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Abstract

In this work, we apply our newly proposed perturbative expansion technique to a quadratic growth FBSDE appearing in an incomplete market with stochastic volatility that is not perfectly hedgeable. By combining standard asymptotic expansion technique for the underlying volatility process, we derive explicit expression for the solution of the FBSDE up to the third order of volatility-of-volatility, which can be directly translated into the optimal investment strategy. We compare our approximation with the exact solution, which is known to be derived by the Cole-Hopf transformation in this popular setup. The result is very encouraging and shows good accuracy of the approximation up to quite long maturities. Since our new methodology can be extended straightforwardly to multi-dimensional setups, we expect it will open real possibilities to obtain explicit optimal portfolios or hedging strategies under realistic assumptions.

Keywords : FBSDE, optimal portfolio, incomplete markets, quadratic growth, perturbative expansion, asymptotic expansion

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1 Introduction

In the last couple of decades, forward-backward stochastic differential equations (FBSDE) have attracted significant academic interests. They were first introduced by Bismut (1973) [1], and then later extended by Pardoux and Peng (1990) [13] for general non-linear cases. They were found particularly relevant for optimal portfolio and indifference pricing issues in incomplete and/or constrained markets. Their financial applications are discussed in details in, for example, El Karoui, Peng and Quenez (1997) [5], Ma and Yong (2000) [12] and a recent book edited by Carmona (2009) [2] . Various topics regarding recursive utilities are thoroughly reviewed in the article written by Skiadas (2008) [14] and references therein.

FBSDEs have become also relevant in practical problems, too. Intensive research on counterparty credit risk, collateral cost, funding rate asymmetry has made clear that one has to handle complicated FBSDEs for these problems (See, for example, [4, 6, 3].). Furthermore, forthcoming regulations on the balance sheets of financial firms and increasing demand of cash collateral both for centrally-cleared and OTC trades are expected to constrain trader's position severely, and may even turn a part of financial products effectively nontradable. These new developments in the financial market will make deeper understanding of FBSDEs a more pressing issue in the coming years.

In the previous work [7], we have presented a simple analytical approximation scheme for generic non-linear FBSDEs. By treating the interested system as the linear decoupled FBSDE perturbed by a non-linear driver and feedback terms, the problem of each order of approximation turns out to be equivalent to those for pricing of standard European contingent claims. In this work, we consider its application to a particular type of FBSDEs with a quadratic growth driver. This type of system is receiving strong attention because it appears in the optimal portfolio problems for very popular utilities of exponential and power forms. In particular, we study the optimal portfolio problem in an incomplete market with one risky asset whose stochastic volatility is not perfectly hedgeable. We derive the explicit solution for the backward components of the corresponding FBSDE up to the third order of volatility of volatility (vol-of-vol). It allows us to have the explicit expression of the optimal strategy, which is of great importance for practical applications.

In the particular setup we use in this paper, a special transformation of variable known as *the Cole-Hopf transformation* gives the exact solution [19], which allows us to test accuracy of the perturbative expansion. We shall see that the comparisons to the exact solution are quite encouraging. Since our approximation scheme is easily extended to multi-dimensional setups, we expect it will open real possibilities to obtain explicit optimal portfolios or hedging strategies in more realistic situations, which is so far limited to very simplistic models.

2 Setup

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the augmented filtration generated by two dimensional Brownian motion (B_1, B_2). The market consists of one risk-free money market account with zero interest rate, and one risky asset with stochastic volatility. The

SDEs of the risky asset S and its volatility X are assumed to follow

$$dS_t/S_t = \mu dt + \sqrt{X_t}\Big(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}\Big) \tag{2.1}$$

$$dX_t = k\big(m - X_t\big)dt + c\sqrt{X_t}dB_{1t} \tag{2.2}$$

where $\rho \in (-1,1)$ is a constant correlation parameter and μ, k, m and c are all positive constants. Let us denote π_t is the invested amount to the risky asset. Then, the investor's wealth dynamics follows

$$dW_t^\pi = \mu\pi_t dt + \pi_t\sqrt{X_t}\Big(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}\Big) \tag{2.3}$$

with the initial endowment w_0. We assume that the utility of an agent is given by the exponential form with risk aversion parameter $\gamma > 0$ and only dependent on the terminal wealth at time T. Let us denote a function U as

$$U(x) = -\exp\Big(-\gamma x\Big) \ , \tag{2.4}$$

and then the agent's problem is given by

$$J(w_0) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[U(W_T^\pi)\Big] \tag{2.5}$$

where $\mathcal{A}$ is the set of all the admissible strategies.

It is well known that the above problem can be represented by a quadratic growth FBSDE. Particularly simple and clear derivation of the relevant FBSDE are given in Hu, Imkeller and Müller (2005) [8] for exponential and power utilities, and in Horst et al. (2011) [9] for generic form of utilities. It can be shown that the optimal strategy π^* is specified by

$$\pi_t^* = \frac{1}{\gamma X_t}\Big(\mu - \gamma\rho\sqrt{X_t}Z_t\Big) \tag{2.6}$$

where Z is a solution of the following FBSDE:

$$\begin{aligned} dV_t &= -f(Z_t, X_t)dt + Z_t dB_{1t} \\ V_T &= 0 \end{aligned} \tag{2.7}$$

with a quadratic growth driver:

$$f(Z_t, X_t) = -\frac{\gamma}{2}(1-\rho^2)Z_t^2 - \frac{\mu}{\sqrt{X_t}}\rho Z_t + \frac{1}{2\gamma}\frac{\mu^2}{X_t} \ . \tag{2.8}$$

One can concentrate on the FBSDE system composed by X and V since the dynamics of S itself drops off from the system. In the following, we denote B_t instead of B_{1t} for simplicity.

3 Perturbative Expansion

We now introduce a perturbative expansion parameter ϵ to render the original system linear decoupled FBSDE in each order of ϵ. We write

$$dV_t^{(\epsilon)} = -\frac{\mu^2}{2\gamma X_t}dt - \epsilon g(Z_t^{(\epsilon)}, X_t)dt + Z_t^{(\epsilon)}dB_t \tag{3.1}$$

$$V_T^{(\epsilon)} = 0 \tag{3.2}$$

where

$$g(z,x) = -\frac{\gamma}{2}(1-\rho^2)z^2 - \frac{\mu\rho}{\sqrt{x}}z \ . \tag{3.3}$$

We suppose that the solution is given by a perturbative expansion in terms of ϵ as

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{3.4}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots \ . \tag{3.5}$$

Although it is possible to eliminate the linear term of z from the driver function $g(z,x)$ by using the change of probability measure, we treat it directly here since it is not always a practical method in the presence of complicated state dependencies in its coefficient in more realistic situations.

