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Imprimitivity in Decomposable Economies

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Abstract

This note extends the analysis of imprimitive indecomosable economies (e.g.~Nikaido, 1968, 1970) to the case of economies represented by a decomposable matrix. Using graph theory we show that imprimitivity leads to cyclical production lags also in decomposable economies, although in such a case the property must not be referred to the matrix representing the economy but to its indecomposable sub-matrices along the block-diagonal. The structure of the overall flow of commodities depends on both the number of imprimitive sub-matrices and their imprimitivity index.

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1 Introduction

Indecomposable matrices constitute a class of non-negative square matrices which occupy a paramount position in the linear multi-sectoral approach to the economic modeling of production (e.g. Leontiev, 1941; Von Neumann, 1945; Sraffa, 1960). Indecomposable matrices may be either primitive or imprimitive. Solow (1952) and Nikaido (1968, 1970) provided an economic interpretation of matrix imprimitivity in terms of cyclical flows of commodities in an indecomposable economy. Such flows have been shown to actually take place in real economies (e.g. Defourny and Thorbecke, 1984; Sonis *et al.*, 1993; Sonis *et al.*, 1997).[1]

The contribution of this note is to provide an extension of these results to the case where a decomposable matrix has a number of indecomposable imprimitive sub-matrices along its block-diagonal. Graph theory is applied to facilitate economic intuition. Although graph theory has long ago become a standard tool to study the properties of linear multi-sectoral models (e.g. Rosenblatt, 1957; Weil, 1968), the specific case under consideration has not yet been given a simple interpretation in terms of the flows of commodities among groups of industries.

Intuitively, the imprimitivity of an indecomposable sub-matrix along the block-diagonal means that there exists a cyclical structure of flows of commodities among the industries which are identified by the sub-matrix. In particular, the index of imprimitivity – i.e. the number of eigenvalues of maximum modulus – determines the number of industry sub-groups which compose the cycle and, hence, the period of the cycle. Combining such cyclical flows of commodities with the one-directional flow which goes from raw materials to consumer products we get a pattern of input and output flows which is composed by a series of nested cycles whose number is equal to the number of imprimitive sub-matrices and whose period is equal to the index of imprimitivity of each imprimitive sub-matrix.

The note is organized as follows. Section 2 explores the case under consideration providing both abstracts characterizations and easily understandable examples. Section 3 discusses the proposed interpretation and adds some final remarks. All mathematical preliminaries which are not strictly necessary are given in the Appendix.

2 Cyclical Production Lags

Let $A \equiv \{a_{ij}\}_{i,j=1}^{n}$ be a square matrix of order n. Element a_{ij} represents the amount of commodity i required to produce one unit of commodity j.[2] Hence, row indices may be interpreted as commodity indices and column indices as industry indices. Joint production is not considered. Finally, we label a single

[1] See also Kurz and Salvadori (1995) for a brief introduction to these issues which is entirely based on matrix theory and linear programming.

[2] This interpretation implicitly requires that i) returns to scale are constant and ii) wages are already taken into account by matrix A. However, these requirements are not crucial for the issues discussed. All characterizations easily extend to the case where we have given quantity produced and explicit wages. The present formulation is applied because it highlights the (logical) time dimension which is tied to the suggested interpretation of imprimitivity.

round of production activity as *production lag*.[3]

Let $g(N, E)$ be a directed graph, where N is the set of nodes and E the set of directed edges. The shorthand ij indicates the directed edge going from node i to node j. By a *path* of length $m \geq 1$ connecting i to j it is meant a finite sequence of m directed edges $\{e\}_{h=1}^m \equiv \{(r_h s_h)\}_{h=1}^m$ where $e_h \in E$, and $r_h \in N$ and $s_h \in N$ indicate, respectively, the root (or starting point) and the sink (or ending point) of each directed edge.[4] When there is no need to specify the edges composing a path, a path of length m connecting node i to node j is denoted by $p^m(i, j)$. Moreover, a graph is *connected* if and only if there exists $i \in N$ such that, for every $j \neq i$ there exists $p^m(i, j)$ for some m. Finally, a graph is *strongly connected* if and only if for any pair $i, j \in N$, $i \neq j$, there exists $p^m(i, j)$ for some m.

