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Continuous Time Models of Interest Rate: Testing the Mexican Data (1998-2006)

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Abstract

Distinct parametric models in continuous time for the interest rates are tested by means of a comparative analysis of the implied parametric and nonparametric densities. In this research the statistic developed by Ait-Sahalia (1996a) has been applied to the Mexican CETES (28 days) interest rate in the period 1998-2006. With this technique, the discrete approximation to the continuous model is unnecessary even when the data are discrete. The results allow to affirm that the models of interest rate shown in this paper are unable to describe the data of the Mexican CETES.

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1. Introduction

In this note we applied the methodology developed in Ait-Sahalia (1996a) to evaluate some models for the Mexican interest rate data (CETES) and test if these approaches describe its empirical evolution. There are many processes which have been proposed to explain the evolution of the interest rate through time (see for example, Vasicek, (1977), Cox, Ingersoll and Ross (1985), Courtadon (1982) and Chan (1992), among others). One of the most important applications of these models is the pricing of interest rate derivative securities; unfortunately the prices of these derivatives are different, depending on the short rate dynamics (Ait-Sahalia, 1996b, Jiang, 1998).

The typical dynamics specified is the stationary diffusion processes

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

Where $\{W_t, t \geq 0\}$ is a standard Brownian motion .The functions $\mu(X_t)$ and $\sigma^2(X_t)$ are respectively, the drift and the diffusion functions of the process. The parametrization can be written as

$$\mu(x) = \mu(x,\theta), and, \sigma^2(x) = \sigma^2(x,\theta) \text{ where } \theta \in \Theta \subset R^k \tag{2}$$

There is no financial theoretical rationale for the election of the parametric drift and diffusion. Therefore, we will test different models in the literature. After parameterizing the drift and diffusion, a common method to estimate (1) consists in discretizing the model (see for example Hansen (1982), Chan (1992)), and it is assumed that the frequency of data is an important issue,i.e., more data means higher frequency data. Ait-Sahalia (1992) developed an estimator designed to take into account the discrete character of the data without use of the discrete approximation of the continuous time model (in this paper the null hypothesis state that there exist parameters of the type (2)). As we choose specific parameters, we define a density function for the interest rate observations. The statistic test applied compares the density implied by the parametric model and a nonparametric estimator (which is always consistent). This test is valid even if the parametric model is misspecified.

The paper is organized as follows. Section 2 presents the statistic test developed by Ait-Sahalia (1996a) and its assumptions. Section 3 presents the application and results from the empirical research .Sections 4 presents conclusions.

2. The test for the parametric specifications of the interest rate dynamics.

We consider processes of the interest rate which are univariate diffusions, strictly stationary with the Markov property, where the zero and infinity are attainable in a finite number of expected steps and that the discrete data are mixing at a sufficiently fast rate (i.e.,the classical asymptotic theory can be applied). When we take a specific parameterization of the process, we are working with the joint parametric family:

$$P \equiv \left\{ (\mu(.,\theta), \sigma^2(.,\theta) \middle| \theta \in \Theta \right\} \tag{3}$$

Where Θ is a compact subset of R^K. The null and alternative hypothesis are written as

$$\begin{Bmatrix} H_0 : \exists \theta_0 \in \Theta \middle| \mu(.,\theta_0) = \mu_0, \sigma^2(.,\theta_0) = \sigma_0^2 \\ H_1 : (\mu_0(.), \sigma_0^2(.)) \notin P \end{Bmatrix} \tag{4}$$

Where P is described by (3). And more generally, given $\mu(.,\theta)$ and $\sigma(.,\theta)$ in P, there is a correspondence with a parameterization of the marginal and transitional densities:

$$\Pi \equiv \left\{ (\pi(.,\theta), p(.,.,|.,.,\theta)) \middle| (\mu(.,\theta), \sigma^2(.,\theta)) \in P, \theta \in \Theta \right\}$$

Where $\pi(x,\theta)$ is the marginal density at x and $p(s, y|t, x, \theta)$ is the transition probability density from x at time t to y at time s. The estimation of the densities explicitly takes into account the discreteness of the data. The marginal density corresponding to the pair (μ, σ^2) is given by

$$\pi(x,\theta) = \frac{\xi(\theta)}{\sigma^2(x,\theta)} \exp\left\{ \int_{x_0}^{x} \frac{2\mu(u,\theta)}{\sigma^2(u,\theta)} du \right\} \tag{5}$$