Once we obtain the solution up to the certain order of ϵ, then putting $\epsilon = 1$ will provide a reasonable approximation as long as the contribution from $g(z,x)$ is small enough. In economic terms, the above approximation corresponds to an expansion of the optimal strategy around the myopic mean-variance portfolio. It is expected to be naturally fit to our perturbative assumption as long as the hedging contribution is only sub-dominant. In the reminder of this work, we consider the expansion up to the third order of ϵ.

Proposition 1 *$(V^{(i)}, Z^{(i)})$ with $i = \{0,1,2,3\}$ follow the linear FBSDEs given below:*

$$dV_t^{(0)} = -\frac{\mu^2}{2\gamma}\frac{1}{X_t}dt + Z_t^{(0)}dB_t \tag{3.6}$$

$$dV_t^{(1)} = -g(Z_t^{(0)}, X_t)dt + Z_t^{(1)}dB_t \tag{3.7}$$

$$dV_t^{(2)} = -\partial_z g(Z_t^{(0)}, X_t)Z_t^{(1)}dt + Z_t^{(2)}dB_t \tag{3.8}$$

$$dV_t^{(3)} = -\left\{\partial_z g(Z_t^{(0)}, X_t)Z_t^{(2)} + \frac{1}{2}\partial_z^2 g(Z_t^{(0)}, X_t)(Z_t^{(1)})^2\right\}dt + Z_t^{(3)}dB_t, \tag{3.9}$$

where the terminal values are all zero, $V_T^{(i)} = 0$ with $i \in \{0,1,2,3\}$, and ∂_z denotes partial derivative with respect to the first argument of function $g(z,x)$.

Proof: It follows from a straightforward application of the method given in [7]. ■

From Proposition 1, one can see that each pair of $(V^{(i)}, Z^{(i)})$ is a solution of a linear decoupled FBSDE and thus easy to integrate. One obtains
zeroth order:

$$V_t^{(0)} = \frac{\mu^2}{2\gamma}\int_t^T \mathbb{E}\left[\frac{1}{X_u}\bigg|\mathcal{F}_t\right] du \tag{3.10}$$

$$Z_t^{(0)} = \frac{\mu^2}{2\gamma}\int_t^T \mathbb{E}\left[\mathcal{D}_t\left(\frac{1}{X_u}\right)\bigg|\mathcal{F}_t\right] du \tag{3.11}$$

first order:

$$V_t^{(1)} = \int_t^T \mathbb{E}\Big[g(Z_u^{(0)}, X_u)\Big|\mathcal{F}_t\Big] du \tag{3.12}$$

$$Z_t^{(1)} = \int_t^T \mathbb{E}\Big[\mathcal{D}_t g(Z_u^{(0)}, X_u)\Big|\mathcal{F}_t\Big] du \tag{3.13}$$

second order:

$$V_t^{(2)} = \int_t^T \mathbb{E}\Big[\partial_z g(Z_u^{(0)}, X_u) Z_u^{(1)}\Big|\mathcal{F}_t\Big] du \tag{3.14}$$

$$Z_t^{(2)} = \int_t^T \mathbb{E}\Big[\mathcal{D}_t\Big(\partial_z g(Z_u^{(0)}, X_u) Z_u^{(1)}\Big)\Big|\mathcal{F}_t\Big] du \tag{3.15}$$

third order:

$$V_t^{(3)} = \int_t^T \mathbb{E}\left[\partial_z g(Z_u^{(0)}, X_u) Z_u^{(2)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0)}, X_u)(Z_u^{(1)})^2\bigg|\mathcal{F}_t\right] du \tag{3.16}$$

$$Z_t^{(3)} = \int_t^T \mathbb{E}\left[\mathcal{D}_t\Big(\partial_z g(Z_u^{(0)}, X_u) Z_u^{(2)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0)}, X_u)(Z_u^{(1)})^2\Big)\bigg|\mathcal{F}_t\right] du \tag{3.17}$$

respectively, where $\mathcal{D}_t$ is a Malliavin derivative with respect to B.

4 Asymptotic Expansion

Although, in the previous section, we have formally expanded the original non-linear FBSDE in terms of a series of linear decoupled FBSDEs, we need to explicitly evaluate the involved expectations to obtain a quantitative result. As explained in [7], this can be done by making use of standard asymptotic expansion technique, which is now widely used for pricing of various European contingent claims and also for computation of the optimal portfolio in complete markets (See, for examples [11, 15, 16, 17, 18] and references therein for concrete examples.).

We introduce a different parameter δ to expand the forward component X in terms of the vol-of-vol, i.e., c:

$$dX_u^{(\delta)} = k(m - X_u^{(\delta)})du + \delta c\sqrt{X_u^{(\delta)}}dB_u \ . \tag{4.1}$$

We expand X up to the third order of δ as

$$X_u^{(\delta)} = X_u^{(0)} + \delta D_{tu} + \frac{\delta^2}{2} E_{tu} + \frac{\delta^3}{3!} F_{tu} + o(\delta^3) \tag{4.2}$$

$$X_t^{(\delta)} = x_t \tag{4.3}$$

where each term is defined by

$$D_{tu} = \left. \frac{\partial X_u^{(\delta)}}{\partial \delta} \right|_{\delta=0}, \qquad E_{tu} = \left. \frac{\partial^2 X_u^{(\delta)}}{\partial \delta^2} \right|_{\delta=0}, \qquad F_{tu} = \left. \frac{\partial^3 X_u^{(\delta)}}{\partial \delta^3} \right|_{\delta=0}. \tag{4.4}$$

The relevant formulas regarding the above expansions are summarized in Appendix A.

