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Linearity, Slutsky symmetry, and a conjecture of Hicks

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Abstract

Hicks (1956) conjectured that Slutsky symmetry should hold for discrete as well as infinitesimal price changes if demand functions are globally linear. This paper proves this conjecture using the LES utility function and the Slutsky compensation for price changes. More importantly, in sharp contrast to previous doubts expressed by Hicks, Samuelson and others, this paper provides the first formal demonstration that compensated cross price effects can indeed be symmetric for discrete changes in prices.

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1. Introduction

It has long been understood that the symmetry of the compensated cross price effects first demonstrated by Slutsky (1915) will in general hold exactly only for infinitesimal changes in prices. Samuelson (1947a) appears to have been one of the first to argue that Slutsky's symmetry result seems to apply only for differential size changes in prices. He noted that the symmetry condition reflects "differential properties of our demand functions which are hard to visualize and hard to refute." (Samuelson 1947a, p. 107) He noted that he had "tried, but thus far without success, to deduce implications of our integrability conditions which can be expressed in finite form …" (Samuelson 1947a, p. 107)

Following Samuelson's initial contribution, Samuelson (1953), Hicks (1956), and Heady (1986) all argued in essence that for finite or discrete changes in prices, Slutsky symmetry is an approximation which should hold with greater accuracy as changes in prices become smaller. However, in his discussion of the Slutsky symmetry condition, Hicks (1956) went a bit further by claiming, without providing a proof, that symmetry will necessarily hold even for arbitrary discrete changes in prices as long as compensated demand curves are linear in prices. According to Hicks' conjecture, if compensated demand curves are globally linear, then symmetry should hold exactly, not merely as an approximation, for arbitrarily large discrete changes in prices as well as for infinitesimal changes in prices.

This paper uses the linear expenditure system (LES) utility function first studied by Klein and Rubin (1947), Samuelson (1947b), Geary (1950), and Stone (1954) to prove Hicks' conjecture. As Samuelson (1947b) and Geary (1950) showed, if demand curves are to be linear in income and the prices of other goods, then the utility function must take the specific functional form considered here. Thus, we can use this functional form to investigate the connection between linearity of demand curves in prices and Slutsky symmetry for discrete price changes. Beyond proving Hicks' conjecture, this paper's main contribution is to provide the first demonstration that compensated cross price effects can be symmetric even for non-infinitesimal changes in prices.

2. Discrete Price Changes LES Utility

This section discusses compensated demand curves derived for the linear expenditure system utility function. Thus, recall that the LES utility function defined over n market goods, $x_1, \ldots x_n$, is given by:

$$U(x_1,\ x_2,\ \ldots,\ x_n) = \sum_{i=1}^{n} \alpha_i \ln(x_i - \beta_i)\ ; \quad \alpha_i > 0, \quad \beta_i > 0, \quad \forall\ i \sum_{i=1}^{n} \alpha_i = 1; \tag{1}$$

where the α_i's and β_i's are parametric constants. Since this functional form reduces to the Cobb-Douglas functional form in the special case where $\beta_i = 0$ for all i, the results derived here necessarily hold for compensated demand functions derived from the Cobb-Douglas utility function.

Since we will be concerned here with discrete changes in prices, it is important to remember that when price changes are assumed to be discrete, there are two different types of compensated demand curves. Along the Hicksian demand curve the household is compensated for changes in prices so as to hold utility constant as prices change. In contrast, along the Slutsky demand curve, the household is compensated for changes in prices so as to permit the household to continue purchasing the initial bundle of goods as prices change. Mosak (1942) first showed that these two compensated demand curves are only locally equivalent, that is, equivalent only for infinitesimal changes in prices. It is straightforward if slightly tedious to show that while the LES utility function yields Hicksian demand functions which are non-linear functions of all prices, it yields a Slutsky demand function for good i which is non-linear in p_i only. The Slutsky demand function is linear in the prices of all other goods. (The Appendix provides a proof that the Hicksian demand functions are non-linear in all prices.) That is, for a given own price p_j, the Slutsky demand function for good x_j is a linear function of p_i for $i \neq j$.

To derive the Slutsky demand functions, it is simplest to begin with the Marshallian demand functions. The Marshallian demand function for good j shows the quantity of good j demanded as a function of income, I, which is assumed to be strictly positive, and the prices of the goods, $p_1, \dots p_n$, which are assumed to be strictly non-negative. It is straightforward to show that for the utility function in (1), the Marshallian demand curve for good j is given by:[1]

$$x_j(p_1,\ p_2,\ \dots,\ p_n,\ I) = \beta_j + \frac{\alpha_j I}{p_j} - \frac{\alpha_j}{p_j}\sum_{i=1}^{n}\beta_i p_i . \qquad (2)$$

Silberberg and Suen (2001) derive equation (2).