We show a sketch of the proof of (5), (based on Karlin and Taylor, 1981): The Forward-Kolmogorov equation is defined as

$$\frac{\partial \psi(t,y)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[\sigma^2(y) \psi(t,y) \right] - \frac{\partial}{\partial t} \left[\mu(y) \psi(t,y) \right]$$

It is assumed that the distribution is stationary for $\psi(t, y)$. Therefore

$$0 = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[\sigma^2(y) \psi(t,y) \right] - \frac{\partial}{\partial t} \left[\mu(y) \psi(t,y) \right]$$

Integrating we have:

$$\frac{\mathrm{d}}{\mathrm{dy}} \left[\frac{\sigma^2(y)}{2} \psi(t,y) \right] - \mu(y) \psi(t,y) = \frac{1}{2} C_1$$

Where C_1 is the integration constant. So

$$\frac{\mathrm{d}}{\mathrm{dy}} \left[\sigma^2(y) \psi(t,y) \right] = 2\mu(y) \psi(t,y) + C_1 \tag{6}$$

On the other hand, we know that:

$$\frac{\partial}{\partial y} \left[s(y) \sigma^2(y) \psi(t,y) \right] = C_1 s(y)$$

$$\Leftrightarrow \quad \frac{1}{s(y)} \left[s(y) \frac{\partial \left[\sigma^2(y) \psi(t,y) \right]}{\partial y} + \sigma^2(y) \psi(t,y) \frac{\partial [s(y)]}{\partial y} \right] = C_1$$

To get the equality between this equation with equation (6), it is necessary:

$$\frac{\sigma^2(y)\psi(y)}{s(y)}\frac{\partial[s(y)]}{\partial y} = -2\mu(y)\psi(y)$$

$$\Rightarrow \quad \frac{1}{s(y)}\frac{\partial[s(y)]}{\partial y} = -\frac{2\mu(y)\psi(y)}{\sigma^2(y)\psi(y)}$$

$$\Rightarrow \quad \frac{\partial}{\partial y}\ln[s(y)] = -\frac{2\mu(y)}{\sigma^2(y)}$$

$$\Rightarrow \quad \ln[s(y)] = -\int\frac{2\mu(y)}{\sigma^2(y)}$$

$$\Rightarrow \quad s(y) = \exp\left\{-\int\frac{2\mu(y)}{\sigma^2(y)}\right\}$$

The integral $S(x) = \int^x s(\varepsilon)\mathrm{d}\varepsilon$, is called the scale function and the speed of the density is

$$m(x) = \frac{1}{\sigma^2(x)s(x)}.$$

If the density is stationary, we have

$$\psi(x) = m(x)[C_1 s(x) + C_2] \tag{7}$$

If $\psi(x) > 0$ in the interval (l, r), then:
1) $C_1 = 0$, and
2) there exists C_2 such thah $1 = \int_l^r \psi(x)\mathrm{d}x$.

Substituting the value of $m(x)$ and $s(x)$ in (7), we have

$$\begin{aligned}\psi(x) &= \frac{1}{\sigma^2(x)s(x)}[C_1 s(x) + C_2] \\ &= \frac{C_1}{\sigma^2(x)} + \frac{C_2}{\sigma^2(x)}[s(x)]^{-1} \\ &= \frac{C_2}{\sigma^2(x)}\exp\left\{\int\frac{2\mu(y)}{\sigma^2(y)}dy\right\}\end{aligned}$$

Changing ψ by π and C_2 by ε we have

$$\pi(x,\theta) = \frac{\xi(\theta)}{\sigma^2(x,\theta)}\exp\left\{\int\frac{2\mu(y,\theta)}{\sigma^2(y,\theta)}dy\right\} \tag{8}$$

which is the expected result. The density (8) is used for each parametric model, and depending on the expression we have to use a particular method of integration.