Now, in each order of ϵ, we try to expand the backward components in terms of δ. More concretely, we are going to approximate each pair of $(V^{(i)}, Z^{(i)})$ with $i \in \{0, 1, 2, 3\}$ as

$$V_t^{(i,\delta)} = V_t^{(i,0)} + \delta V_t^{(i,1)} + \frac{\delta^2}{2} V_t^{(i,2)} + \frac{\delta^3}{3!} V_t^{(i,3)} + o(\delta^3) \tag{4.5}$$

$$Z_t^{(i,\delta)} = Z_t^{(i,0)} + \delta Z_t^{(i,1)} + \frac{\delta^2}{2} Z_t^{(i,2)} + \frac{\delta^3}{3!} Z_t^{(i,3)} + o(\delta^3) \tag{4.6}$$

As we shall see, the required calculation to obtain $V^{(i,j)}$ is to take expectation value of a polynomial function of $X^{(k)}$ with $k \in \{1, 2, 3\}$. Since each $X^{(k)}$ is given by a multiple Wiener integral, the evaluation of the expectation for $V^{(i,j)}$ can be easily calculated. Once $V^{(i,j)}$ is obtained explicitly in terms of x_t, simple application of Itô's formula gives us the expression of $Z^{(i,j+1)}$ by

$$Z_t^{(i,j+1)} = (j+1) c \sqrt{x_t} \frac{\partial}{\partial x_t} V_t^{(i,j)}(x_t) . \tag{4.7}$$

It is easy to see that $Z^{(i,0)}$ is zero. As long as the vol-of-vol (or c) is small relative to the other parameters, putting $\delta = 1$ is expected to give a reasonable approximation to the original model.

4.1 Asymptotic Expansion of $V^{(0,\delta)}$

In the zero-th order of ϵ, we want to expand

$$V_t^{(0,\delta)}(x_t) = \frac{\mu^2}{2\gamma} \int_t^T \mathbb{E}\left[v_u^{(\delta)} \middle| \mathcal{F}_t \right] du \tag{4.8}$$

in terms of δ, where

$$v_u^{(\delta)} = \frac{1}{X_u^{(\delta)}} . \tag{4.9}$$

One can show that

$$v_u^{(\delta)} = v_u^{(0)} + \delta v_u^{(1)} + \frac{\delta^2}{2} v_u^{(2)} + \frac{\delta^3}{3!} v_u^{(3)} + o(\delta^3) \tag{4.10}$$

where each term is given by

$$
\begin{aligned}
v_u^{(0)} &= (X_u^{(0)})^{-1} && (4.11)\\
v_u^{(1)} &= -(X_u^{(0)})^{-2} D_{tu} && (4.12)\\
v_u^{(2)} &= 2(X_u^{(0)})^{-3} D_{tu}^2 - (X_u^{(0)})^{-2} E_{tu} && (4.13)\\
v_u^{(3)} &= -6(X_u^{(0)})^{-4} D_{tu}^3 + 6(X_u^{(0)})^{-3} D_{tu} E_{tu} - (X_u^{(0)})^{-2} F_{tu}\ . && (4.14)
\end{aligned}
$$

Let us define

$$v_u^{(i)}(x_t) := \mathbb{E}\left[v_u^{(i)}\middle|\mathcal{F}_t\right] \tag{4.15}$$

then, from the results of Appendix, one can check that

$$v_u^{(1)}(x_t) = v_u^{(3)}(x_t) = 0 \tag{4.16}$$

and also

$$
\begin{aligned}
v_u^{(0)}(x_t) &= (X_u^{(0)}(x_t))^{-1} && (4.17)\\
v_u^{(2)}(x_t) &= 2(X_u^{(0)}(x_t))^{-3} D_{tu}^2(x_t)\ . && (4.18)
\end{aligned}
$$

Integration in (4.8) can be performed explicitly as

$$V_t^{(0,\delta)}(x_t) = V_t^{(0,0)}(x_t) + \frac{\delta^2}{2} V_t^{(0,2)}(x_t) + o(\delta^3) \tag{4.19}$$

where

$$
\begin{aligned}
V_t^{(0,0)}(x_t) &= -\frac{\mu^2}{2\gamma}\frac{1}{km}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right) && (4.20)\\
V_t^{(0,2)}(x_t) &= -\frac{\mu^2}{2\gamma}\frac{c^2}{k^2}\left\{\frac{(1-Y_{tT})\big[m(1-Y_{tT})+2Y_{tT}x_t\big]}{2m(X_T^{(0)}(x_t))^2} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right)\right\}\ . && (4.21)
\end{aligned}
$$

The relevant definitions of variables are given in Appendix.

4.2 Asymptotic Expansion of $Z^{(0,\delta)}$

Although we have considered the dynamics of Malliavin derivative $\mathcal{D}_t X_u^{(\delta)}$ directly in [7], it is easier to simply apply Itô's formula to the result of $V^{(0,\delta)}$, since we already have its explicit expression in terms of x_t. One can easily confirm that

$$Z_t^{(0,\delta)}(x_t) = \delta Z_t^{(0,1)}(x_t) + \frac{\delta^3}{3!} Z_t^{(0,3)}(x_t) + o(\delta^3) \tag{4.22}$$

where

$$
\begin{aligned}
Z_t^{(0,1)}(x_t) &= -\frac{\mu^2 c}{2\gamma k}\frac{1-Y_{tT}}{\sqrt{x_t}(X_T^{(0)}(x_t))} && (4.23)\\
Z_t^{(0,3)}(x_t) &= -\frac{3\mu^2 c^3}{2\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^3}\Big[m(1-Y_{tT}) + 2Y_{tT}\ x_t\Big]\ . && (4.24)
\end{aligned}
$$

4.3 Asymptotic Expansion of $V^{(1,\delta)}$

In the first order of ϵ, we need to expand

$$V_t^{(1,\delta)}(x_t) = \int_t^T \mathbb{E}\Big[g(Z_u^{(0,\delta)}, X_u^{(\delta)})\Big|\mathcal{F}_t\Big]du \tag{4.25}$$

$$= -\frac{\gamma}{2}(1-\rho^2)\int_t^T \mathbb{E}\Big[(Z_u^{(0,\delta)})^2\Big|\mathcal{F}_t\Big]du - \mu\rho\int_t^T \mathbb{E}\Big[(X_u^{(\delta)})^{-\frac{1}{2}}Z_u^{(0,\delta)}\Big|\mathcal{F}_t\Big]du\ . \tag{4.26}$$

