Individual and Aggregate Money Demands[*]

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Abstract

I construct a model in which money and bond holdings are consistent with individual decisions and aggregate variables such as production and interest rates. The agents are infinitely-lived, have constant-elasticity preferences, and receive a fraction of their income in money. Each agent solves a Baumol-Tobin money management problem. Markets are segmented because financial frictions make agents trade bonds for money at different times. Trading frequency, consumption, government decisions and prices are mutually consistent. An increase in inflation, for example, implies higher trading frequency, more bonds sold to account for seigniorage, and lower real balances.

JEL Codes: E3, E4, E5.

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1. Introduction

Aggregate variables such as the money-income ratio depend on individual decisions. Here, I combine the general equilibrium Baumol-Tobin models of Jovanovic (1982) and Romer (1986) with the market segmentation models of Grossman and Weiss (1983), Rotemberg (1984), and Grossman (1987) to connect individual decisions to variables used in monetary policy. The objective is to create a framework to analyze consumption, prices, and money taking into account the changes in the individual demands for money.

I use two features from the models above. I obtain the demands for money from an inventory model of Baumol (1952) and Tobin (1956) in general equilibrium, as Jovanovic and Romer (other general equilibrium Baumol-Tobin models are in Fusselman and Grossman 1989, Heathcote 1998, Chiu 2007, and Rodriguez-Mendizabal 2006). And I express individual optimization problems as in the market segmentation models of Grossman and Weiss, and Rotemberg. As a result, agents trade bonds for money at different times, now with the trading frequency obtained in equilibrium.

In addition to combining the two frameworks, I make two changes from the models above. First, the model has infinitely-lived agents and consumption smoothing while Jovanovic assumes constant consumption and Romer assumes zero intertemporal discount and overlapping generations. I consider consumption smoothing because it affects the demand for money and the welfare cost of inflation. Infinite-lived agents, on the other hand, remove the influence of the length of life of each generation on equilibrium variables. In particular, consumption and money over time after policy changes are not affected by the length of each generation. Infinite lives and consumption smoothing, moreover, facilitate comparison with cash-in-advance models such as the models of Lucas and Stokey (1987), and Cooley and Hansen (1989, 1991).

Second, I let agents receive a fraction of income in money within holding periods. I

2

allow a fraction of income in money because market segmentation implies large holding periods to match data on velocity (Edmond and Weill 2008), holding periods of six months or larger. As traditional Baumol-Tobin models implicitly assume that agents receive their income in interest-bearing bonds (Karni 1973), large holding periods would make agents separated from their income for a long period. Therefore, I follow Alvarez et al. (2009) and Khan and Thomas (2010) and assume that agents receive part of their income in bonds and the remaining in money. The fraction of income in money is thought to be substantial, sixty percent for example, and interpreted as labor income.

The result is a monetary model in which trading periods, consumption and the distribution of money holdings are consistent with individual decisions and aggregate variables. An increase in inflation, for example, implies higher trading frequency, more bonds sold to account for seigniorage, and lower real balances. Silva (2009, forthcoming) uses the model to study the effects of interest rate shocks and the welfare cost of inflation. Even with the modifications made here, the model allows its steady state to be characterized analytically.

Having the frequency of trades chosen optimally, as in the model, implies a better fit with the data on money and interest rates. The demand for money, for example, has an interest elasticity of -0.5 and semi-elasticity of -12.5. The interest elasticity is approximately zero, in contrast, with fixed holding periods (Romer 1986 and Grossman 1987). The choice of the interval between trades makes easier for agents to change their demand for money.

2. The Model

There is a continuum of infinitely-lived agents with measure one. There is an asset market and a goods market. The asset market concentrates trades between bonds and money and the goods market concentrates the trades between goods and money.

Only money can be used to buy goods. The government sets government consumption and taxes and controls the supply of money through open market operations.

The financial frictions appear when agents transfer resources between the asset market and the goods market. Each agent has a brokerage account and a bank account, as in Alvarez et al. (2002, 2009). The brokerage account is used to manage the activities in the asset market and the bank account to manage the activities in the goods market. The financial frictions are represented by a transfer cost Γ in real terms that the agents need to pay whenever they transfer resources between the brokerage account and the bank account. The transfer cost is paid with the resources in the brokerage account and it does not depend on the volume transferred. Γ represents a fixed cost of portfolio adjustment.

Time is continuous, $t \geq 0$. Time is continuous to avoid integer constraints on the decision of the time to make transfers. At $t = 0$, each agent has M_0 in money in the bank account and B_0 in bonds in the brokerage account. There is a given distribution F of M_0 and B_0. Index agents by their initial holdings of money and bonds, $s = (M_0, B_0)$.

Each agent is composed of three participants, a worker, a trader, and a shopper, as in Lucas (1990). At the beginning of each period, the worker engages in the production and sales of the consumption good, the trader goes to the asset market to manage the brokerage account, and the shopper goes to the goods market to buy consumption goods. At the end of each period, the three participants rejoin to share the consumption good.

The flow of funds occurs in the following way. The worker produces $Y(t)$ goods and sells the production for money to other agents in the goods market by the price $P(t)$. After the sale, the worker transfers $aP(t)Y(t)$ to the bank account and $(1 - a)P(t)Y(t)$ to the brokerage account. The trader trades bonds and money with the resources of the brokerage account. The trades can be made with other

traders or with the government in open market operations. If it is necessary to make a transfer from the brokerage account to the bank account, the trader sells the necessary quantity of bonds and makes the transfer. In the same way, the trader can make transfers from the bank account to the brokerage account. As the money deposited in the brokerage account cannot be used to buy goods and does not receive interest, the money in the brokerage account is immediately used to buy bonds. The shopper uses the available money in the bank account to buy goods in the goods market. The shopper then brings the goods purchased to the other participants to be shared among then in the end of the period.