Suppose matrix A represents an economy composed of n industries. We define the graph induced by A as $g_A(N, E)$ where $N \equiv \{1, 2, ..., n\}$ and $ij \in E$ whenever $a_{ij} > 0$. Moreover, A is decomposable if and only if g_A is not strongly connected (see lemma 1 in the Appendix).

2.1 Imprimitive Economies

Primitive matrices are indecomposable matrices characterized by the spectrum being strictly dominated by a single eigenvalue of multiplicity one (see definition 3 in the Appendix). Imprimitive matrices are indecomposable matrices which are not primitive.

A useful interpretation of imprimitivity is easily obtained by considering that A is primitive if and only if there exists a positive integer k such that $A^k > 0$ (see proposition 1 in the Appendix). Recall that A^k represents the quantities of inputs that are required to produce 1 unit of each commodity in exactly k production lags – each column referring to the inputs required by the corresponding industry. Hence, an economy is imprimitive whenever it is impossible to find a number of production lags for which the output of each industry shows up (indirectly) as input of any other industry.

In terms of graph representation, imprimitivity of A means that g_A shows a cyclical flow of commodities among industries whose period is the index of imprimitivity (Nikaido, 1968).[5] The following corollary makes this statement rigorous[6]

Corollary 1 *If A is an imprimitive indecomposable matrix of order n and index of imprimitivity $h > 1$. Then, there exist a partition $\Pi \equiv \{N_1, ..., N_h\}$ of the set N such that $g_A(N, E)$ satisfies*

i) $\forall i, j \in N_k$, $ij \notin E$, $k = 1, ..., h$

[3] Production lags have been studied with standard linear programming techniques in Dorfman *et al.* (1958).

[4] A path can have edges repeated as long as it is consistent with the definition (for instance $\{ij, ji, ij\}$ or $\{ij, jj, jj, jk\}$).

[5] Note that A is primitive if and only if g_A has n^2 paths of length k, for some k, connecting any two (not necessarily distinct) nodes (see corollary 3 in the Appendix). See for instance Gale(1956) or Ando *et al.* (1963) on unweighted graphs. For an analysis which puts more emphasis on weighted graphs see for instance Lantner (1974) and Gazon (1979).

[6] The corollary follows by checking the properties of g_A where A is in the form of point ii) of Proposition 2 (see the Appendix).

ii) $\forall N_k \in \Pi$, $\exists N_l \in \Pi$:
$\big(\forall i \in N_k, \exists j \in N_l : ij \in E\big) \wedge \big(\forall i \in N_k, j \notin N_l \Rightarrow ij \notin E\big)$

A brief illustration of the corollary will later help to better understand the case of decomposable economies. Consider an economy represented by an indecomposable matrix with index of imprimitivity h. Then, its industries can be sorted in h groups of $N_1, ..., N_h$ such that the output of each group constitute all direct inputs of industries belonging to one and only one other group. In particular, there exists a cyclic linkage among the h groups of industries whose period is h. Without loss of generality, suppose that N_k directly produces the inputs of N_{k+1} and $h+1 \equiv 1$. Hence, some output of the industries in the group N_k is indirectly used as input by industries in the group N_{k+r} every r production lags, with $1 < r \leq h$. However, the output of a single industry $i \in N_k$ is indirectly used as an input by $j \in N_{k+r}$ every $r + hz > 1$ productions lags, where $z \geq 1$ is an integer. Indeed, the shortest path between $i \in N_k$ and $j \in N_{k+r}$ may not be r because, although i must be linked by a path of length h to some $u \in N_{k+r}$, the latter may differ from j. So, it could take one or more additional full rounds of length h to get to j.

2.2 Decomposable Economies

Decomposable matrices are neither primitive nor imprimitive. However, decomposable matrices contain indecomposable sub-matrices which may or may not be primitive. By looking at the index of imprimitivity of a particular set of these sub-matrices, we can gather information about inter-industry relationships.