From the previous results, we have

$$Z_u^{(0,\delta)} = \delta Z_u^{(0,1)}(X_u^{(\delta)}) + \frac{\delta^3}{3!}Z_u^{(0,3)}(X_u^{(\delta)}) + o(\delta^3) \tag{4.27}$$

and hence both of the integrands in (4.26) can be explicitly written as a function of $X_u^{(\delta)}$. Therefore, we can follow the same procedures in Section 4.1: Firstly apply ∂_δ, i.e. partial derivative with respect to δ, and then express the integrand as a function of $X_u^{(0)}$, D_{tu} etc.. The evaluation of its expectation is now easily performed using the results given in Appendix. After straightforward but lengthy calculation, we obtain

$$V_t^{(1,\delta)}(x_t) = \delta V_t^{(1,1)}(x_t) + \frac{\delta^2}{2}V_t^{(1,2)}(x_t) + \frac{\delta^3}{3!}V_t^{(1,3)}(x_t) + o(\delta^3) \tag{4.28}$$

where

$$V_t^{(1,1)}(x_t) = -\frac{\rho\mu^3 c}{2\gamma k^2}\left\{\frac{(1-Y_{tT})}{mX_T^{(0)}(x_t)} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.29}$$

$$V_t^{(1,2)}(x_t) = (1-\rho^2)\frac{\mu^4 c^2}{4\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}\ x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.30}$$

$$V_t^{(1,3)}(x_t) = \frac{3\rho\mu^3 c^3}{2\gamma k^3}\left\{\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^2}\big[m(1-Y_{tT}) - 2Y_{tT}\ x_t\big] - \frac{2}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right.$$
$$\left. - \frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^3}\Big[5m^2(1-Y_{tT})^2 + 9m(1-Y_{tT})(Y_{tT}x_t) + 2(Y_{tT}x_t)^2\Big]\right\}\ . \tag{4.31}$$

4.4 Asymptotic Expansion of $Z^{(1,\delta)}$

By applying Itô's formula to the expanded $V^{(1,\delta)}$, one obtains the volatility component easily as before:

$$Z_t^{(1,\delta)}(x_t) = \frac{\delta^2}{2}Z_t^{(1,2)}(x_t) + \frac{\delta^3}{3!}Z_t^{(1,3)}(x_t) + o(\delta^3) \tag{4.32}$$

where

$$Z_t^{(1,2)}(x_t) \quad = \quad -\frac{\rho\mu^3 c^2}{\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^2} \tag{4.33}$$

$$Z_t^{(1,3)}(x_t) \quad = \quad (1-\rho^2)\frac{3\mu^4 c^3}{4\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \ . \tag{4.34}$$

4.5 Asymptotic Expansion of $V^{(2,\delta)}$

In the second order of ϵ, we have to evaluate

$$V_t^{(2,\delta)}(x_t) = \int_t^T \mathbb{E}\Big[\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)}) Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big] du \tag{4.35}$$

$$= -\gamma(1-\rho^2)\int_t^T \mathbb{E}\Big[Z_u^{(0,\delta)} Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big] du - \mu\rho\int_t^T \mathbb{E}\Big[(X_u^{(\delta)})^{-\frac{1}{2}} Z_u^{(1,\delta)}\Big|\mathcal{F}_t\Big] du \ . \tag{4.36}$$

Following the same arguments in Section 4.3, we can express the above expectation explicitly. After tedious calculation, one obtains

$$V_t^{(2,\delta)}(x_t) = \frac{\delta^2}{2}V_t^{(2,2)}(x_t) + \frac{\delta^3}{3!}V_t^{(2,3)}(x_t) + o(\delta^3) \tag{4.37}$$

where

$$V_t^{(2,2)}(x_t) \quad = \quad -\frac{\rho^2\mu^4 c^2}{\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \tag{4.38}$$

$$V_t^{(2,3)}(x_t) \quad = \quad \rho(1-\rho^2)\frac{9\mu^5 c^3}{4\gamma k^4}\left\{\frac{(1-Y_{tT})\big[11m^2(1-Y_{tT})^2 + 15m(1-Y_{tT})(Y_{tT}x_t) + 6(Y_{tT}x_t)^2\big]}{6m^3(X_T^{(0)}(x_t))^3} + \frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} \ . \tag{4.39}$$

4.6 Asymptotic Expansion of $Z^{(2,\delta)}$

As before, simple application of Itô's formula yields

$$Z_t^{(2,\delta)}(x_t) = \frac{\delta^3}{3!}Z_t^{(2,3)}(x_t) + o(\delta^3) \tag{4.40}$$

where

$$Z_t^{(2,3)}(x_t) = -\frac{3\rho^2\mu^4 c^3}{\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \ . \tag{4.41}$$

4.7 Asymptotic Expansion of $(V^{(3,\delta)}, Z^{(3,\delta)})$

We have

$$V_t^{(3,\delta)}(x_t) = \int_t^T \mathbb{E}\left[\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)}) Z_u^{(2,\delta)} + \frac{1}{2}\partial_z^2 g(Z_u^{(0,\delta)}, X_u^{(\delta)})\big(Z_u^{(1,\delta)}\big)^2 \middle| \mathcal{F}_t\right] du \quad (4.42)$$

and we can easily confirm that the contribution of $O(\delta^3)$ comes only from the first term. The result is

$$V_t^{(3,\delta)}(x_t) = \frac{\delta^3}{3!} V_t^{(3,3)}(x_t) + o(\delta^3) \quad (4.43)$$

where

$$V_t^{(3,3)}(x_t) = -\frac{3\rho^3\mu^5 c^3}{\gamma k^4}\left\{\frac{(1-Y_{tT})\Big[11m^2(1-Y_{tT})^2 + 15m(1-Y_{tT})(Y_{tT}x_t) + 6(Y_{tT}x_t)^2\Big]}{6m^3(X_T^{(0)}(x_t))^3} + \frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\} . \quad (4.44)$$

It is obvious to conclude

$$Z_t^{(3,\delta)}(x_t) = o(\delta^3) . \quad (4.45)$$

4.8 Asymptotic Expansion of $(V^{(i,\delta)}, Z^{(i,\delta)})$ with $(i \geq 4)$

Let us consider what happens when we proceed further to a higher order of ϵ. In the fourth order, we see that $V^{(4,\delta)}$ has contributions from

$$\partial_z g(Z_u^{(0,\delta)}, X_u^{(\delta)}) Z_u^{(3,\delta)} \quad (4.46)$$
$$\partial_z^2 g(Z_u^{(0,\delta)}, X_u^{(\delta)}) Z_u^{(1,\delta)} Z_u^{(2,\delta)} \quad (4.47)$$
$$\partial_z^3 g(Z_u^{(0,\delta)}, X_u^{(\delta)})\big(Z_u^{(1,\delta)}\big)^3 \quad (4.48)$$

which are all $o(\delta^3)$ and hence $V^{(4,\delta)} = o(\delta^3)$. Thus, we obviously have $Z^{(4,\delta)} = o(\delta^3)$. By repeating the same arguments, we can conclude

$$V_t^{(i,\delta)} = Z_t^{(i,\delta)} = o(\delta^3) \quad (4.49)$$

for all $i \geq 4$.