Let the true marginal density of the process be $\pi_0(x,\theta) = \frac{\xi_0}{{\sigma_0}^2}\exp\left\{\int \frac{2\mu_0(u)}{{\sigma_0}^2(u)}dy\right\}$ and the space of possible density functions corresponding to the pairs (μ, σ^2) in P is

$$\Pi_M = \{\pi(.,\theta) \big| (\mu(.,\theta), \sigma^2(.,\theta)) \in P, \theta \in \Theta\}.$$

Therefore the null and alternative hypotheses are:

$$\begin{cases} H_{M0} : \exists \theta_0 \in \Theta \big| \pi(.,\theta_0) = \pi_0(.) \\ H_{M1} : \pi_0(.) \notin \Pi_M \end{cases}$$

Whether or not the parametric model is correctly specified, a nonparametric estimator of the density will converge to the true density. The parametric model of the density will converge to the true density only if it is correctly specified. Ait-Sahalia (1996a) proposed a measure of the distance M between the two densities estimates, where the null hypothesis to be tested is that the parametric specification is correct. The proposed statistic is

$$\hat{M} \equiv nb_n \min_{\theta \in \Theta} \frac{1}{n}\sum_{i=1}^{n}(\pi(r_i,\theta) - \pi_0(r_i))^2 \tag{9}$$

The nonparametric estimator is calculated using the kernel estimator of the marginal density

$$\hat{\pi}_0(u) \equiv \frac{1}{n}\sum_{i=0}^{n}\frac{1}{b_n}K\left(\frac{u-r_i}{b_n}\right)$$

And the distributions of the parameters are

$$n^{1/2}\left\{\hat{\theta}_M - \theta_0\right\} \xrightarrow{d} N(0, \Omega_M)$$

$$h_n^{-1/2}\left\{\hat{M} - E_M\right\} \xrightarrow{d} N(0, V_M)$$

The test is

$$reject \quad H_0 \quad when \quad \hat{M} \geq \hat{c}(\alpha) = \hat{E}_M + h_n^{1/2} z_{1-\alpha} / \left\{\hat{V}_M\right\}^{1/2} \tag{10}$$

with

$$\hat{E}_M \equiv \left(\int_{-\infty}^{+\infty} K^2(x)dx\right)\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\pi}_0(r_i)\right)$$

$$\hat{V}_M = 2\left(\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} K(u)K(u+x)du\right\}^2 dx\right)\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\pi}_0^3(r_i)\right)$$

and Ω_M given in Ait-Sahalia (1996a, p.421), and we use the Gaussian kernel

$$K(u) = \exp(-u^2/2)/\sqrt{2\pi}$$

3. Testing Rate Models

The interest rate used is the daily Mexican CETES of 28 days (1998-2006). A time series plot and summary statistics are provided in figure 1 (see appendix).

3.1 The parametric specifications of the short rate process.

The general specification is

$$\begin{cases} \mu(r,\theta) = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \alpha_3 / r \\ \sigma^2(r,\theta) = \beta_0 + \beta_1 r + \beta_2 r^{\beta_3} \end{cases}$$

The constraints which are consequence of the properties 1 through 5 are (Ait-Sahalia, 1996a):

$$\beta_0 \geq 0 (\beta_2 > 0 \quad \mathit{if}\ \beta_0 = 0 \quad \mathit{and} \quad 0 < \beta_3 < 1, or, \beta_1 > 0 \quad \mathit{if} \quad \beta_0 = 0, and, \beta_3 > 1)$$

$$\beta_2 > 0 \quad \mathit{if} \quad \mathit{either} \quad \beta_3 > 1, or, \beta_1 = 0, and, \beta_1 > 0, \mathit{if}, \mathit{either}, 0 < \beta_3 < 1, or, \beta_2 = 0$$

$$\alpha_2 \leq 0, and, \alpha_1 < 0, \mathit{if}, \alpha_2 = 0$$

$$\alpha_3 > 0, and, 2\alpha_3 \geq \beta_0 \geq 0, or, \alpha_3 = 0, \alpha_0 > 0, \beta_0 = 0, \beta_3 > 1$$

$$and, 2\alpha_0 \geq \beta_1 > 0$$

The list of models is shown in table 1. It can be proved from the assumptions that the drift and diffusion are not directly dependent on time

Insert Table 1 here

Applying (8) we get the analytical expression for some models (table 2). In other cases is necessary to apply numerical integration. The next step is calculating the statistics (9) for each model.

