ECONOMICS BULLETIN

# Using the compensating and equivalent variations to define the Slutsky Equation under a discrete price change

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## *Abstract*

In our experience, all textbook presentations of the Slutsky Equation under a discrete price change use a compensation scheme based on the compensating variation. Our students have sensed this convention is arbitrary in that they have asked, why consider this compensation scheme, and not one based on the equivalent variation? The present paper outlines how one might address this matter analytically, and then discusses how our findings provide a new insight into the Giffen Paradox.

An initial draft of this paper was presented in a departmental seminar at Carleton University, Ottawa, Canada. The authors thank the seminar participants (especially Keith Acheson, Stephen Ferris, Nicholas Rowe, and Frances Woolley) for their comments. Later drafts have benefited from comments offered by Marianne Vigneault, and by two anonymous referees. The authors assume responsibility for all remaining errors and omissions.
**Citation:** Leung, Ambrose and Robert Sproule, (2007) "Using the compensating and equivalent variations to define the Slutsky Equation under a discrete price change." *Economics Bulletin,* Vol. 4, No. 11 pp. 1-9
**Submitted:** January 31, 2007. **Accepted:** April 4, 2007.
**URL:** http://economicsbulletin.vanderbilt.edu/2007/volume4/EB-07D00001A.pdf

# I. Introduction

The Slutsky Equation has a long and venerated history in microeconomics. It was first articulated by Eugene Slutsky (1915) over ninety years ago, and was revisited in such classics as Hicks and Allen (1934), and Hicks (1939).[1] Today, the Slutsky Equation is a staple of most modern microeconomics textbooks [e.g., Luenberger (1995), Roberts and Schulze (1976), Takayama (1993), and Varian (1992 and 2003)], and remains a topic of ongoing research [e.g., Raiklin (1990), Panik (2002), and Weber (2002a and 2002b)].

In our experience, all textbook presentations of the Slutsky Equation under a discrete price change use a compensation scheme based on the compensating variation. Our students have sensed this convention is arbitrary in that they have asked, why consider this compensation scheme, and not one based on the equivalent variation? The present paper outlines how one might address this matter analytically, and then discusses how our findings provide a new insight into the Giffen Paradox.

The rest of this paper is organized as follows. Section II outlines the case of the decision maker without an endowment under an arbitrary utility function, and Section III outlines the case of a Cobb-Douglas utility function. Section IV outlines the case of the decision maker with an endowment, and then it argues that whether a good can be labeled a Giffen Good or not may come down to the analyst's choice of the compensation scheme and the magnitude of the price change. Concluding remarks are offered in Section V.

# II. An Arbitrary Utility Function

Consider a decision maker (DM) in an n-good world. Let $p_i$ denote the price for the ith good (where i = 1,n), let $x_i$ denote the quantity of the ith good, let m denote the DM's income, and let $\sum_{i=1}^{n} p_i x_i \;=\; m$ denote the DM's budget constraint. Let $U \;=\; U(x_1, x_2,\ ..,x_n)$ denote the DM's (well-behaved) utility function, let $x_i^* \;=\; x_i^*(p_1, p_2,..,p_n, m)$ denote the DM's n ordinary demand functions at initial prices, and let $V \;=\; V(p_1, p_2,..,p_n, m)$ denote his associated indirect utility function.

Suppose that the price of the jth good, $p_j$, *is perturbed to* $p_j + \Delta p_j$, where $\Delta p_j \;\neq\; 0$. It follows that the level of demand for the jth good at initial prices and income is $x_j^* \;=\; x_j^*(p_1, p_2,..,p_j,..,p_n, m) \;=\; x_j^*(..,p_j,..,m)$ and that the level of demand at final prices and income is $x_j^* \;=\; x_j^*(p_1, p_2,..,p_j + \Delta p_j,..,p_n, m) \;=\; x_j^*(..,p_j + \Delta p_j,..,m)$.