4.9 Summary of Expansion and its Interpretation

Let us suppose, as we have hypothesized at the beginning, that the perturbative expansions

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \frac{\epsilon^2}{2} V_t^{(2)} + \frac{\epsilon^3}{3!} V_t^{(3)} + \cdots \quad (4.50)$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \frac{\epsilon^2}{2} Z_t^{(2)} + \frac{\epsilon^3}{3!} Z_t^{(3)} + \cdots \quad (4.51)$$

really converges to the true solution. Since, from the previous observation, there is no contribution to the solution of FBSDE from the fourth or higher order terms of ϵ as long as we neglect $o(\delta^3)$ part, the results we have obtained can be interpreted as the asymptotic expansion of the true solution of the FBSDE up to the third order of δ.

As a summary, whole of the discussion in Section 4 leads to the next proposition:

Proposition 2 *The solution* (V, Z) *of the following FBSDE:*

$$dV_t = -\left\{-\frac{\gamma}{2}(1-\rho^2)Z_t^2 - \frac{\mu}{\sqrt{X_t}}\rho Z_t + \frac{1}{2\gamma}\frac{\mu^2}{X_t}\right\}dt + Z_t dB_t;\ V_T = 0, \tag{4.52}$$

$$dX_t = k(m - X_t)dt + c\sqrt{X_t}dB_t;\ X_0 = x \tag{4.53}$$

can be asymptotically expanded in terms of vol-of-vol that is c, as:

$$\begin{aligned} V_t(x_t) &= V_t^{(0,0)}(x_t) + \frac{1}{2}V_t^{(0,2)}(x_t) + V_t^{(1,1)}(x_t) + \frac{1}{2}V_t^{(1,2)}(x_t) + \frac{1}{3!}V_t^{(1,3)}(x_t) \\ &\quad + \frac{1}{2}V_t^{(2,2)}(x_t) + \frac{1}{3!}V_t^{(2,3)}(x_t) + \frac{1}{3!}V_t^{(3,3)}(x_t) + o(c^3) \end{aligned} \tag{4.54}$$

$$\begin{aligned} Z_t(x_t) &= Z_t^{(0,1)}(x_t) + \frac{1}{3!}Z_t^{(0,3)}(x_t) + \frac{1}{2}Z_t^{(1,2)}(x_t) + \frac{1}{3!}Z_t^{(1,3)}(x_t) \\ &\quad + \frac{1}{3!}Z_t^{(2,3)}(x_t) + o(c^3)\ , \end{aligned} \tag{4.55}$$

where each term is given by

$$V_t^{(0,0)}(x_t) = -\frac{\mu^2}{2\gamma}\frac{1}{km}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right)$$

$$V_t^{(0,2)}(x_t) = -\frac{\mu^2}{2\gamma}\frac{c^2}{k^2}\left\{\frac{(1-Y_{tT})\big[m(1-Y_{tT}) + 2Y_{tT}x_t\big]}{2m(X_T^{(0)}(x_t))^2} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(1,1)}(x_t) = -\frac{\rho\mu^3 c}{2\gamma k^2}\left\{\frac{(1-Y_{tT})}{mX_T^{(0)}(x_t)} + \frac{1}{m^2}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(1,2)}(x_t) = (1-\rho^2)\frac{\mu^4 c^2}{4\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT}) + 2Y_{tT}\ x_t]}{2m^2(X_T^{(0)}(x_t))^2} + \frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$\begin{aligned} V_t^{(1,3)}(x_t) &= \frac{3\rho\mu^3 c^3}{2\gamma k^3}\left\{\frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^2}\big[m(1-Y_{tT}) - 2Y_{tT}\ x_t\big] - \frac{2}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right. \\ &\quad \left. - \frac{(1-Y_{tT})}{2m^2(X_T^{(0)}(x_t))^3}\Big[5m^2(1-Y_{tT})^2 + 9m(1-Y_{tT})(Y_{tT}x_t) + 2(Y_{tT}x_t)^2\Big]\right\} \end{aligned}$$

$$V_t^{(2,2)}(x_t) = -\frac{\rho^2\mu^4c^2}{\gamma k^3}\left\{\frac{(1-Y_{tT})[3m(1-Y_{tT})+2Y_{tT}x_t]}{2m^2(X_T^{(0)}(x_t))^2}+\frac{1}{m^3}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(2,3)}(x_t) = \rho(1-\rho^2)\frac{9\mu^5c^3}{4\gamma k^4}\left\{\frac{(1-Y_{tT})\left[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\right]}{6m^3(X_T^{(0)}(x_t))^3}+\frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

$$V_t^{(3,3)}(x_t) = -\frac{3\rho^3\mu^5c^3}{\gamma k^4}\left\{\frac{(1-Y_{tT})\left[11m^2(1-Y_{tT})^2+15m(1-Y_{tT})(Y_{tT}x_t)+6(Y_{tT}x_t)^2\right]}{6m^3(X_T^{(0)}(x_t))^3}+\frac{1}{m^4}\ln\left(\frac{Y_{tT}\ x_t}{X_T^{(0)}(x_t)}\right)\right\}$$

and

$$\begin{aligned}
Z_t^{(0,1)}(x_t) &= -\frac{\mu^2c}{2\gamma k}\frac{1-Y_{tT}}{\sqrt{x_t}(X_T^{(0)}(x_t))} \\
Z_t^{(0,3)}(x_t) &= -\frac{3\mu^2c^3}{2\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^3}\left[m(1-Y_{tT})+2Y_{tT}\ x_t\right] \\
Z_t^{(1,2)}(x_t) &= -\frac{\rho\mu^3c^2}{\gamma k^2}\frac{(1-Y_{tT})^2}{\sqrt{x_t}(X_T^{(0)}(x_t))^2} \\
Z_t^{(1,3)}(x_t) &= (1-\rho^2)\frac{3\mu^4c^3}{4\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3} \\
Z_t^{(2,3)}(x_t) &= -\frac{3\rho^2\mu^4c^3}{\gamma k^3}\frac{(1-Y_{tT})^3}{\sqrt{x_t}(X_T^{(0)}(x_t))^3}\ .
\end{aligned}$$

It then specifies the optimal strategy π_t^* in (2.6) up to the third order of vol-of-vol.

