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Solutions without dummy axiom for TU cooperative games

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Abstract

In this paper we study an expression for all additive, symmetric and efficient solutions, i.e., the set of axioms that traditionally are used to characterize the Shapley value except for the dummy axiom. Also, we obtain an expression for this kind of solutions by including the self duality axiom. These expressions allow us to give an alternative formula for the consensus value, the generalized consensus value and the solidarity solution. Furthermore, we introduce a new axiom called coalitional independence which replaces the symmetry axiom and use it to get similar results.

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1 Introduction

Since the seminal axiomatic contribution of Shapley in 1953, the variations of his novel axioms have played a key role in the literature of transferable-utility cooperative games, mainly trying to understand different applications. Most variations substitute his original axioms either by weaker versions or by different axioms that fit certain application. However, very few papers consider the raw solutions that emerge by taking out one or two of the original axioms. This paper is one of them; it provides a closed form expression of semivalues without the dummy axiom. It also proposes a weakening of the symmetry axiom, and provides a closed form expression of these solutions.

Semivalues without the dummy axiom have applications in problems incompatible with a subsidy-free scenario. For instance, labor unions require employers to pay some minimum compensation even if an employee does not work: a waiter receives a minimum salary when the restaurant is empty; and, in certain countries, workers enjoy unemployment insurance from the government.

This paper is divided in five parts. In part 2 we give an alternative proof for the general formula for semivalues without the dummy axiom. In part 3 we replace symmetry by a coalition independent axiom. In part 4 we relate the previous results with solutions in the literature. In part 5 we remark the extension of previous formulas to non-additive semivalues. All proofs are written in the appendix.

2 Solutions without a dummy axiom

By a game we mean a pair (N, v) where $N \subset \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a real function such that $v(\emptyset) = 0$. Let $G = G^N$ be the set of games with a fixed set of players N. We consider N fixed and $n = |N|$. Let (N, z) be the zero game, i.e., the game defined by $z(T) = 0$ for every $T \subseteq N$.

Definition 1 *By a solution in G we mean a continuous function $\varphi : G \to \mathbb{R}^N$. Let V be the set of solutions in G.*

A solution is a rule to divide the common gain or cost among the players in N. The requirement of continuity, in the definition of a solution, is necessary to obtain all of our results. Also, it is desirable to have a zero payoff for the zero game because in this case no coalition generates gain or cost.

In this section, we obtain an expression for all additive, symmetric and efficient solutions: that is, all the axioms that traditionally characterize the Shapley value except the dummy axiom. A similar, albeit different expression of linear, symmetric and efficient semivalues can be found in the appendix of Ruiz et al. (1998) and Hernández et al. (2007).

Proposition 1 *The solution φ satisfies additivity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{1}$$

for some $n-1$ *real numbers* $\{\beta_s\}_{s=1}^{n-1}$.

Proof of this result, and the ones that follow, are given in the Appendix.

We denote by φ^β the solution φ with parameters $\{\beta_s\}_{s=1}^{n-1}$ when we need to refer to its parameters. Every additive, symmetric and efficient solution is of the form (1) and for every set of real numbers $\{\beta_s\}_{s=1}^{n-1}$ in (1) we get a solution that satisfies these three axioms. Furthermore, notice that for different sets of real numbers $\{\beta_s\}_{s=1}^{n-1}$ we get different solutions, therefore, the dimension of this (affine) subspace of solutions is $n-1$. Moreover, the Shapley value corresponds to the numbers $\beta_s = \frac{(s-1)!(n-s-1)!}{n!}$, $s = 1, \ldots, n-1$.

We may interpret (1) as follows. With respect to an efficient solution, a game v has the following information: $v(N)$ tells us the total amount to be shared and $v(S)$, $S \neq N$, the amount that coalition S claims for itself. So, we start with the egalitarian solution, i.e., we give $\frac{v(N)}{n}$ to each player. We keep going with one transference from $N\backslash S$ to S for each coalition $S \neq N, \emptyset$: Every player in $N\backslash S$ pays $s\beta_s v(S)$ and every player in S receives $(n-s)\beta_s v(S)$. Notice that we use the same factor β_s for coalitions with equal cardinality s. At the end, player i has an amount $\varphi_i(v)$ given by (1).

3 Solutions satisfying the coalitional independence axiom

In this section we replace the symmetry axiom in the previous proposition with a new one that we call coalitional independence axiom. This axiom looks like the fair ranking axiom of Chun (1989).

Definition 2 *We say that the two games* (N, v) *and* (N, w) *only differ in* S *if and only if* $v(T) = w(T)$ *for every coalition* $T \neq S$.

The coalitional independence axiom requests that the solution changes equally for any two players in S or any two players in $N\backslash S$ for every two games that only differ in S.