After Hicks (1956), the compensating variation (CV) is defined as "the maximum amount of income that could be taken from someone who gains from a particular change while still leaving him no worse off than before the change", and the equivalent variation (EV) is defined as "the minimum amount that someone who gains from a particular change would be willing to accept to forego the change" [Pearce (1992, p. 78)]. Therefore, associated with the price change, $\Delta p_j$, the CV may be restated as:

[1] For a recent history of the Slutsky Equation, see Chipman and Lenfant (2002).

$$V_1(.., p_j, .., m) \;=\; V_1(.., p_j + \Delta p_j, .., m + \Delta m)\,|_{\Delta m = CV_j} \tag{1}$$

and the EV may be restated as:

$$V_2(.., p_j, .., m + \Delta m)\,|_{\Delta m = EV_j} \;=\; V_2(.., p_j + \Delta p_j, .., m) \tag{2}$$

The DM's ordinary demand function for the jth good under a compensation scheme defined by the CV is:

$$\begin{aligned} x_j^* \;&=\; x_j^*(p_1, p_2, .., p_j + \Delta p_j, .., p_n, m + \Delta m)\,|_{\Delta m = CV_j} \\ &=\; x_j^*(.., p_j + \Delta p_j, .., m + \Delta m)\,|_{\Delta m = CV_j} \end{aligned} \tag{3}$$

Likewise, the DM's ordinary demand function for the jth good under a compensation scheme defined by the EV is:

$$\begin{aligned} x_j^* \;&=\; x_j^*(p_1, p_2, .., p_j, .., p_n, m + \Delta m)\,|_{\Delta m = EV_j} \\ &=\; x_j^*(.., p_j, .., m + \Delta m)\,|_{\Delta m = EV_j} \end{aligned} \tag{4}$$

These ideas lead to two propositions, which serve as the basis for two alternative definitions of the Slutsky Equation:

**Proposition 1 [The Slutsky Equation Under The CV-Type Compensation Scheme]:** If $U \;=\; U(x_1, x_2,\; .., x_n)$, if $p_j$ *is perturbed to* $p_j + \Delta p_j$, and if the DM is compensated for $\Delta p_j \;\neq\; 0$ by $\Delta m \;=\; CV_j \neq\; 0$, then the associated Slutsky Equation is:

$$\begin{aligned} &\frac{x_j^*(.., p_j + \Delta p_j, .., m) \;-\; x_j^*(.., p_j, .., m)}{\Delta p_j} \\ &\qquad =\; \frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \;-\; x_j^*(.., p_j, .., m)}{\Delta p_j}\,\Big|_{\Delta m = CV_j} \\ &\qquad\qquad -\; x_j^*.\frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \;-\; x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m}\,\Big|_{\Delta m = CV_j} \end{aligned}$$

where

$$\mathit{Total\ Effect}\ (TE) \;=\; \frac{x_j^*(.., p_j + \Delta p_j, .., m) \;-\; x_j^*(.., p_j, .., m)}{\Delta p_j}$$

$$Substitution\ Effect\ (SE_1) \ = \ \frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j}\Big|_{\Delta m = CV_j}$$

$$Income\ Effect\ (IE_1) \ = \ - \ x_j^*.\frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j+\Delta p_j,..,m)}{\Delta m}\Big|_{\Delta m = CV_j}$$

**Proof:** By definition,

$$\begin{aligned}
&\frac{x_j^*(..,p_j+\Delta p_j,..,m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j} \\
&\equiv \ \frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j} \\
&\quad - \ \frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j+\Delta p_j,..,m)}{\Delta p_j} \qquad (5)
\end{aligned}$$

Assume $\Delta m \ = \ CV_j \neq \ 0$. Since $\Delta p_j x_j^* \ = \ \Delta m$, thus $\frac{1}{\Delta p_j} \ = \ \frac{x_j^*}{\Delta m}$, and by Equation (5):

$$\begin{aligned}
&\frac{x_j^*(..,p_j+\Delta p_j,..,m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j} \\
&= \ \frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j}\Big|_{\Delta m = CV_j} \\
&\quad - \ x_j^*.\frac{x_j^*(..,p_j+\Delta p_j,..,m+\Delta m) \ - \ x_j^*(..,p_j+\Delta p_j,..,m)}{\Delta m}\Big|_{\Delta m = CV_j} \ \bullet
\end{aligned}$$