For the sake of exposition the analysis is limited to the case where there exists a group of industries whose outputs are used, either directly or indirectly, as inputs in any other industry (e.g. Solow, 1952; Sraffa, 1960). Let A represent such an economy. In terms of the graph g_A, we have that there exists a set $B \subseteq N$ such that, for any $i \in B$ and $j \in N$, there exists $p^k(i,j)$ for some $k > 0$.[7] Without loss of generality, we assume that A is a matrix already in normal form (see definition 2 in the Appendix). Then, we have the following:

Corollary 2 *Let A be a decomposable matrix of order n and $h_1, ..., h_s$ be the indices of imprimitivity of the indecomposable blocks along the block-diagonal of A. Then, there exist a partition $\Pi \equiv \{N_{1,1}, \ldots, N_{1,h_1}, \ldots, N_{s,1}, \ldots, N_{1,h_s}\}$ of the set N such that $g_A(N, E)$ satisfies*

i) $h_s > 1 \Rightarrow \forall i, j \in N_{s,k}$, $ij \notin E$, $k = 1, \ldots, h_s$

ii) $\forall N_{s,k} \in \Pi$, $\exists N_{s,l} \in \Pi$:
$(\forall i \in N_{s,k}, \exists j \in N_{s,l} : ij \in E) \wedge (\forall i \in N_{s,k}, j \in N_{s,t}, t \neq l \Rightarrow ij \notin E)$

Proof. Each square matrix A_i along the block-diagonal identifies a set $N_i \subseteq N$ of industries. For each N_i, consider the partition $\Pi_i \equiv \{N_{i,1}, ..., N_{i,h_i}\}$ of the type described in Corollary 1. Since $\cup_{i=1}^{n} N_i = N$, the partitions $\Pi_1 ..., \Pi_s$ induce the partition $\Pi \equiv \{N_{1,1}, ..., N_{1,h_1}, ..., N_{1,1}, ..., N_{1,h_s}\}$ on N. Applying corollary 1 gives the result.■

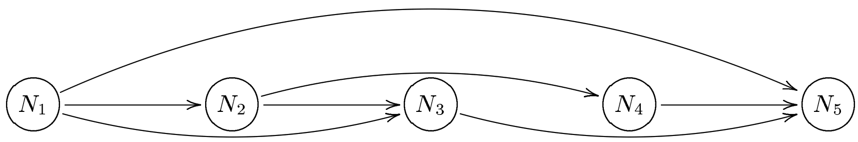
[7]The extension to the case where there exist disjoint $B_t \subseteq N$, $t = 1, \ldots, q$, such that for any $i \in B_t$ and $j \in N \setminus B_t$, there exists $p^k(i,j)$ for some $k > 0$ is straightforward.

Example 1. A decomposable economy without cyclical production lags

Matrix representation

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & 0 & A_{1,5} \\ 0 & A_{2,2} & A_{2,3} & A_{2,4} & 0 \\ 0 & 0 & A_{3,3} & 0 & A_{3,5} \\ 0 & 0 & 0 & A_{4,4} & A_{4,5} \\ 0 & 0 & 0 & 0 & A_{5,5} \end{bmatrix}$$

Graph representation of the quotient set



The sets $N_1, \ldots, N_5$ identify a partition of N according to the indecomposable matrices $A_{1,1}, \ldots, A_{5,5}$ along the block-diagonal. The flows of commodities among these sets of industries are given by the non-negative matrices $A_{1,2}$, $A_{1,3}$, $A_{1,5}$, $A_{2,3}$, $A_{2,4}$, $A_{3,5}$ and $A_{4,5}$. The absence of imprimitive matrices among $A_{1,1}, \ldots, A_{5,5}$ guarantees the absence of cycles.

In order to better focus on the flows of commodities occurring among groups of industries, it is useful to define the equivalence relation $\sim$ such that, for any $i, j \in N$, $i \sim j$ if and only if $i, j \in N_{k,l}$. Then, we can apply the quotient graph $g_{A|\sim}$ to study the qualitative structure of commodity flows. Since there are s indecomposable sub-matrices we can have up to s cycles which may be connected among themselves in various ways. For each set N_k, industries belonging to the same $N_{k,l} \subseteq N_k$ produce inputs for some industries in $N_{k,l+1}$ and use products of some industries in $N_{k-1,l}$. Moreover, industries in $N_{k,l}$ may directly use the products of any industry belonging to N_q, $q < k$, or produce the direct inputs for any industry in N_r, $r > k$.

