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Seasonal fractional integration with structural break. An application to the German GNP data

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Abstract

This paper deals with the analysis of the German nominal GNP quarterly data (1973q1 – 1996q4) using a new approach based on seasonal fractional integration that allows us to incorporate a structural break that is endogenously determined by the model. The results show that the break occurs at 1990q2, the time of the German re-unification, and the order of integration is slightly above 1 before the break, and strictly smaller than 1 (though highly persistent) after the unification.

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1. Introduction

Modelling seasonality in macroeconomic time series is a matter that still remains controversial. Deterministic models based on seasonal dummy variables have been employed for many years. However, in many cases, the seasonal component changes or evolves over time, and seasonal unit root models have been preferred. Test statistics developed for testing seasonal unit roots are among others those of Dickey, Hasza and Fuller (1984), Hylleberg, Engle, Granger and Yoo (1990), Tam and Reimsel (1997), etc. The seasonal unit root model, however, is merely one very specialized case of a more general class of model, called seasonal fractional integration, where the number of seasonal differences required to get stationary I(0) disturbances may not necessarily be an integer value but a fractional one, (see, e.g., Gil-Alana, 2002). Lildholdt (2002) provides both theoretical and Monte Carlo evidence that seasonal fractional integration may be generated by: a) cross-sectional aggregation of seasonal data; b) aggregation of seasonal duration models, and c) regime-switching if the underlying Markov process possesses seasonal dependencies.

In this paper we apply a seasonal fractional integration model to the quarterly nominal GNP data in Germany. Moreover, given the structure of the series and the time period considered (1975q1 – 1996q4) we also allow for a structural break that is endogenously determined by the model.

The outline of the article is as follows: In Section 2 we describe the statistical model and present a simple procedure for estimating the coefficients associated to the deterministic terms and the fractional differencing parameters at each subsample along with the time of the break. In Section 3 the procedure is applied to the German GNP data while Section 4 contains some concluding comments.

2. The statistical model

We suppose that y_t is the observed seasonal time series, with a periodicity s = 4, generated by the model:

$$y_t = \alpha_1 + \beta_1 t + x_t; \quad (1 - L^4)^{d_1} x_t = u_t, \quad t = 1, \ldots, T_b \tag{1}$$

$$y_t = \alpha_2 + \beta_2 t + x_t; \quad (1 - L^4)^{d_2} x_t = u_t, \quad t = T_b + 1, \ldots, T, \tag{2}$$

where the α's and the β's are the coefficients corresponding respectively to the intercept and the linear trend; d_1 and d_2 may be real values, u_t is I(0) and T_b is the time of the break that is supposed to be unknown. Note that the model in equations (1) and (2) can also be written as:

$$(1 - L^4)^{d_1} y_t = \alpha_1 \tilde{1}_t(d_1) + \beta_1 \tilde{t}_t(d_1) + u_t, \quad t = 1, \ldots, T_b, \tag{3}$$

$$(1 - L^4)^{d_2} y_t = \alpha_2 \tilde{1}_t(d_2) + \beta_2 \tilde{t}_t(d_2) + u_t, \quad t = T_b + 1, \ldots, T, \tag{4}$$

where $\tilde{1}_t(d_i) = (1 - L^4)^{d_i} 1$, and $\tilde{t}_t(d_i) = (1 - L^4)^{d_i} t$, i = 1, 2.

The procedure presented here is based on the residuals sum square principle. First we choose a grid for the values of the fractionally differencing parameters d_1 and d_2, for example, $d_{io} = 0, 0.01, 0.02, \ldots, 1$, $i = 1, 2$. Then, for a given partition $\{T_b\}$ and given d_1, d_2-values, (d_{1o}, d_{2o}), we estimate the α's and the β's by minimizing the sum of squared residuals,[1]

$$\min_{w.r.t.\{\alpha_1,\alpha_2,\beta_1,\beta_2\}} \sum_{t=1}^{T_b} \left[(1-L^4)^{d_{1o}} y_t - \alpha_1 \, \tilde{1}_t(d_{1o}) - \beta_1 \, \tilde{t}_t(d_{1o})\right]^2 + \sum_{t=T_b+1}^{T} \left[(1-L^4)^{d_{2o}} y_t - \alpha_2 \, \tilde{1}_t(d_{2o}) - \beta_2 \, \tilde{t}_t(d_{2o})\right]^2$$