**Proposition 2 [The Slutsky Equation Under The EV-Type Compensation Scheme]:** If $U \ = \ U(x_1,x_2, \ ..,x_n)$, if $p_j$ *is perturbed to* $p_j+\Delta p_j$, and **i**f the DM is compensated for $\Delta p_j \ \neq \ 0$ by $\Delta m \ = \ EV_j \neq \ 0$, then the associated Slutsky Equation is:

$$\begin{aligned}
&\frac{x_j^*(..,p_j+\Delta p_j,..,m) \ - \ x_j^*(..,p_j,..,m)}{\Delta p_j} \\
&= \ \frac{x_j^*(..,p_j+\Delta p_j,..,m) \ - \ x_j^*(..,p_j,..,m+\Delta m)}{\Delta p_j}\Big|_{\Delta m = EV_j} \\
&\quad - \ x_j^*.\frac{x_j^*(..,p_j,..,m) \ - \ x_j^*(..,p_j,..,m+\Delta m)}{\Delta m}\Big|_{\Delta m = EV_j}
\end{aligned}$$

where

$$SE_2 \ = \ \frac{x_j^*(..,p_j+\Delta p_j,..,m) \ - \ x_j^*(..,p_j,..,m+\Delta m)}{\Delta p_j}\Big|_{\Delta m = EV_j}$$

$$IE_2 = - x_j^*.\frac{x_j^*(..,p_j,..,m) - x_j^*(..,p_j,..,m+\Delta m)}{\Delta m}|_{\Delta m = EV_j}$$

**Proof:** The proof here would be the mirror image of the proof for Proposition 1. •

**Remark 1:** It would seem that Proposition 1 serves as the sole analytical basis for graphical, textbook presentations of the Slutsky Equation, in that we are not aware of any textbook that presents a graph from the vantage point of Proposition 2. The dominance of one vantage point over the other is arbitrary and (as shown in Section IV below) potentially misleading.

**Remark 2:** The difference between the SEs under the CV- and EV-type compensation schemes, and the difference between the IEs under the same two compensation schemes, may increase as $\Delta p_j$ increases, a fact that underscores the need for caution in electing one compensation scheme over the other.[2] Conversely, any differences may tend to zero as the discrete price change, $\Delta p_j$, goes to zero.[3]

## III. A Cobb-Douglas Utility Function

The objective here is to show that the SE (or alternatively, the IE) under one compensation scheme may not be identical to its counterpart under the other, and this by applying Propositions 1 and 2 to the case of the Cobb-Douglas utility function. To develop these results, we require function-specific definitions of the ordinary demand function, the indirect utility function, the CV, and the EV. In particular,

**Lemma 1:** If $U = \prod_{i=1}^{n} x_i^{\alpha_i} \;\; s.t. \;\; \alpha_i \geq 0 \;\; and \; \sum_{i=1}^{n} \alpha_i = 1$, then the ordinary demand function for the jth good is:

$$x_j^* = x_j^*(p_1, p_2,.., p_j,.., p_n, m) = x_j^*(.., p_j,.., m) = \alpha_j.\frac{m}{p_j},$$

and the associated indirect utility function is:

$$V(p_1, p_2,.., p_n, m) = m \; . \prod_{i=1}^{n}\left(\frac{\alpha_i}{p_i}\right)^{\alpha_i} \tag{6}$$

**Lemma 2:** If $U = \prod_{i=1}^{n} x_i^{\alpha_i} \;\; s.t. \;\; \alpha_i \geq 0 \;\; and \; \sum_{i=1}^{n} \alpha_i = 1$, then:

(a) $CV_j = \Delta m = m.\left(\left(\frac{p_j + \Delta p_j}{p_j}\right)^{a_j} - 1\right)$, and

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[2] Because the CV and EV are identical under a quasi-linear utility function [Varian (2003, Chapter 14)], the word, "may", is used.

[3] This last result echoes a similar claim by Mosak (1942), in the case of a comparison of the SEs and IEs under a Slutsky and under a Hicksian decomposition. [For more, see Cornes (1992, p. 102).]