5 Numerical Comparison to the Exact Solution

In [19], it is shown that the Cole-Hopf transformation allows the exact solution for our problem. We define $K_t = e^{\eta V_t}$ with some constant $\eta \in \mathbb{R}$. Then, the dynamics of K is given by

$$dK_t/K_t = \left(\frac{\gamma\eta}{2}(1-\rho^2)+\frac{\eta^2}{2}\right)Z_t^2dt + \left\{\frac{\mu\eta}{\sqrt{X_t}}\rho Z_t - \frac{\mu^2\eta}{2\gamma}\frac{1}{X_t}\right\}dt + \eta Z_t dB_t\ . \tag{5.1}$$

Thus, by choosing $\eta^* = -\gamma(1-\rho^2)$ one can eliminate the quadratic term. By defining $Q_t = \eta^* K_t Z_t$, the above equation becomes

$$dK_t = \left(\frac{\mu\rho}{\sqrt{X_t}}Q_t - \frac{\mu^2\eta^*}{2\gamma}\frac{K_t}{X_t}\right)dt + Q_t dB_t \ , \tag{5.2}$$

which is a linear FBSDE with terminal value $K_T = 1$.

Now, let us introduce a new measure $\mathbb{P}^*$ for which Brownian motion is related to that in the original measure $\mathbb{P}$ by

$$dB_t^* = dB_t + \frac{\mu\rho}{\sqrt{X_t}}dt \ . \tag{5.3}$$

Then, we have

$$dK_t = \frac{\mu^2(1-\rho^2)}{2X_t}K_t dt + Q_t dB_t^* \ , \tag{5.4}$$

which can be integrated easily. Thus, the solution of the original FBSDE is given by

$$V_t = -\frac{1}{\gamma(1-\rho^2)}\ln\left\{\mathbb{E}^{\mathbb{P}^*}\left[\exp\left(-\frac{\mu^2}{2}(1-\rho^2)\int_t^T \frac{ds}{X_s}\right)\middle|\mathcal{F}_t\right]\right\} \tag{5.5}$$

where X follows

$$dX_t = k(n - X_t)dt + c\sqrt{X_t}dB_t^* \tag{5.6}$$

under the new measure, where the adjusted mean n denotes $n = m - \rho\mu c/k$.

Remark: Note that, the Cole-Hope transformation cannot always be used to derive exact solutions in more generic situations, such as, cases including multi-dimensional risk factors, time or state dependent correlation parameters, e.t.c.. Our scheme can be extended easily, at least in principle, for these cases, too.

5.1 Numerical Comparison

Since there is no closed expression for the exact solution (5.5), we have to estimate it by Monte Carlo (MC) simulation. In order to guarantee the positivity of X, we use the implicit Milstein scheme [10]:

$$X(t_n) = \frac{X(t_{n-1}) + kn\Delta t + c\sqrt{X(t_{n-1})}\xi_n\sqrt{\Delta t} + \frac{1}{4}c^2\Delta t\big(\xi_n^2 - 1\big)}{1 + k\Delta t} \tag{5.7}$$

where $(t_n)_{n\geq 1}$ is equally spaced time grids and $\Delta t = t_n - t_{n-1}$. $(\xi_n)_{n\geq 1}$ is a sequence of independent random variable with standard normal distribution $\mathbf{N}(0,1)$. We have run 1-million plus 1-million antipathetic scenarios with step size $\Delta t = 0.005$ for all the simulations. In the following tables, we have compared the MC simulation of the exact solution of V_0 and the result of our asymptotic expansion up to the third order of vol-of-vol. The each column represents the maturity T, the result of MC simulation, its standard deviation, ϵ-0th, ϵ-1st, ϵ-2nd and ϵ-3rd order approximation, respectively. All the parameters used are provided in a caption under the each table.

In Table 1, for example, we have used $m = 6.25\%$ and $c = 5\%$, which corresponds to roughly $\sqrt{m} = 25\%$ implied volatility of the risky asset with $c/\sqrt{m} = 20\%$ vol-of-vol in log-normal terms. One can see that the approximation is quite accurate even for 10-year maturity. In the second example, the vol-of-vol is relatively smaller than the first case, corresponding to $\sqrt{m} = 34.6\%$ and $c/\sqrt{m} = 14.4\%$, and the approximation becomes even more accurate. In Tables 3 and 4, we have provided examples where the initial value of asset volatility X is far from its mean. In the latter case, for example, $m = 15\%$ but $x_0 = 5\%$ with $c = 6\%$. This means asset implied volatility increases roughly 22% to 39% where the vol-of-vol decreases roughly 27% to 15%. One can see the good accuracy of our approximation also for these examples.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	23.061	0.0003	23.539	23.035	23.049	23.049
2	45.844	0.0008	47.769	45.671	45.787	45.783
3	68.197	0.0013	72.510	67.691	68.086	68.067
4	90.067	0.0016	97.630	88.997	89.919	89.868
5	111.455	0.0018	123.031	109.560	111.313	111.207
6	132.397	0.0018	148.639	129.398	132.317	132.128
7	152.938	0.0019	174.401	148.552	152.987	152.685
8	173.128	0.0023	200.278	167.076	173.377	172.932
9	193.011	0.0031	226.239	185.028	193.537	192.918
10	212.630	0.0041	252.263	202.468	213.508	212.686

Table 1: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 6.25\%, k = 15\%, c = 5\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	17.030	0.0001	16.827	17.026	17.028	17.028
2	34.718	0.0004	33.929	34.684	34.702	34.702
3	52.955	0.0008	51.246	52.862	52.908	52.912
4	71.636	0.0012	68.734	71.464	71.549	71.559
5	90.667	0.0015	86.355	90.411	90.538	90.559
6	109.967	0.0017	104.081	109.636	109.803	109.841
7	129.472	0.0017	121.889	129.086	129.284	129.345
8	149.129	0.0018	139.763	148.717	148.930	149.024
9	168.898	0.0019	157.688	168.494	168.704	168.836
10	188.748	0.0025	175.653	188.387	188.572	188.752