Insert Table 2 here

3.2 The results

The table 3 shows the estimation of the statistic (9) (with a 5% of significance). In the same table the application of the statistical criterion (10) allows to affirm that all these continuous models are rejected in order to describe the dynamics of the Mexican CETES data. The result has an important financial implication: the application of these models is incorrect to the study of the Mexican CETES.

Table 3
Results

	VASICEK	CIR	Brennan & Schwartz	Chan	General Drift	MERTON
Min	2,565.40747	2,565.87317	645.43314	612.37889	2,565.43618	2,225.27921
M	74.45160	74.46512	18.73135	17.77207	74.45244	64.58062
Result	Rejected	Rejected	Rejected	Rejected	Rejected	Rejected
	CIR SR	Dothan	CIR VR	GBM	CEV	Brennan & Schwartz
min	2,564.47988	908.92776	2,564.52981	2,565.87224	2,565.92205	2,565.87205
M	74.42469	26.37832	74.42613	74.46509	74.46654	74.46509
Result	Rejected	Rejected	Rejected	Rejected	Rejected	Rejected

4. Conclusions

The election of an adequate stochastic differential equation that explains the dynamics of the empirical interest rate is still an unsolved problem. With this fact in mind, it has tested a group of known models, observing an important empirical fact: the interest rate models shown in this paper are unable to describe the dynamics of the Mexican CETES data.

Even though, these models have been used by practitioners with different goals in mind, but the use in pricing interest rate derivative securities has negative consequences over the measuring of some important variables (for example in risk management). As an alternative, it can include jumps in the stochastic differential equation (an alternative to be studied by the authors in the future), or the use of multivariate models.

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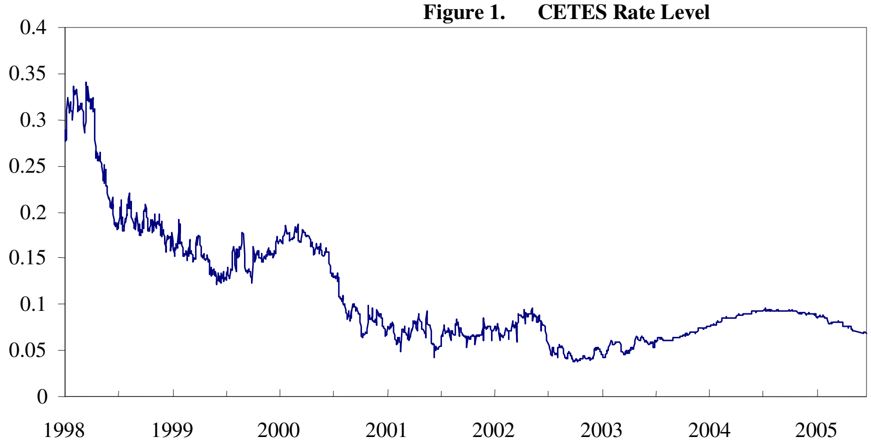
Appendix

Figure 1. CETES Rate Level



	Mean	Standard Deviation	Skewness	Kurtosis	First Autocorrelation
r_t	0.11118	0.06296	1.48273	5.07239	0.99551

Figure 1. The time series of the level CETES rate, November 1998 to April 2006 and Summary Statistics

Table 1Models

Parametric Model	$\mu(x,\theta)$	$\sigma^2(x,\theta)$
Vasicek (1977)	$\alpha_0+\alpha_1 x$	β_0
Cox, Ingersoll y Ross (1985)	$\alpha_0+\alpha_1 x$	$\beta_1 x$
Brennan & Schwartz (1982)	$\alpha_0+\alpha_1 x$	$\beta_2 x^2$
Chan (1992)	$\alpha_0+\alpha_1 x$	$\beta_2 x^{\beta_3}$
General Drift, CEV.	$\alpha_0+\alpha_1 x+\alpha_2 x^2+\alpha_3/x$	$\beta_0+\beta_1 x+\beta_2 x^{\beta_3}$
*** Merton**	α_0	β_0
*** Dothan**	-	$\beta_1 x$
*** CIR VR**	-	$\beta_1 x^{3/2}$
*** CIR SR**	$\alpha_0+\alpha_1 x$	$\beta_1 x^{1/2}$
*** GBM**	$\alpha_1 x$	$\beta_1 x$
*** CEV**	$\alpha_1 x$	$\beta_2 x^{\gamma}$
*** Brennan & Schwartz**	$\alpha_0+\alpha_1 x$	$\beta_1 x^{3/2}$