(b) $EV_j = \Delta m = m.\left( \left( \frac{p_j}{p_j + \Delta p_j} \right)^{a_j} - 1 \right).$

**Proof:** Parts (a) and (b) follow directly from Equations (1), (2), and (6). •

With Lemma 2 in place, we are in a position to present the function-specific counterparts to Propositions 1 and 2. These are presented next:

**Proposition 3:** If $U = \prod_{i=1}^{n} x_i^{\alpha_i} \ s.t. \ \alpha_i \geq 0 \ and \ \sum_{i=1}^{n} \alpha_i = 1$, and if compensation is defined by the CV, then the associated SE is:

$$SE_1 = \frac{\alpha_j.\frac{m + CV_j}{p_j + \Delta p_j} - \alpha_j.\frac{m}{p_j}}{\Delta p_j} = \frac{\alpha_j.\left( \frac{m + CV_j}{p_j + \Delta p_j} - \frac{m}{p_j} \right)}{\Delta p_j}$$

and the associated IE is:

$$IE_1 = -\frac{\alpha_j.\frac{m + CV_j}{p_j + \Delta p_j} - \alpha_j.\frac{m}{p_j + \Delta p_j}}{CV_j}.\frac{CV_j}{\Delta p_j} = -\frac{\frac{\alpha_j}{p_j + \Delta p_j}.CV_j}{\Delta p_j}$$

**Proposition 4:** If $U = \prod_{i=1}^{n} x_i^{\alpha_i} \ s.t. \ \alpha_i \geq 0 \ and \ \sum_{i=1}^{n} \alpha_i = 1$, and if compensation is defined by the EV, then the associated SE is:

$$SE_2 = \frac{\alpha_j.\frac{m}{p_j + \Delta p_j} - \alpha_j.\frac{m + EV_j}{p_j}}{\Delta p_j} = \frac{\alpha_j.\left( \frac{m}{p_j + \Delta p_j} - \frac{m + EV_j}{p_j} \right)}{\Delta p_j}$$

and the associated IE is:

$$IE_2 = -\frac{\alpha_j.\frac{m}{p_j} - \alpha_j.\frac{m + EV_j}{p_j}}{EV_j}.\frac{EV_j}{\Delta p_j} = \frac{\frac{\alpha_j}{p_j}.EV_j}{\Delta p_j}$$

**Remark 3:** As a numerical example of the above, suppose that the DM resides in a two-good world, and he has a Cobb-Douglas utility function of the following form, $U(x_1, x_2) = \sqrt{x_1.x_2}$. Suppose too that $p_1 = 1$, $\Delta p_1 = 1$, and $m = 100$. His associated numerical values are reported in Table 1 below.

## IV. Is This Good A Giffen Good or Not? It Depends ..

If the DM's initial endowment, $(\omega_1, \omega_2, .., \omega_n)$, is added to the choice-theoretical framework, then the Slutsky Equation under the CV-type compensation scheme is:

$$\frac{x_j^*(.., p_j + \Delta p_j, .., m) \ - \ x_j^*(.., p_j, .., m)}{\Delta p_j}$$
$$\equiv \ \frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \ - \ x_j^*(.., p_j, .., m)}{\Delta p_j}\Big|_{\Delta m = CV_j}$$
$$+ \ \left(\omega_j - x_j^*\right).\frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \ - \ x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m}\Big|_{\Delta m = CV_j}$$

where

$$\left(\omega_j - x_j^*\right).\frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \ - \ x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m}\Big|_{\Delta m = CV_j} \qquad (7)$$

denotes the associated combined income effect (CIE). Likewise, the Slutsky Equation under the EV-type compensation scheme is:

$$\frac{x_j^*(.., p_j + \Delta p_j, .., m) \ - \ x_j^*(.., p_j, .., m)}{\Delta p_j}$$
$$\equiv \ \frac{x_j^*(.., p_j + \Delta p_j, .., m) \ - \ x_j^*(.., p_j, .., m + \Delta m)}{\Delta p_j}\Big|_{\Delta m = EV_j}$$
$$+ \ \left(\omega_j - x_j^*\right).\frac{x_j^*(.., p_j, .., m) \ - \ x_j^*(.., p_j, .., m + \Delta m)}{\Delta m}\Big|_{\Delta m = EV_j}$$

where

$$\left(\omega_j - x_j^*\right).\frac{x_j^*(.., p_j, .., m) \ - \ x_j^*(.., p_j, .., m + \Delta m)}{\Delta m}\Big|_{\Delta m = EV_j} \qquad (8)$$

denotes the associated CIE.