Table 2: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 12\%, k = 12\%, c = 5\%, x_0 = m, \mu = 20\%, \rho = 30\%, \gamma = 1$.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	11.998	0.0001	11.815	11.995	11.998	11.998
2	25.573	0.0005	24.766	25.524	25.549	25.550
3	40.812	0.0011	38.848	40.636	40.712	40.719
4	57.752	0.0019	54.038	57.345	57.510	57.533
5	76.374	0.0027	70.298	75.637	75.925	75.981
6	96.601	0.0035	87.579	95.469	95.904	96.019
7	118.311	0.0039	105.819	116.774	117.361	117.575
8	141.352	0.0040	124.953	139.465	140.190	140.548
9	165.566	0.0041	144.907	163.439	164.264	164.823
10	190.798	0.0048	165.611	188.589	189.452	190.277

Table 3: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 5\%, k = 15\%,\ c = 5\%, x_0 = 10\%, \mu = 15\%, \rho = 40\%, \gamma = 1$.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	20.814	0.0004	20.277	20.769	20.782	20.782
2	38.851	0.0010	37.228	38.695	38.746	38.750
3	54.882	0.0013	52.008	54.596	54.698	54.709
4	69.454	0.0016	65.294	69.040	69.196	69.218
5	82.940	0.0018	77.508	82.409	82.619	82.654
6	95.602	0.0019	88.922	94.962	95.225	95.274
7	107.625	0.0020	99.726	106.884	107.198	107.263
8	119.143	0.0021	110.053	118.309	118.672	118.753
9	130.259	0.0022	120.001	129.335	129.746	129.845
10	141.048	0.0023	129.643	140.040	140.496	140.613

Table 4: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 15\%, k = 15\%,\ c = 6\%, x_0 = 5\%, \mu = 15\%, \rho = 35\%, \gamma = 1$.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	23.780	0.0013	24.746	23.551	23.614	23.612
2	47.900	0.0032	52.004	46.632	47.179	47.148
3	71.330	0.0044	80.948	68.134	70.016	69.887
4	93.790	0.0049	111.020	87.670	92.054	91.724
5	115.320	0.0050	141.849	105.214	113.444	112.785
6	136.070	0.0053	173.185	120.908	134.385	133.260
7	156.190	0.0064	204.862	134.962	155.056	153.327
8	175.790	0.0085	236.766	147.600	175.595	173.130
9	194.990	0.0110	268.824	159.036	196.098	192.776
10	213.860	0.0138	300.983	169.464	216.628	212.341

Table 5: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 6.25\%, k = 20\%,\ c = 10\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

maturity (yr)	MC (%)	std err (%)	ϵ-0th (%)	ϵ-1st (%)	ϵ-2nd (%)	ϵ-3rd (%)
1	24.340	0.0020	25.461	23.896	23.992	23.988
2	49.550	0.0048	54.541	47.232	48.090	48.035
3	73.840	0.0061	86.046	68.269	71.261	71.038
4	96.840	0.0066	119.177	86.407	93.441	92.870
5	118.640	0.0066	153.398	101.609	114.899	113.761
6	139.490	0.0072	188.350	114.097	135.960	134.016
7	159.560	0.0093	223.792	124.189	156.901	153.913
8	179.030	0.0125	259.561	132.223	177.919	173.660
9	198.030	0.0161	295.551	138.520	199.144	193.404
10	216.650	0.0200	331.688	143.364	220.643	213.235

Table 6: A comparison to the MC simulation and asymptotic expansion with parameters: $m = 6.25\%, k = 20\%, c = 12\%, x_0 = m, \mu = 17\%, \rho = -30\%, \gamma = 1$.

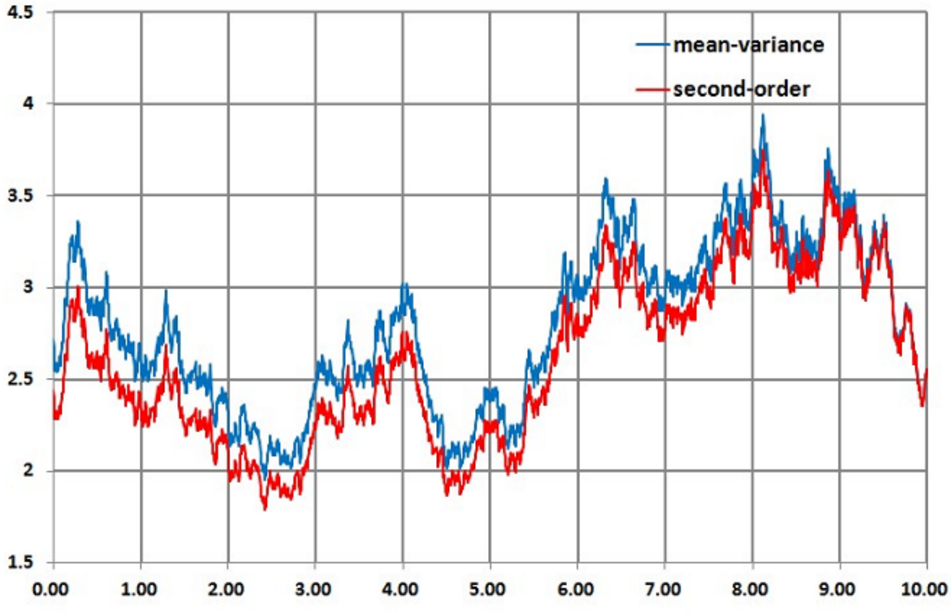

Figure 1: A sample path each for the mean-variance portfolio and approximated (ϵ-2nd order) optimal portfolio weight. The used parameters are $m = 6.25\%$, $k = 15\%$, $c = 5\%$, $x_0 = m$, $\mu = 17\%$, $\rho = -35\%$ and $\gamma = 1$.

In Tables 5 and 6, we studied a bit harsh situations where the vol-of-vol has comparable or larger size relative to parameters included in the drift process of X. For example, in Table 6, we have used $m = 6.25\%$ and $c = 12\%$, which corresponds to $\sqrt{m} = 25\%$ and $c/\sqrt{m} = 48\%$. Even in these examples, ϵ-2nd and 3rd order approximations provide much better estimation than the one obtained from ϵ-0th order, which is equivalent to the approximation using a mean-variance portfolio. For the last two examples, it would be better to introduce δ also in the drift term of X to avoid the appearance of small parameters in denominators of the resultant formulas. These potential improvements of the asymptotic expansion technique will be pursued in different opportunities.