Depending on both the number of imprimitive matrices along the block-diagonal and their index of imprimitivity, $g_{A|\sim}$ can take quite different shapes. On one extreme, if $h_1 = ... = h_s = 1$, then $g_{A|\sim}$ is composed by s nodes and one or more paths of the type $\{(r_j, s_j)\}_{j=1}^{m}$ where $m \leq s$ and $r_j < s_j$. This is the case of no cyclicity at all. Example 1 illustrates it for a decomposable economy composed by five indecomposable sub-groups of industries.

On the other extreme, if $h_i > 1$ for all $i = 1, ..., s$, then $g_{A|\sim}$ is composed by s cycles which have, respectively, $h_1, ..., h_s$ nodes. Such cycles are encompassed by paths of the type $\{(r_j, s_j)\}_{j=1}^{m}$ where $r_j \geq s_j$. This is the case of maximal cyclicity as indirect inputs flow according to regular periods both within and across the s blocks of industries. Example 2 illustrates it for a decomposable economy composed by thirteen sub-groups of industries divided in four indecomposable groups whose index of imprimitivity are, respectively, 3, 4, 2, and 4.

Any other case is just a combination of the two extremes described.

3 Comment

We have provided a straightforward economic interpretation of a previously overlooked case: linear multi-sectoral economies represented by decomposable matrices which have indecomposable imprimitive sub-matrices along their block-diagonal. Imprimitivity of a block-diagonal sub-matrix means that there exists a cyclical structure of flows of commodities among the industries which are identified by the sub-matrix. The index of imprimitivity gives the number of industry sub-groups which compose the cycle and, hence, the period of the cycle. When we consider such cyclical flows together with the usual one-directional flow which goes from raw materials to consumer products we obtain a pattern of input and output flows which is composed by a series of nested cycles. In particular, the number of cycles is equal to the number of imprimitive sub-matrices and the period of each cycle is equal to the index of imprimitivity of the associated sub-matrix.

Incidentally, we have also established that matrix imprimitivity leads to cyclical production lags also in decomposable economies, although in such a case the property must not be referred to the matrix representing the economy but to its indecomposable sub-matrices along the block-diagonal.

A final remark is worth doing. If a non-negative indecomposable matrix has at least a non-null element along its diagonal, then it is primitive (e.g. Solow, 1952; Nikaido, 1968). Indeed, if a part of industry i's product at time t is used as a direct input to produce i's output at time $t+1$ then, from a certain period

Example 2. A decomposable economy with cyclical production lags

Matrix representation

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & 0 \\ 0 & A_{2,2} & A_{2,3} & 0 \\ 0 & 0 & A_{3,3} & A_{3,4} \\ 0 & 0 & 0 & A_{4,4} \end{bmatrix}$$

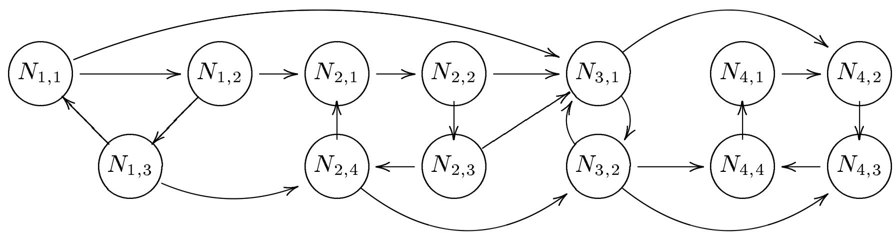
$$A_{1,1} = \begin{bmatrix} 0 & B_{2,1} & 0 \\ 0 & 0 & B_{2,3} \\ B_{3,1} & 0 & 0 \end{bmatrix} \quad A_{2,2} = \begin{bmatrix} 0 & B_{4,5} & 0 & 0 \\ 0 & 0 & B_{5,6} & 0 \\ 0 & 0 & 0 & B_{6,7} \\ B_{7,4} & 0 & 0 & 0 \end{bmatrix}$$