Let $\hat{\alpha}(T_b;\, d_{1o}^{(1)}, d_{2o}^{(1)})$ and $\hat{\beta}(T_b;\, d_{1o}^{(1)}, d_{2o}^{(1)})$ denote the resulting estimates for partition $\{T_b\}$ and initial values $d_{1o}^{(1)}$ and $d_{2o}^{(1)}$. Substituting these estimated values on the objective function, we have RSS(T_b; $d_{1o}^{(1)}$, $d_{2o}^{(1)}$), and minimizing this expression across all values of d_{1o} and d_{2o} in the grid we obtain:

$$RSS(T_b) \;=\; \arg\min_{\{i,j\}} \; RSS(T_b; d_{1o}^{(i)}, d_{2o}^{(i)}).$$

Then, the estimated break date, $\hat{T}_k$, is such that

$$\hat{T}_k \;=\; \arg\min_{\,i=1,\ldots,m} RSS\,(T_i)\,,$$

where the minimization is taken over all partitions $T_1, T_2, \ldots, T_m$, such that $T_i - T_{i-1} \geq |\varepsilon T|$. Then, the regression parameter estimates are the associated least-squares estimates of the estimated k-partition, i.e. $\hat{\alpha}_i = \hat{\alpha}_i(\{\hat{T}_k\})$; $\hat{\beta}_i = \hat{\beta}_i(\{\hat{T}_k\})$, and their corresponding differencing parameters, $\hat{d}_i = \hat{d}_i(\{\hat{T}_k\})$, for i = 1 and 2.

The model can easily be extended to the case of multiple breaks. Thus, we can consider the model,

$$y_t \;=\; \alpha_j \;+\; \beta_j\, t \;+\; x_t; \;\; (1 \,-\, L^4)^{d_j} x_t \;=\; u_t, \quad t \;=\; T_{j-1}+1,\ldots,T_j,$$

for $j = 1, \ldots, m+1$, $T_0 = 0$ and $T_{m+1} = T$. Then, the parameter m is the number of changes. The break dates $(T_1, \ldots, T_m)$ are explicitly treated as unknown and for $i = 1, \ldots, m$, we have $\lambda_i = T_i/T$, with $\lambda_1 < \ldots < \lambda_m < 1$. Note that this model is similar to the one proposed by Bai and Perron (1998) and Boutahar and Jouini (2004) for the case of stationary AR processes, though we extend it to the case of I(d) models with the singularity or pole in the spectrum not restricted to the zero frequency.

Following the same lines as in the previous case, for each j-partition, $\{T_1, \ldots T_j\}$, denoted $\{T_j\}$, the associated least-squares estimates of α_j, β_j and the d_j are obtained by minimizing the sum of squared residuals in the d_i-differenced models, i.e.,

[1] In case of autocorrelated disturbances the coefficients can be estimated by GLS.

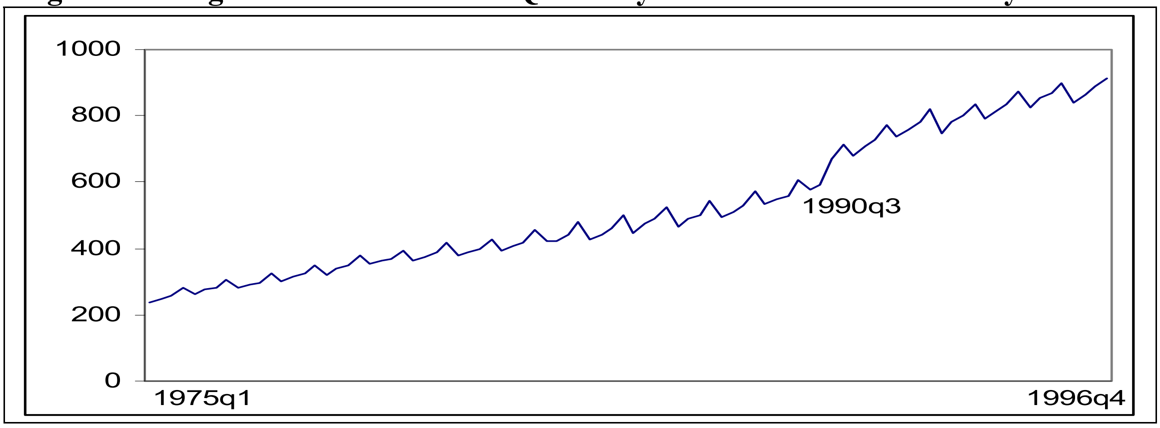
$$\sum_{j=1}^{m+1} \sum_{t=T_{j-1}+1}^{T_j} (1 - L^4)^{d_i} (y_t - \alpha_i - \beta_i t)^2,$$