**Table 1**

| Compensation Method | The Amount of Compensation | SE | IE | TE |
|---|---|---|---|---|
| $\Delta m \ = \ CV$ | 41.421 | -14.645 | -10.355 | -25.000 |
| $\Delta m \ = \ EV$ | - 29.289 | -10.355 | -14.645 | -25.000 |

**Remark 4:** The component parts of the CIE in Equations (7) and (8) can be viewed two ways: (a) The CIE can be seen as the sum of the ordinary income effect (OIE), and the endowment income effect (EIE). For example, in the case of Equation (7),

$$OIE_1 \; = \; - \; x_j^* . \frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \; - \; x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m} \big|_{\Delta m = CV_j}$$

and

$$EIE_1 \; = \; \omega_j . \frac{x_j^*(.., p_j + \Delta p_j, .., m + \Delta m) \; - \; x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m} \big|_{\Delta m = CV_j}$$

The same sort of statement can be made for Equation (8), in which $OIE_2$ and $EIE_2$ would be defined. (b) In addition, the CIE can be seen to be the product of two terms. For example, in the case of Equation (7), if

$$A_1 \; = \; \left( \omega_j - x_j^* \right)$$

and

$$B_1 \; = \; \left( \frac{x_j^*(.., p_j \; + \Delta p_j, .., m + \Delta m) \; - \; x_j^*(.., p_j + \Delta p_j, .., m)}{\Delta m} \right)$$

then $CIE_1 = A_1 . B_1$. The same sort of statement can be made for Equation (8), in which $A_2$, $B_2$, and $CIE_2$ would be defined.

**Remark 5:** The importance of Remark 4 is this: (a) If sign($A_k$) = - sign($B_k$) where k = 1,2, then $CIE_k < 0$. Since the $SE_k < 0$, then sign($A_k$) = - sign($B_k$) implies $TE_k < 0$. (b) However, if sign($A_k$) = sign($B_k$), then the sign($CIE_k$) depends on the relative magnitudes, $| SE_k |$ and $| CIE_k |$. Stated differently, since the $SE_k < 0$, then sign($A_k$) = sign($B_k$) implies $TE_k < 0$, if and only if $| SE_k | \; - \; | CIE_k | > \; 0$. Alternatively, if sign($A_k$) = sign($B_k$), and if $| SE_k | \; - \; | CIE_k | < \; 0$, then $TE_k > 0$, all of which serves to define a Giffen Good.

**Remark 6:** In Sections II and III, we argued that both the choice of compensation scheme and the magnitude of the price change, $\Delta p_j$, can affect the magnitudes of the $SE_k$ and $IE_k$ (or alternatively defined, the $OIE_k$). It follows from this that if sign($A_k$) = sign($B_k$), then the choice of the compensation scheme, and of the magnitude of $\Delta p_j$, may affect the sign of the difference, $| SE_k | \; - \; | OIE_k + EIE_k |$, and hence the sign of the $TE_k$.

**Remark 7:** In a sense, the discussion in the present section offers the discrete counterpart to Berg's (1987) analysis of the choice-theoretical foundations of the Giffen Paradox, and to some of the arguments in the prior debate [e.g., Dougan (1982), and Dooley (1983a, 1983b, and 1985)]. But in another sense, this paper in the aggregate offers a new twist, which is: whether a good can be labeled a Giffen Good or not depends in part upon the interplay of two factors -- the analyst's choice of the compensation scheme and the magnitude of the price change, $\Delta p_j$.

## V. Concluding Remarks

We began this paper by noting that the textbook presentation of the Slutsky Equation under a discrete price change uses a compensation scheme based on the compensating variation. The question was then posed, how to define the Slutsky Equation when the compensation scheme is based on the equivalent variation? This paper then outlined analytically how the Slutsky Equation might be defined under both schemes.

This paper then outlined how the present analysis extends our common understanding of the Giffen Paradox under a discrete price change.

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