$$A_{3,3} = \begin{bmatrix} 0 & B_{8,9} \\ B_{9,8} & 0 \end{bmatrix} \quad A_{2,2} = \begin{bmatrix} 0 & B_{10,11} & 0 & 0 \\ 0 & 0 & B_{11,12} & 0 \\ 0 & 0 & 0 & B_{12,13} \\ B_{13,10} & 0 & 0 & 0 \end{bmatrix}$$

$$A_{1,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & B_{2,5} & 0 & 0 \\ B_{3,4} & 0 & 0 & 0 \end{bmatrix} \quad A_{1,3} = \begin{bmatrix} 0 & B_{2,9} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{2,3} = \begin{bmatrix} 0 & 0 \\ 0 & B_{5,9} \\ 0 & B_{6,9} \\ B_{7,8} & 0 \end{bmatrix}$$

$$A_{3,4} = \begin{bmatrix} B_{8,10} & 0 & 0 & B_{8,13} \\ 0 & 0 & B_{9,12} & 0 \end{bmatrix}$$

Graph representation of the quotient set



There are 4 groups of industries encompassed by a cycle, one for each block along the diagonal of the normal form: $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5, 6, 7\}$, $N_3 = \{8, 9\}$, $N_4 = \{10, 11, 12, 13\}$. These are in turn constituted by the sub-groups $N_{1,1}$, $N_{1,2}$, $N_{1,3}$, $N_{2,1}$, $N_{2,2}$, $N_{2,3}$, $N_{2,4}$, $N_{3,1}$, $N_{3,2}$, $N_{4,1}$, $N_{4,2}$, $N_{4,3}$, $N_{4,4}$ which identify a partition of $N = \{1, \ldots, 13\}$.

$t' \leq t+1$ onwards, i's product at time t is used as an indirect input in the production of all commodities. This suggests two things. First, there are good reasons to expect that both the number of cycles and their periods increase when we move from industries producing raw materials to those producing final goods. In fact, output reuse seems particularly likely for the former and less and less likely as we approach the latter along the chain of production. Second, both the number of cycles and their periods are likely to increase in the level of disaggregation of production. In other words, as we adopt a finer definition of industry it is more likely that we find cyclical production lags.

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Appendix: mathematical preliminaries

Where not stated differently, definitions and propositions are borrowed from Gantmacher (1959).

Definition 1 *A non-negative square matrix* $A \equiv \{a_{ij}\}_{i,j=1}^{n}$ *is decomposable if the indices* $1, 2, ..., n$ *can be divided into two disjoint non-empty sets* $I \equiv \{i_1, i_2, ..., i_l\}$ *and* $J \equiv \{j_1, j_2, ..., j_m\}$ *with* $l + m = n$*, such that* $a_{i_\alpha j_\beta} = 0$ *for any* $i_\alpha \in I$ *and* $j_\beta \in J$*. Otherwise the matrix* A *is indecomposable.*

The following characterization of matrix indecomposability is based on graph theory (e.g. Rosenblatt, 1957).

Lemma 1 *Let* A *be a non-negative square matrix. Then,* A *is decomposable if and only if* g_A *is not strongly connected.*

Proof. Suppose A is of order n. If A is decomposable then there exist two disjoint sets of indices $I \equiv \{i_1, i_2, ..., i_u\}$ and $J \equiv \{j_1, j_2, ..., j_v\}$ with $u + v = n$, such that $a_{i_\alpha j_\beta} = 0$ for any $i_\alpha \in I$ and $j_\beta \in J$. It is straightforward to see that $p^m(i_\alpha, j_\beta)$ does not exist for any $i_\alpha \in I$ and $j_\beta \in J$ and $m \geq 1$.

Conversely, suppose that there are no paths of any length going from node i to node j, $i \neq j$. Define $I \equiv \{i\} \cup \{k \in N : p^m(i,k), m \geq 1\}$ and $J \equiv N \setminus I$. Notice that J contains at least j. By construction, if $r \in I$ then, for any $m \geq 1$ and $s \in J$, $p^m(r,s)$ does not exists, otherwise there would exist $p^{m+l}(i,s)$ for some $l \geq 1$ and s would belong to I; in particular, $p^1(r,s)$ does not exist. Therefore, we have that $a_{rs} = 0$ whenever $r \in I$ and $s \in J$.■