where $\hat{\alpha}_i(T_j)$, $\hat{\beta}_i(T_j)$ and $\hat{\mathrm{d}}(\mathrm{T_j})$ denote the resulting estimates. Substituting them in the new objective function and denoting the sum of squared residuals as $RSS_T(T_1, \ldots, T_m)$, the estimated break dates $(\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_m)$ are obtained by: $\min_{(T_1, T_2, \ldots, T_m)} RSS_T(T_1, \ldots, T_m)$, where the minimization is again obtained over all partition $(T_1, \ldots, T_m)$.[2]

3. The empirical application

The time series data analysed in this section correspond to the quarterly, seasonally unadjusted, nominal German Gross National Product, GNP, for the time period 1975q1 - 1996q4, obtained from the Deutsches Institut für Wirtschaftsforschung, Volkswirtschaftliche Gesamtrechnung. Until 1990q2 the data refer to West Germany only, and all of Germany is included after the re-unification. The time series data are displayed in Figure 1.

Figure 1: Original time series data: Quarterly nominal GNP in Germany



We observe in this figure a clear seasonal pattern that is changing across time. Thus, the series seems to be nonstationary, and a structural break seems to occur due to the unification. Moreover, the inclusion of seasonal dummies produced insignificant coefficients in practically all cases.

We now present the results of the procedure described in Section 2. We assume that there is a single break across the sample and consider the cases of white noise, AR(1) and seasonal AR(1) disturbances. In the latter case, we suppose that u_t follows a process of form: $u_t = \alpha\, u_{t-4} + \varepsilon_t$. The results are displayed in Table I.

Assuming that the disturbance term u_t is white noise or AR(1), the break date occurs at 1990q3, which corresponds to the first quarter after the unification. However, if u_t is modelled throughout a seasonal AR process, the break takes place two periods before (1990q1). Starting with the case of white noise u_t, the orders of integration are 1.23 and

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[2] Note that the deterministic structure in (1) and (2) can also include seasonal dummies and the procedure can be carried out exactly in the same lines as the one reported here.

0.73 respectively for each subsample, and the coefficients associated to the time trends are statistically significant in the two cases. If we permit an AR(1) structure for u_t, the order of integration is slightly smaller in the first subsample, though still above 1 ($d_1 = 1.03$), while $d_2 = 0.85$. Finally, if u_t is seasonally AR, the structure completely changes, with $d_1 = 0.06$ and $d_2 = 0.54$. The low orders of integration in the latter case are clearly due to the competition between them and the AR coefficients in describing the seasonal nonstationarity.[3] Note that for the first subsample d_1 is very close to 0, though the AR coefficient is then very close to the unit circle ($\alpha_1 = 0.908$).

Table I: Estimation based on seasonal fractional integration with a linear trend and a single break

Estimation based on seasonal fractional integration with a linear trend and a single break									
u_t	T_b	First subsample				Second subsample			
		d_1	α_1	β_1	AR_1	d_2	α_2	β_2	AR_2
White noise	1990q3	1.23	242.63 (65.51)	5.911 (14.04)	---	0.73	156.56 (3.820)	8.383 (14.20)	---
AR (1)	1990q3	1.03	259.27 (43.27)	5.921 (11.31)	0.645	0.85	157.23 (2.254)	8.310 (8.09)	0.426
Seasonal AR (1)	1990q1	0.06	24.366 (0.30)	7.903 (8.775)	0.908	0.54	423.23 (10.94)	5.711 (12.68)	0.020

t-values in parenthesis.

4. Concluding comments

In this article we have applied a procedure for estimating linear trends and orders of integration in the context of seasonal fractional integration with a structural break to the nominal German GNP quarterly data. The procedure is based on the residuals sum square principle for a grid of values of the fractional differencing parameters and time-breaks. The method correctly detects the break at 1990q2, the time of the German re-unification, and finds that the two subsamples are nonstationary (d_1, $d_2 > 0.5$) with a higher degree of persistence before the break. In fact, d_1 seems to be above 1 in the first subsample, while d_2 is strictly below 1 (and thus showing mean reversion) after the unification.

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[3] Note that if d = 1 in the model $(1-L^4)^d x_t = u_t$ and white noise u_t, a very similar model, though with very different statistical properties can be obtained if d = 0 and u_t is of form: $u_t = \alpha u_{t-1} + \varepsilon_t$, with α close to 1.

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