Axiom 1 (Coalitional independence) *We say that* φ *satisfies the coalitional independence axiom if*

$$\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w)$$

for every two games (N, v) *and* (N, w) *that only differ in* S *and* $i,\, j \in S$ *or* $i,\, j \in N\backslash S$.

Remark 1 *Notice that additivity and symmetry imply coalitional independence, but additivity and coalitional independence do not imply symmetry. Indeed, let* φ *be the solution given by,*

$$\varphi_i(v) = \sum_{j \in N} j v(\{j\}).$$

Clearly φ *satisfies additivity and coalitional independence but not symmetry. Now, suppose that* φ *is an additive and symmetric solution. Let* v,$w \in G$ *be two games that only differ in*

S, and take either $i,\, j \in S$ or $i,\, j \in N \backslash S$. Let θ be the permutation of N that interchanges i and j. Then $\theta^*(v-w) = v - w$, so, $\varphi(v-w) = \varphi(\theta^*(v-w)) = \theta^*\varphi(v-w)$.

Therefore, $\varphi_i(v-w) = \varphi_j(v-w)$. Hence, φ satisfies coalitional independence.

Proposition 2 *The solution φ satisfies additivity, coalitional independence and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N\backslash S} v(N\backslash S)] \tag{2}$$

for some set of $2^n - 2$ real numbers $\{\beta_S\}_{\emptyset \neq S \subsetneq N}$.

We get a similar expression as that of proposition 1, except that now we have one β_S for each non empty coalition $S \neq N$. Again, every solution that satisfies the additive, symmetry and efficiency axioms is of the form (2) and for every set of real numbers $\{\beta_S\}_{\emptyset \neq S \subsetneq N}$ we get a solution that satisfies these axioms. Now, the next corollary includes the self duality axiom as part of its hypotheses **2**. Again, including this axiom roughly halves the dimension of the space of solutions.

Axiom 2 (Self duality) *We say that the solution φ is self dual if $\varphi(v) = \varphi(v^*)$ for every game $v \in G$.*

Corollary 1 *The solution φ satisfies additivity, coalitional independence, efficiency and self duality axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N\backslash S} v(N\backslash S)] \tag{3}$$

for some set of $2^{n-1} - 1$ real numbers $\{\beta_S\}_{S \subseteq N}$ such that $\beta_S = \beta_{N\backslash S}$.

We conclude this section with a characterization of the Shapley value, we replace the symmetry axiom with the coalitional independence axiom.

Proposition 3 *The Shapley value is the unique solution that satisfies additivity, coalitional independence, dummy and efficiency axioms.*

4 Some special cases

In this section we briefly see some special solutions of the form (1) that do not satisfy the dummy axiom (i.e. different from Shapley's value). A first example is the Equal Surplus solution:

$$\varphi_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}$$

which we get when we choose $\beta_1 = \frac{1}{n}$ and $\beta_s = 0$ for $s \neq 1$ in (1). Another self dual solution with a simple expression is

$$\varphi_i(v) = \sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}$$

where $\beta_s = \frac{1}{s(n-s)}$ in (1).

4.1 The Consensus Value

The convex linear combination (rather than the linear combination) of two solutions of the form (1) gives another solution of the same form. Moreover, the parameters of the new solution are just the convex combination of those of the two original solutions. More precisely, for any two solutions φ^β and φ^γ, and a real number $\theta \in [0,1]$:

$$(1-\theta)\varphi^\beta + \theta\varphi^\gamma = \varphi^{(1-\theta)\beta+\theta\gamma}.$$

In this sense, Ju et al. (2007) prove that the consensus value is the middle point between the Equal Surplus solution and the Shapley value. Thus an expression for the consensus value is:

$$\frac{v(N)}{n} + \frac{1}{2}\left[v(\{i\}) - \frac{\sum_{k\neq i} v(\{k\})}{n-1}\right] + \frac{1}{2}\sum_{S\ni i, |S|\neq n, n-1, 1} (n-s)[\frac{(s-1)!(n-s-1)!}{n!}(v(S) - v(N\backslash S))].$$

In the same way, we could generate an expression for any generalized consensus value, i.e., we would only need to replace $\beta_1 = \frac{1-\theta}{n} + \frac{\theta}{n(n-1)}$ and $\beta_s = \frac{\theta(s-1)!(n-s-1)!}{n!}$ for $s = 2, .., n-1$, in (1).