We can convert this Marshallian demand function into a Slutsky demand function, which holds total expenditure on all goods constant as prices change, as follows: Assume that instead of holding income, I, constant as prices change, we adjust income in response to a change in any price so as to permit the household to consume the given bundle, $[x_1^*\ x_2^*\ \dots\ x_n^*]$, at all sets of prices. That is, we adjust income as prices change so that at all price combinations, $I = f(p_1, p_2, \dots, p_n) = \Sigma_i p_i x_i^*$ with x_i^* treated as being parametrically given for all i. Substituting this income function into (2) yields the Slutsky demand function:

$$x_j\left(p_1,\ p_2,\ \dots,\ p_n,\ \sum_{i=1}^{n} p_i x_i^*\right) = \beta_j + \frac{\alpha_j \sum_{i=1}^{n} p_i x_i^*}{p_j} - \frac{\alpha_j}{p_j}\sum_{i=1}^{n}\beta_i p_i$$

$$= \beta_j + \frac{\alpha_j}{p_j}\sum_{i=1}^{n} p_i\left(x_i^* - \beta_i\right) . \qquad (3)$$

Since inspection of (3) reveals that this demand function is linear in all of the prices except the own price, p_j, we can easily calculate the impact of discrete changes in prices other than p_j on the demand for good j. For example, for a discrete change in p_k, $j \neq k$, the functional form in (3) implies that we have:

$$\Delta x_j = \frac{\alpha_j}{p_j}\left(x_k^* - \beta_k\right)\Delta p_k, \qquad or:$$

$$\frac{\Delta x_j}{\Delta p_k} = \frac{\alpha_j}{p_j}\left(x_k^* - \beta_k\right) . \qquad (4)$$

Similarly, the effect of a discrete change in p_j on the demand for good k with $j \neq k$, is given by:

$$\Delta x_k = \frac{\alpha_k}{p_k}\left(x_j^* - \beta_j\right)\Delta p_j, \qquad or:$$

$$\frac{\Delta x_k}{\Delta p_j} = \frac{\alpha_k}{p_k}\left(x_j^* - \beta_j\right) . \qquad (5)$$

To prove symmetry, we must show that if the consumption bundle, [x_1* x_2* … x_n*], was optimal at the initial set of prices, then the expressions in (4) and (5) must be equal. To see that this is indeed the case, recall that since the LES utility function is quasiconcave and twice differentiable in the x_i's, at the initial set of prices the first order conditions for maximizing the utility function in (1) subject to the usual budget constraint include the budget constraint and:

$$\frac{\partial U}{\partial x_i} = \frac{\alpha_i}{x_i^* - \beta_i} = \lambda p_i \quad \forall\ i, \qquad (6)$$

where λ is the marginal utility of income, that is, the Lagrange multiplier for the income constraint in the household's constrained utility maximization problem. At the initial price vector, the Lagrange multiplier and the n x_i*'s solve the budget constraint and the n versions of equation (6). Finally, observe that the n f.o.c.'s in (6) imply:

$$\frac{\alpha_j}{p_j(x_j^* - \beta_j)} = \lambda = \frac{\alpha_k}{p_k(x_k^* - \beta_k)}. \qquad (7)$$

Multiplying equation (7) through by (x_j* - β_j)(x_k* - β_k), we see immediately that at the optimum, the expressions for the Slutsky cross price effects on the righthand sides of (4) and (5) must both equal λ(x_j* - β_j)(x_k* - β_k), and hence that they must equal each other. Since this shows that the cross price effects in (4) and (5) are equal even for discrete changes in prices, we have verified Hicks' conjecture: If the (Slutsky compensated) demand function for each good is linear in the prices of all other goods, then compensated cross price effects must be equal for discrete as well as infinitesimal changes in prices. As was mentioned in the introduction, this seems to be the first formal demonstration that for some preferences, the Slutsky symmetry condition holds even for discrete changes in prices.

3. Conclusion

In the process of proving Hicks' conjecture, we have also shown that historical pessimism concerning the possibility that cross price effects might be symmetric for arbitrarily large discrete changes in prices (see, e.g., Samuelson (1947a, 1953), Hicks (1956), and Heady (1986)) has been at least partially misplaced. That is, for at least one class of preferences, compensated cross price effects will indeed be symmetric even for arbitrary non-infinitesimal changes in prices if the compensation follows Slutsky rather than Hicks.

