tilde{f}(a, \mu).$$

By a theorem of Aliprantis et al. (2006), the function $\iint f \, d\mu_a d\mu_b$ is jointly continuous in the product topology on $\triangle(A) \times \triangle(B)$. Recall that a sequence converges in the product topology if and only if it converges pointwise. But δ_{a_n} to δ_a in $\triangle(A)$ if and only if $a_n \to a$ in A; so (δ_{a_n}, μ_n) converges pointwise to (δ_a, μ) on $\triangle(A) \times \triangle(B)$ if and only if $a_n \to a$ and $\mu_n \to \mu$, that is, $(a_n, \mu_n) \to (a, \mu)$ pointwise. The continuity of $\tilde{f}$ in the product topology on $A \times \triangle(B)$ now follows from the theorem of Aliprantis et al. ■

Proof of Lemma 1. Let $B_{-i} = \prod_{N_{-i}} B_j$. The assumption that A_i and each B_j is compact and metrizable together with N countable guarantees that $A_i \times B_{-i}$, when endowed with the product topology, is also compact and metrizable. By assumption π_i is bounded and jointly continuous on $A_i \times A_{-i}$ and so also on the subspace $A_i \times B_{-i}$ (Armstrong (1983), Theorem 2.8). Apply lemma 2 to establish that $\tilde{\pi}_i$ is jointly continuous on $A \times \triangle(B_{-i})$.

($\Rightarrow$) Suppose a^* is a never-best-reply given B_{-i}. Following Pearce (1984), define the function:

$$u(a_i, \mu) := \tilde{\pi}_i(a_i, \mu) - \tilde{\pi}_i(a^*, \mu)$$

Since $\tilde{\pi}_i(a^*, \mu)$ is a constant for each $\mu \in A_{-i}$, the function u is quasiconcave on A_i for fixed μ. Since a^* is a never-best-reply, for each $\mu \in \triangle(B_{-i})$, there exists an action $a_i \neq a^*$ such that $u(a_i, \mu) > 0$. We claim:

Claim 1. There is a point $(\hat{a}, \hat{\mu}) \in A_i \times \triangle(B_{-i})$ such that $u(\hat{a}, \hat{\mu}) = \inf_{\mu \in \triangle(B_{-i})} \sup_{a \in A_i} u(a, \mu) > 0$.

Proof of claim. For given $\mu \in \triangle(B_{-i})$, consider the problem:

$$\max_{a \in A_i} u(a, \mu),$$

and let $g(\mu) := \max_{a \in A_i} u(a, \mu)$ and $\phi(\mu) := \arg\max_{a \in A_i} u(a, \mu)$ be the set of maximizers of u for given μ. Since A_i is compact and u is continuous, by the Weierstrass theorem u attains a maximum, thus g is a well defined function and, $\phi(\mu)$ non-empty for each μ. As u is continuous on $A_i \times \triangle(B_{-i})$, by the Maximum Theorem (Berge (1997), p. 116), g is a continuous function. Since $\triangle(B_{-i})$ is compact (see Aliprantis and Border (2006), Theorem 15.11), g attains a minimum on $\triangle(B_{-i})$. Let $\hat{\mu}$ be a point that minimizes g, and let $\hat{a} \in \phi(\mu)$. Then $u(\hat{a}, \hat{\mu}) = \inf_{\mu \in \triangle(B_{-i})} \sup_{a \in A_i} u(a, \mu)$. For each μ, there is some a such that $u(a, \mu) > 0$, and so $u(\hat{a}, \hat{\mu}) = g(\hat{\mu}) > 0$. □

Claim 2. $\inf_{\mu \in \triangle(B_{-i})} \sup_{a \in A_i} u(a, \mu) = \sup_{a \in A_i} \inf_{\mu \in \triangle(B_{-i})} u(a, \mu)$.

Proof of claim. Suppose to the contrary that for some c we have

$$\sup_{a \in A_i} \inf_{\mu \in \triangle(B_{-i})} u(a, \mu) < c < \inf_{\mu \in \triangle(B_{-i})} \sup_{a \in A_i} u(a, \mu). \tag{4}$$

First note that $\inf_{a_{-i}\in B_{-i}} u(a,a_{-i}) = \inf_{\mu\in\triangle(B_{-i})} u(a,\mu)$, due to the linearity of u on $\triangle(B_{-i})$. Hence:

$$\sup_{a\in A_i} \inf_{a_{-i}\in B_{-i}} u(a,a_{-i}) < c \quad \text{if and only if} \quad \sup_{a\in A_i} \inf_{\mu\in\triangle(B_{-i})} u(a,\mu) < c \tag{5}$$

We will now derive a contradiction to the inequality (4). The derivation is based on Sion's minimax theorem (1958), with only very minor adaptations to the present framework.

For $a_{-i} \in B_{-i}$ let $Y_{a_{-i}} := \{a_i \in A_i \mid u(a_i,a_{-i}) < c\}$. These are open sets, and by (5) for each $a_i \in A_i$ there is a $a_{-i} \in B_{-i}$ such that $u(a_i,a_{-i}) < c$; so the $Y_{a_{-i}}$'s are an open cover of A_i. A_i is compact so there is a finite subcover. That is, there is a finite set $Y \subseteq B_{-i}$ such that for each $a_i \in A_i$, there exists $y \in Y$ such that $u(a_i,y) < c$. The finite set Y is closed in B_{-i}, and by theorem 15.19 in Aliprantis and Border (2006), $\triangle(Y)$ is a so called closed face of $\triangle(B_{-i})$, *viz.*, the set $\triangle(Y)$ is the set of probability distributions with support in the finite set Y, and $\triangle(Y)$ is compact. For $a_i \in A_i$, let $X_{a_i} := \{\mu \in \triangle(Y) \mid u(a_i,\mu) > c\}$. By (4), for each $\mu \in \triangle(B_{-i})$ there exists a_i such that $u(a_i,\mu) > c$, so the X_{a_i}'s cover $\triangle(Y)$. Since $\triangle(Y)$ is compact, we can choose a finite set $X \subseteq A_i$ such that for each $\mu \in \triangle(Y)$ we have $a_i \in X$ such that $u(a_i,\mu) > c$.