4.2 Solidarity value

Nowak and Radzik (1997) introduce the solidarity value. They define, for any non-empty coalition T and any game $v \in G$, $A^v(T) = \frac{1}{t}\sum_{k\in T}[v(T) - v(T\backslash\{k\})]$. Then, they define the solidarity value for player i as,

$$\psi_i(v) = \sum_{T \ni i} \frac{(n-t)!(t-1)!}{n!} A^v(T) \tag{4}$$

They characterized this value with the efficiency, additivity, symmetry and A-null player axioms, so the solidarity value must be a special case of (1). Indeed, if we expand (4) we get that the coefficient of $v(S)$, for a coalition T which does not contain i, is $\frac{(n-s-1)!s!}{n!}\frac{1}{s+1}$. Thus, this coefficient corresponds to $s\beta_s$ in (1), and therefore

$$\beta_s = \frac{(n-s-1)!(s-1)!}{(s+1)n!}$$

which gives us an alternative expression for (4):

$$\psi_i(v) = \frac{v(N)}{n} + \sum_{S\ni i, S\neq N} \frac{(n-s)!(s-1)!}{n!}\left[\frac{v(S)}{s+1} - \frac{v(N\backslash S)}{n-s+1}\right].$$

Observe that the solidarity value is not self dual since $\beta_s \neq \beta_{n-s}$.

4.3 Least Square Prenucleolus

Lastly, Ruiz et al. (1996) introduce the Least Square Prenucleolus solution,

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\left[\sum_{S\ni i}(n-s)v(S) - \sum_{S\not\ni i} sv(S)\right]$$

This solution is also of the form (1). The corresponding parameters are $\beta_s = \frac{1}{n2^{n-2}}$.

Remark 2 *Current literature does not say much about non-additive semivalues. This paper does not explore this topic; however, the non-additive and non-dummy semivalues can be easily characterized by modifying (1). Indeed, replace the constant β_i by any function $\beta_i^* : G^N \to \mathbb{R}$ that is symmetric in G. This function is clearly efficient and symmetric. It will be additive when β_i^* is constant, and by construction the dummy axiom does not hold. Similar formulas can be given by replacing symmetry by the coalition independence axiom.*

5 Appendix

Lemma 1 *A solution satisfies the additivity axiom if and only if it is linear.*

Before we continue with the proofs, we need to define a game χ_S for every $S \subseteq N$,

$$\chi_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2 *The solution φ satisfies the additivity and symmetry axioms if and only if there exist real numbers $\{\beta_s\}_{s=1}^{n} \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$ such that*

$$\varphi_i(v) = \sum_{S\ni i} \beta_s v(S) + \sum_{S\not\ni i} \widetilde{\beta}_s v(S) \text{ for every } i \in N.$$

Proof. Let φ be an additive and symmetric solution, from Lemma 1, φ is linear. Clearly $\{\chi_S\}_{S\subset N}$ is a basis for G, and for every game $v \in G$, $v = \sum_{S\subset N} v(S)\chi_S$. Set $\beta_S^i = \phi_i(\chi_S)$, then

$$\varphi_i(v) = \sum_{S\subseteq N} v(S)\beta_S^i$$

for every $i \in N$. On the other hand, let $U, V \subseteq N$ be such that $|U| = |V|$, $k \in U$, $l \in V$ and θ a permutation of N such that $\theta(U) = V$ and $\theta(k) = l$. Since $\theta(\chi_U) = \chi_V$ then $\varphi_k(\chi_U) = \varphi_l(\chi_V)$ by symmetry. Therefore, $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \in U$, $l \in V$. Similarly, we can conclude that $\beta_U^k = \beta_V^l$ if $|U| = |V|$, $k \notin U$, $l \notin V$. Thus

$$\varphi_i(v) = \sum_{S \ni i} \beta_s v(S) + \sum_{S \not\ni i} \widetilde{\beta}_s v(S)$$

for some constants $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. The proof in the other direction is straightforward. $\square$

Proof of Proposition 1. By Lemma 2, $\varphi_i(v) = \sum_{S \ni i} \beta_s v(S) + \sum_{S \not\ni i} \widetilde{\beta}_s v(S)$ for some numbers $\{\beta_s\}_{s=1}^n \cup \{\widetilde{\beta}_s\}_{s=1}^{n-1}$. Since φ is efficient, we have that $\sum_{i \in N} \varphi_i(\chi_S) = s\beta_s + (n-s)\widetilde{\beta}_s = 0$ for every $S \subsetneq N$, and $\sum_{i \in N} \varphi_i(\chi_N) = n\beta_n = 1$. Therefore, $\widetilde{\beta}_s = -\frac{s}{n-s}\beta_s$ for $s = 1, 2, ..., n-1$ and $\beta_n = \frac{1}{n}$. Thus

$$\begin{aligned} \varphi_i(v) &= \textstyle\sum_{S \ni i} \beta_s v(S) - \sum_{S \not\ni i} \frac{s}{n-s} \beta_s v(S) \\ &= \textstyle\sum_{S \ni i} \beta_s v(S) - \sum_{S \ni i} \frac{n-s}{s} \beta_{n-s} v(N \backslash S) \end{aligned}$$

for some numbers $\{\beta_s\}_{s=1}^{n-1}$. Now, if we replace $q_s = \frac{\beta_s}{n-s}$ we get