Table 2
Analytical expressions

Parametric Model	Marginal Density $\pi(x,\theta)$	Method of integration
Vasicek (1977)	$\frac{\varepsilon(\theta)}{\beta_0}\exp\left\{\frac{2\alpha_0}{\beta_0}x+\frac{\alpha_1}{\beta_0}x^2\right\}$	**Change of variable**
CIR -Cox, Ingersoll y Ross- (1985)	$\frac{\varepsilon(\theta)}{\beta_1\cdot x}\cdot x^{\left(\frac{2\alpha_0}{\beta_1}-1\right)}\cdot\exp\left\{-\frac{2\alpha_0}{\beta_1}+\frac{2\alpha_1}{\beta_1}x\right\}$	**Change of variable**
Brennan & Schwartz (1982)	$\frac{\varepsilon(\theta)}{\beta_2\cdot x_o^{2\alpha_1/\beta_2}}\cdot x^{\left(\frac{2\alpha_1}{\beta_2}-2\right)}\cdot\exp\left\{\frac{2\alpha_0}{\beta_2\cdot x_0}-\frac{2\alpha_0}{\beta_2\cdot x}\right\}$	**Numerical**
Chan (1992)	$\frac{\varepsilon(\theta)}{\beta_2\cdot x^{\beta_3}}\cdot\exp\left\{\frac{2}{\beta_2}\cdot x^{-\beta_3}\cdot\left(\frac{\alpha_0\cdot x}{1-\beta_3}+\frac{\alpha_1\cdot x^2}{2-\beta_2}\right)\right\}$	**Numerical**
General Drift, CEV.	$\frac{\varepsilon(\theta)}{\beta_2\cdot x^{\beta_3}}\cdot\exp\left\{\frac{2}{\beta_2}\cdot x^{-\beta_3}\cdot\left(\frac{\alpha_0\cdot x}{1-\beta_3}+\frac{\alpha_1\cdot x^2}{2-\beta_3}+\frac{\alpha_2\cdot x^3}{3-\beta_3}-\frac{\alpha_3}{\beta_3}\right)\right\}$	**Numerical**
Merton	$\frac{\varepsilon(\theta)}{\beta_0}\exp\left\{\frac{2\alpha_0}{\beta_0}x\right\}$	**Change of variable**
Dotan	$\frac{\varepsilon(\theta)}{\beta_1}x^{-1}$	**Direct**
CIR VR	$\frac{\varepsilon(\theta)}{\beta}x^{-3/2}$	**Direct**
CIR SR	$\frac{\varepsilon(\theta)}{\beta_2\cdot x^{1/2}}\cdot\exp\left\{\frac{4\alpha_0}{\beta_1}x^{1/2}+\frac{4\alpha_1}{3\beta_1}x^{3/2}\right\}$	**Numerical**
GBM	$\frac{\varepsilon(\theta)}{\beta_1}x^{-1}\cdot\exp\left\{\frac{2\alpha_1}{\beta_1}x\right\}$	**Change of variable**
CEV	$\frac{\varepsilon(\theta)}{\beta_1}x^{-\gamma}\cdot\exp\left\{\frac{2\alpha_1}{\beta_1\cdot(2-\gamma)}x\right\}$	**Change of variable and fractions**
Brennan & Schwartz	$\frac{\varepsilon(\theta)}{\beta_1\cdot x^{3/2}}\cdot\exp\left\{\frac{4\alpha_1}{\beta_1}x^{1/2}-\frac{4\alpha_0}{\beta_1}x^{-1/2}\right\}$	**Numerical**