Definition 2 *Let* A *be a square matrix of order* n*. Its normal form is*

$$\hat{A} = \begin{bmatrix} A_{1,1} & \dots & 0 & A_{1,g+1} & \dots & A_{1,s} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & A_{g,g} & A_{g,g+1} & \dots & A_{g,s} \\ 0 & \dots & 0 & A_{g+1,g+1} & \dots & A_{g+1,s} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & A_{s,s} \end{bmatrix}$$

where sub-matrices $A_{1,1}, \dots, A_{s,s}$ *are indecomposable and in each row* $f = g + 1, ..., s$ *at least one of the matrices* $A_{f,1}, ..., A_{f,f-1}$ *is different from zero.*

The normal form of a matrix is unique up to permutations of the blocks of indices. More precisely, the blocks $1, ..., g$ can be always be permuted without modifying the normal form while permutations of the blocks $g + 1, ..., s$ are allowed only in certain cases. See Gantmacher (1959) for a detailed proof. Clearly, if A is indecomposable then $s = 1$ while if A is completely reducible then $g = s$.

Definition 3 *Let* A *be an indecomposable matrix and* S *the set of its eigenvalues where each eigenvalue appears a number of times equal to its multiplicity as a root of the characteristic polynomial of* A*. Define* $\lambda^* \equiv \max_{\lambda \in S} |\lambda|$ *and* $h \equiv ||\{\lambda \in S : |\lambda| = \lambda^*\}||$*. If* $h = 1$ *then* A *is primitive; otherwise,* A *is imprimitive and* h *is its index of imprimitivity.*

Proposition 1 *An indecomposable matrix A is primitive if and only if there exists a positive integer k such that $A^k > 0$.*

Corollary 3 *An indecomposable matrix A is primitive if and only if there exists an integer $k > 0$ such that, in the graph g_A, there exists $p^k(i,j)$ for every $i \neq j$.*

Proof. Let $\underline{a}_i$ and $\underline{a}^j$ be, respectively, the i-th row and the j-th column of matrix A. Let $\{a_{ij}\}^k$ denote the entry of matrix A^k belonging to the i-th row and j-th column. Corollary follows from Proposition 1 once it is proved that $\{a_{ij}\}^k > 0$ if and only if there exists a path $p^k(i,j)$ in g_A. I show this by induction. The condition is trivially satisfied for $k = 1$. Suppose it holds for $k > 1$. Since $\{a_{ij}\}^{k+1} = \{\underline{a}_i\}^k \underline{a}^j$ we have that $\{a_{ij}\}^{k+1} > 0$ if and only if there exists some $h \in N$ such that $\{a_{ih}\}^k a_{hj} > 0$. Then, by the inductive hypothesis, $\{a_{ij}\}^{k+1} > 0$ if and only if there exists $p^k(i,h)$ and $p^1(h,j)$ in g_A for some $h \in N$, which in turn implies the existence of $p^{k+1}(i,j)$ in g_A.$\blacksquare$

If A is an indecomposable matrix and S and λ^* are defined as in Definition 3, we know that there exists a positive real eigenvalue $\tilde{\lambda} \in S$ which is equal to λ^* (Perron, (1907); Frobenius, 1912). However, A may have negative or complex eigenvalues with a modulus equal to λ^*. The following proposition from Frobenius (1912) proves that whenever an indecomposable matrix is imprimitive it shows a cyclical structure whose period is equal to the index of imprimitivity. A proof can be found in Gantmacher (1959).

Proposition 2 *Let A be an indecomposable matrix and S the set of its eigenvalues where each eigenvalue appears a number of times equal to its multiplicity as a root of the characteristic polynomial of A. Define $\lambda^* \equiv \max_{\lambda \in S} |\lambda|$ and $h \equiv ||\{\lambda \in S : |\lambda| = \lambda^*\}||$. Then,*

i) $\lambda_1, ..., \lambda_h \in S$ are distinct solutions of the equation $\lambda^h - \lambda^* = 0$,

ii) if $h > 1$ then A is imprimitive and there exists a permutation of both rows' and columns' indices such that A is reduced to the following cyclic form with zero square blocks along the diagonal

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ 0 & 0 & 0 & \dots & A_{h-1,h} \\ A_{h,1} & 0 & 0 & \dots & 0 \end{bmatrix}$$