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)(q_s v(S) - q_{n-s} v(N \backslash S)). \square$$

Lemma 3 *The solution φ satisfies additivity and coalitional independence axioms if and only if there exist numbers $\{\beta_S\}_{S \subseteq N} \cup \{\widetilde{\beta}_S\}_{S \subsetneq N}$ such that*

$$\varphi_i(v) = \sum_{S \ni i} \beta_S v(S) + \sum_{S \not\ni i} \widetilde{\beta}_S v(S) \text{ for every } i \in N.$$

Proof. Note that, for every S, χ_S and the zero game differ only on S, thus if φ is any linear and coalitional independent solution

$$\varphi_i(\chi_S) = \varphi_j(\chi_S), \text{if } i, j \in S \text{ or if } i, j \in N \setminus S.$$

For every S define $\beta_S := \varphi_i(S)$ for any $i \in S$; similarly, for every $S \neq N$, define $\widetilde{\beta_S} := \varphi_j(S)$ for any $j \notin S$. Then, for every game v

$$\varphi_i(v) = \varphi_i(\sum_S v(S)\chi_S) = \sum_{S \in i} \beta_S v(S) + \sum_{S \not\ni i} \widetilde{\beta_S} v(S).$$

It is straightforward to check the converse. $\square$

Proof of Proposition 2. Keeping the same notation as above, we know that if φ is in addition an efficient solution, then, for every $S \neq N$

$$0 = \chi_S(N) = \sum_i \varphi_i(\chi_S) = s\beta_S + (n-s)\widetilde{\beta_S};$$

thus $\widetilde{\beta_S} = -\frac{s}{n-s}\beta_S$, $S \neq N$. Similarly, $\beta_N = \frac{1}{n}$. Hence,

$$\begin{aligned} \varphi_i(v) &= \sum_{S \ni i} \beta_S v(S) - \sum_{S \not\ni i} \tfrac{s}{n-s}\beta_S v(S) \\ &= \sum_{S \ni i} \beta_S v(S) - \sum_{S \ni i} \tfrac{n-s}{s}\beta_{N\setminus S} v(N\setminus S) \end{aligned}$$

Let us take $q_S = \frac{\beta_S}{n-s}$ then

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s) \left[q_S v(S) - q_{N\setminus S} v(N\setminus S) \right]$$

for some numbers $\{q_S\}_{S \subsetneq N}$. □

Proof of Corollary 1. Assume $i \in N$, then by Proposition **2** there exists a set of numbers $\{\beta_S\}_{S \subsetneq N}$ such that

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N\setminus S} v(N\setminus S)].$$

For every $S \neq N$ we define a game ξ_S as follows

$$\xi_S(T) = \begin{cases} 0 & \text{si } T \neq S \text{ and } T \neq N\setminus S \\ 1 & T = S \text{ or } T = N\setminus S \end{cases} .$$

Therefore

$$(n-s)[\beta_{N\setminus S} - \beta_S] = \varphi(\xi_S^*) = \varphi(\xi_S) = (n-s)[\beta_S - \beta_{N\setminus S}],$$

so, $\beta_S = \beta_{N\setminus S}$. Thus

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)\beta_S[v(S) - v(N\setminus S)].$$

It is easy to show that solutions of this kind satisfy the self duality axiom. □

Proof of proposition 3. We leave to the reader to verify that Shapley's value satisfies the coalitional independence axiom.

Now, let φ be a solution that satisfies additivity, coalitional independence, dummy and efficiency axioms. Fix a player i and consider coalitions $T \subset N \setminus \{i\}$ but $T \neq N \setminus \{i\}$. First of all, notice that i is a dummy player for $\chi_{T\cup\{i\}} + \chi_T$, therefore -using the expression for coalitional independent solutions from Proposition 3-

$$0 = \varphi_i(\chi_{T\cup\{i\}} + \chi_T) = (n-t-1) \cdot \beta_{T\cup\{i\}} - t \cdot \beta_T.$$

Thus, $\beta_T = \frac{n-t-1}{t}\beta_{T\cup\{i\}}$ for each $T \subsetneq N\setminus\{i\}$.

From this, it follows that the solution is determined by the single number $\beta_{N\setminus\{1\}}$. Moreover, from

$$0 = \varphi_1(\chi_N + \chi_{N\setminus\{1\}}) = \frac{1}{n} - (n-1)\beta_{N\setminus\{1\}},$$

we conclude that the solution is unique. Since the Shapley value satisfies these axioms, $\varphi = Sh$. □

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