It is already understood that the own substitution effect is negative for discrete as well as infinitesimal price changes. Similarly, when the income compensation takes the Slutsky (1915) rather than the Hicks (1946) form, the Slutsky equation and the implication that the price weighted sum of the compensated cross price effects is zero all still hold for discrete price changes. This paper has added a fourth comparative statics implication of the utility maximization hypothesis (under a suitable restriction on preferences) which continues to hold for discrete changes in prices. It remains an important topic for future research to determine conditions under which other comparative statics results originally derived only for infinitesimal changes in parameters continue to hold for discrete changes as well.

APPENDIX

This appendix demonstrates that Hicksian demand functions derived from the LES utility function will be non-linear functions of all prices. To see this, simply substitute the Marshallian demand function in equation (2) back into the utility function in (1) to derive the indirect utility function, V(p_1, p_2, …, p_n, I):

$$V(p_1,\ p_2,\ \ldots,\ p_n,\ I) \ = \ \sum_{j=1}^{n} \ln\left(\frac{\alpha_j I}{p_j} \ - \ \frac{\alpha_j}{p_j}\sum_{i=1}^{n}\ln\beta_i p_i\right)^{\alpha_j} \ = \ \ln\left[\left(I \ - \ \sum_{i=1}^{n}\beta_i p_i\right)\prod_{j=1}^{n}\left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}\right],$$

where the second equality makes use of the normalization $\Sigma_j\alpha_j$=1. Setting utility equal to the constant V^0, exponentiating, and rearranging slightly then yields the following equivalent representation of the household's indirect preferences, V*(p_1, p_2, …, p_n, I):

$$V^*(p_1,\ p_2,\ \ldots,\ p_n,\ I) \ = \ e^{V^0}\prod_{j=1}^{n}\alpha_j^{-\alpha_j} \ = \ \left(I \ - \ \sum_{i=1}^{n}\beta_i p_i\right)\prod_{j=1}^{n}p_j^{-\alpha_j}.$$

Then invert the indirect utility function to obtain the expenditure function:

$$E(p_1,\ p_2,\ \ldots,\ p_n,\ V^0) \ = \ I \ = \ e^{V^0}\prod_{j=1}^{n}\alpha_j^{-\alpha_j}\prod_{j=1}^{n}p_j^{\alpha_j} \ + \ \sum_{i=1}^{n}\beta_i p_i.$$

By virtue of Hotelling's lemma, the Hicksian compensated demand function for good j is the derivative of the expenditure function with respect to p_j. Since none of the α_j's equals 2, the expenditure function is not quadratic in any price. This in turn implies that the Hicksian compensated demand functions must be non-linear functions of all prices. Hence, we cannot test Hicks' conjecture on the Hicksian demand functions, but only on Slutsky demand functions. Silberberg and Suen (2001) provide a similar derivation of the Hicksian demand for the LES utility function.

REFERENCES

Geary, R.C. (1950) "A Note on 'A Constant-utility Index of the Cost of Living'" *Review of Economic Studies* **18**, 65-66.

Heady C. (1986) "A New Look at the Symmetry of the Slutsky Matrix" Unpublished working paper, University College, London.

Hicks, J.R. (1946) *Value and Capital*, 2nd ed., Oxford University Press: Oxford.

Hicks, J.R. (1956) *A Revision of Demand Theory* Oxford University Press: Oxford.

Klein, L.R., and H. Rubin (1947) "A Constant-utility Index of the Cost of Living" *Review of Economic Studies* **15**, 84-87.

Mosak, J. (1942) "On the Interpretation of the Fundamental Equation of Value Theory" In *Studies in Mathematical Economics and Econometrics in Memory of Henry Schultz* by O. Lange, F. McIntyre, and T. Yntema, Eds., University of Chicago Press: Chicago, 69-74.

Samuelson, P.A. (1947a) *Foundations of Economic Analysis*, Harvard University Press: Cambridge.

Samuelson, P.A. (1947b) "Some Implications of 'Linearity'" *Review of Economic Studies* **15**, 88-90.

Samuelson, P.A. (1953) "Consumption Theorems in Terms of Overcompensation Rather than Indifference Comparisons" *Economica* **20**, 1-9.

Silberberg, E., and W. Suen (2001) *The Structure of Economics: A Mathematical Analysis*, Irwin McGraw-Hill: New York.

Slutsky, E.E. (1915) "Sulla teoria del bilancio del consumatore" *Giornale degli Economisti* **51**, 1-26; reprinted as "On the Theory of the Budget of the Consumer" in *A.E.A. Readings in Price Theory* by J. Stigler and K.E. Boulding, Eds. (1952) Richard D. Irwin, Inc. Chicago.

Stone, R. (1954) "Linear Expenditure Systems and Demand Analysis: An Application to the Pattern of British Demand" *Economic Journal* **64**, 511-527.