Lastly, in Figure 1, we give a sample path each for the mean-variance and the approximated ϵ-2nd order optimal portfolio weight π^* with parameters $m = 6.25\%$, $k = 15\%$, $c = 5\%$, $x_0 = m$, $\mu = 17\%$, $\rho = -35\%$ and $\gamma = 1$ for a 10-year investment. We give only the ϵ-2nd order result here since it is difficult to distinguish from the others in the graph. Note that the ϵ-3rd oder contribution $Z^{(3,\delta)}$ is $o(\delta^3)$ and hence it has no contribution in the current analysis. One can see that the optimal amount of investment is smaller than that of the mean-variance strategy due to the hedging demand. This relationship flips the sign when the positive correlation ρ is used. The difference between the mean-variance and optimal strategies becomes gradually smaller as the time comes closer to the maturity as expected.

6 Conclusion

In this work, we have studied the optimal portfolio problem in an incomplete market with stochastic volatility that is not perfectly hedgeable. We have applied the newly developed perturbative methodology combined with standard asymptotic expansion technique and derived the explicit solution of the corresponding quadratic growth FBSDE up to the third order of vol-of-vol. The comparison to the exact solution shows quite encouraging results about its accuracy even for quite long maturities, such as 10 years. As long as we know, the existing numerical techniques, such as regression based Monte Carlo simulations, seem mostly limited to short maturities, say, several months to one year. Furthermore, the great advantage of our method is its ability to provide explicit expressions of the optimal portfolios or hedging strategies, which obviously have great importance for the practical use.

In contrast to the Cole-Hopf transformation, our method can be applied to much more generic setups with multi-dimensional risk factors, which, we expect, will open real possibilities to obtain explicit expressions of optimal portfolios and hedging strategies in incomplete and/or constrained markets with realistic assumptions. This will be addressed in separate works in the future.

A Formulas for X's Asymptotic Expansion

We assume $(u > t)$ throughout this section. The value x_t is defined as the initial condition at time t by

$$x_t = X_t^{(\delta)} \ . \tag{A.1}$$

A.1 δ 0th order

The relevant equation becomes deterministic in this case:

$$dX_u^{(0)} = k(m - X_u^{(0)})du \tag{A.2}$$

and thus

$$X_u^{(0)} = Y_{tu}x_t + m(1 - Y_{tu}) \tag{A.3}$$

where we have defined

$$Y_{tu} = \exp\Big(-k(u-t)\Big) \ . \tag{A.4}$$

A.2 δ 1st order

Since we have

$$d(\partial_\delta X_u^{(\delta)}) = -k(\partial_\delta X_u^{(\delta)})du + \left(c\sqrt{X_u^{(\delta)}} + \frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta X_u^{(\delta)})\right)dB_u \tag{A.5}$$

which yields

$$dD_{tu} = -kD_{tu}du + c\sqrt{X_u^{(0)}}dB_u \tag{A.6}$$

and hence

$$D_{tu} = c\int_t^u Y_{us}\sqrt{X_s^{(0)}}dB_s \ . \tag{A.7}$$

A.3 δ 2nd order

Since we have

$$\begin{aligned} d(\partial_\delta^2 X_u^{(\delta)}) &= -k(\partial_\delta^2 X_u^{(\delta)})du \\ &+ \left\{c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta X_u^{(\delta)}) - \frac{1}{4}\delta c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})^2 + \frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^2 X_u^{(\delta)})\right\}dB_u \end{aligned}$$

which yields

$$dE_{tu} = -kE_{tu}du + c(X_u^{(0)})^{-\frac{1}{2}}D_{tu}dB_u \tag{A.8}$$

and hence

$$E_{tu} = c\int_t^u Y_{us}(X_s^{(0)})^{-\frac{1}{2}}D_{ts}dB_s \ . \tag{A.9}$$

A.4 δ 3rd order

We have

$$
\begin{aligned}
d(\partial_\delta^3 X_u^{(\delta)}) = -k(\partial_\delta^3 X_u^{(\delta)})du \\
+\Big\{-\frac{3}{4}c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})^2 + \frac{3}{2}c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^2 X_u^{(\delta)}) \\
+\frac{3}{8}\delta c(X_u^{(\delta)})^{-\frac{5}{2}}(\partial_\delta X_u^{(\delta)})^3 - \frac{3}{4}\delta c(X_u^{(\delta)})^{-\frac{3}{2}}(\partial_\delta X_u^{(\delta)})(\partial_\delta^2 X_u^{(\delta)}) \\
+\frac{1}{2}\delta c(X_u^{(\delta)})^{-\frac{1}{2}}(\partial_\delta^3 X_u^{(\delta)})\Big\}dB_u
\end{aligned} \tag{A.10}
$$

thus,

$$
dF_{tu} = -kF_{tu}du + \frac{3}{2}c\left\{(X_u^{(0)})^{-\frac{1}{2}}E_{tu} - \frac{1}{2}(X_u^{(0)})^{-\frac{3}{2}}D_{tu}^2\right\}dB_u \tag{A.11}
$$

and then

$$
F_{tu} = \frac{3}{2}c\int_t^u Y_{us}\Big\{(X_s^{(0)})^{-\frac{1}{2}}E_{ts} - \frac{1}{2}(X_s^{(0)})^{-\frac{3}{2}}D_{ts}^2\Big\}dB_s\ . \tag{A.12}
$$

A.5 Relevant expectation values

It is easy to check that

$$
\mathbb{E}\Big[D_{tu}\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[E_{tu}\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[F_{tu}\Big|\mathcal{F}_t\Big] = 0\ . \tag{A.13}
$$

On the other hand, we have

$$
\begin{aligned}
d(D_{tu})^2 &= 2D_{tu}dD_{tu} + d\langle D_{t\cdot}\rangle_u \\
&= -2kD_{tu}^2du + c^2X_u^{(0)}du + 2cD_{tu}\sqrt{X_u^{(0)}}dB_u
\end{aligned} \tag{A.14}
$$

and hence

$$
\begin{aligned}
D_{tu}^2(x_t) &:= \mathbb{E}\Big[D_{tu}^2\Big|\mathcal{F}_t\Big] = c^2\int_t^u e^{-2k(u-s)}X_s^{(0)}(x_t)ds \\
&= \frac{c^2}{2k}(1-Y_{tu})\Big[(1-Y_{tu})m + 2Y_{tu}x_t\Big]\ .
\end{aligned} \tag{A.15}
$$

By following the similar procedures, it is easy to confirm that

$$
\mathbb{E}\Big[D_{tu}^3\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[D_{tu}E_{tu}\Big|\mathcal{F}_t\Big] = 0\ . \tag{A.16}
$$

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