For a set $X' \subseteq A_i$, denote the convex hull of X' by $[X']$. For fixed μ, $u(a_i,\mu)$ is quasiconcave in a_i on $[X']$ by assumption. For fixed a_i and closed $Y' \subseteq B_{-i}$, the set $\{\mu \in \triangle(Y') : u(a_i,\mu) \leq c\}$ is convex, so u is quasiconvex on $\triangle(Y')$ for fixed a_i. Therefore lemmata 3.3 and 3.3′ in Sion (1958) apply, and there exist finite subsets $X^* \subseteq X$ and $Y^* \subseteq Y$ such that: (i) there exists $x^* \in [X^*]$ such that $u(x^*,y) < c$ for each $y \in Y^*$—and by quasiconvexity of u on $\triangle(Y^*)$, $u(x^*,\mu) < c$ for each $\mu \in \triangle(Y^*)$; (ii) there exists $y^* \in \triangle(Y^*)$ such that $u(a_i,y^*) > c$ for each $a_i \in X^*$—and by quasiconcavity, $u(a_i,y^*) > c$ for each $a_i \in [X^*]$. Thus we have $c < u(x^*,y^*) < c$, an absurdity. □

By claim 1 and claim 2, there exists $a_i \in A_i$ such that $u(a_i,\mu) > 0$ for all $\mu \in \triangle(B_{-i})$, in other words, the action a_i purely-dominates a^*.

($\Leftarrow$) If a^* is purely-dominated given B_{-i}, then a^* is a never-best-reply to any $a_{-i} \in B_{-i}$, and *a fortiori* a never-best-reply to any probability distribution over B_{-i}. ■

Proof of the Theorem. The theorem is proved by showing that for each $n \in \mathbb{N}$ and each $i \in N$, the sets U_i^n and R_i^n coincide. The argument is well-known; it can be found, for instance, in Osborne and Rubinstein (1994) and also in Zimper (2005). The key is to prove, using Lemma 1, that $a \in U_i^n$ if and only if $a \in R_i^n$. By the Lemma and the assumptions (1)–(3), this claim is certainly true for $n = 0$. Moreover, we can apply the Lemma inductively, provided we can show in addition that for each $n > 0$, the set R_i^{n-1} (and thus U_i^{n-1}) is compact.

So let $B_i \subseteq A_i$ be compact for all $i \in N$, pick $i \in N$, and let $B_{-i} = \prod_{N_{-i}} B_j$. For each $\mu \in \triangle(B_{-i})$, let $\phi(\mu) := \{a \in A_i \mid \tilde{\pi}_i(a,\mu) \geq \tilde{\pi}_i(a_i,\mu) \text{ for all } a_i \in A_i\}$. The set $\triangle(B_{-i})$ is compact if B_{-i} is. By Berge's maximum theorem, $\phi(\mu)$ is non-empty and compact for each μ, and ϕ is an upper-hemicontinuous correspondence. By lemma 17.8 in Aliprantis and Border (2006), the set $\bigcup_{\mu\in\triangle B_{-i}} \phi(\mu)$ is compact. Now note that $a \in \phi(\mu)$ for some $\mu \in \triangle(B_{-i})$ if and only if a is not a never-best-reply given B_{-i}, so $\bigcup_{\mu\in\triangle B_{-i}} \phi(\mu) = R_i(B_{-i})$. ■

References

Aliprantis, C. D., and K. C. Border (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide,* Springer-Verlag: New York etc.

Aliprantis, C. D., D. Glycopantis, and D. Puzzello (2006) "The Joint Continuity of the Expected Payoff Functions" *Journal of Mathematical Economics* **42**, 121–30.

Apt, K. (2007) "The Many Faces of Rationalizability" *The BE Journal of Theoretical Economics* **7**, article 18.

Armstrong, M. (1983) *Basic Topology,* Springer-Verlag: New York.

Berge, C. (1997) *Topological Spaces: Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity,* Dover Publications Inc: Mineola, New York.

Bernheim, P. D. (1984) "Rationalizable Strategic Behavior" *Econometrica* **52**, 1007–28.

Börgers, T. (1993) "Pure Strategy Dominance" *Econometrica* **61**, 423–30.

Glicksberg, I. (1952) "A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points" *Proceedings of the American Mathematical Society* **3**, 170–4.

Nash Jr, J. (1950) "Equilibrium Points in n-Person Games" *Proceedings of the National Acadamy of Science of the United States of America* **36**, 48–9.

Osborne, M. J., and A. Rubinstein (1994) *A Course in Game Theory,* The MIT Press: Cambridge.

Pearce, D. (1984) "Rationalizable Strategic Behavior and the Problem of Perfection" *Econometrica* **52**, 1029–50.

Sion, M. (1958) "On General Minimax Theorems" *Pacific Journal of Mathematics* **8**, 171–6.

Zimper, A. (2005) "Equivalence Between Best Responses and Undominated Strategies: A Generalization from Finite to Compact Strategy Sets" *Economics Bulletin* **3**, no. 7, 1–6.