In Baumol (1952) and Tobin (1956), the agents have access to money only when they pay the financial costs to convert bonds into money. This case is obtained here with $a = 0$. Then, all sales are converted into bonds and the shopper can only use the sales proceeds to buy goods after a transfer from the brokerage account to the bank account. The introduction of a fraction $a > 0$ allows the shopper to use part of the sales proceeds immediately. To simplify, the transfers of the worker to the trader and to the shopper do not pay the financial costs. Only the transfers between the brokerage account and the bank account pay the financial costs.

Agent s decides consumption $c(t, s)$, the times to make transfers $T_j(s)$, $j = 1, 2, ...,$ money and bond holdings in the bank and brokerage account, $M(t, s)$, $B(t, s)$, and the transfers of money between the two accounts, $z(t, s)$. The worker, the trader, and the shopper are together represented as agent s. Let $T_0(s) \equiv 0$. T_0 is not a decision variable. If an agent decides to make the first transfer at $t = 0$, then $T_1(s) = 0$. A holding period is given by (T_j, T_{j+1}).

Let $r(t)$ denote the nominal interest rate at time t. If there is not a transfer at t, bond holdings in the brokerage account evolve as

$$\dot{B}(t, s) = r(t) B(t) + (1 - a) P(t) Y(t), \, t \geq 0, \, t \neq T_1(s), T_2(s), ..., \quad (1)$$

where $\dot{x}$ is the derivative of x with respect to time. If there are no transfers, the agent simply accumulates the interest rate and the income from sales. Bond holdings in the brokerage account increase.

Let $B^- (T_j (s), s)$ represent bond holdings just before a transfer at $t = T_j$ and $B^+ (T_j (s), s)$ represent bond holdings just after the transfer, $B^- (T_j (s), s) \equiv \lim_{t \to T_j},$ $_{t < T_j} B(t, s)$ and $B^+ (T_j (s), s) \equiv \lim_{t \to T_j, t > T_j} B(t, s)$. At $t = T_j$, the constraint on the brokerage account is

$$z (T_j (s), s) + P (T_j (s)) \Gamma = B^- (T_j (s), s) - B^+ (T_j (s), s), \, t = T_1 (s), T_2 (s), ... \quad (2)$$

If there is a positive transfer to the bank account, $z (T_j (s), s) > 0$, then $B^- (T_j (s), s)$ $> B^+ (T_j (s), s)$. In this case, bond holdings in the brokerage account decrease just after the transfer.

Money holdings in the bank account follow

$$\dot{M} (t, s) = -P (t) c (t, s) + aP (t) Y (t), \, t \geq 0, \, t \neq T_1 (s), T_2 (s), ..., \quad (3)$$

during a holding period. If there are no transfers, money holdings decrease with goods purchases and increase with the income transfers from sales. The shopper can use the money transferred from the worker to buy goods in the same period. Analogously to the definitions for bond holdings, let $M^+ (T_j (s), s)$ and $M^- (T_j (s), s)$ denote money holdings just after a transfer and money holdings just before a transfer. We have $z (T_j (s), s) = M^+ (T_j (s), s) - M^- (T_j (s), s)$. If the transfer is positive then $M^+ (T_j (s), s) - M^- (T_j (s), s) > 0$. When there is a transfer,

$$\dot{M} (T_j (s), s)^+ = -P (T_j) c^+ (T_j (s), s) + aP (t) Y (t), \, t \geq 0, \, t = T_1 (s), T_2 (s), ...,$$
$$(4)$$

where $\dot{M} (T_j (s), s)^+$ is the right derivative of $M (t, s)$ with respect to time at $t = T_j (s)$

and $c^+ (T_j (s), s)$ is consumption just after the transfer. Notice that the government does not distribute money directly to agents with, for example, lump-sum transfers. Only those agents in the asset market, trading bonds for money, have access to the transfers of money from the government.

The agents make transfers so that $M^+ (T_j (s), s)$ covers the purchases during (T_j, T_{j+1}) and any balance $M^- (T_{j+1} (s), s)$, that is, using (3),

$$M^+ (T_j (s), s) = \int_{T_j(s)}^{T_{j+1}(s)} P(t) c(t, s) \, dt - \int_{T_j(s)}^{T_{j+1}(s)} a P(t) Y(t) \, dt + M^- (T_{j+1} (s), s), \tag{5}$$

$j = 1, 2, ...$ Agent $s = (M_0, B_0)$ starts with M_0 in money holdings at $t = 0$. The agent has to use M_0 until the first transfer, at $T_1 (s)$. For the first holding period $(0, T_1 (s))$, we have

$$M_0 = \int_0^{T_1(s)} P(t) c(t, s) \, dt - \int_0^{T_1(s)} a P(t) Y(t) \, dt + M^- (T_1 (s), s). \tag{6}$$

It can be the case that the agent chooses to make the first transfer at $t = 0$. For example, if $a = 0$ and $M_0 = 0$. In this case, $T_1 (s) = 0$, and $M^- (0, s) = 0$.

Let $Q(t)$ denote the price at time zero of a bond that pays one dollar at time t. Given the nominal interest rate $r(t)$, $Q(t) = e^{-R(t)}$, where $R(t) = \int_0^t r(\tau) \, d\tau$. Using (1), $Q(T_{j+1}) B^- (T_{j+1}) = Q(T_j) B^+ (T_j) + \int_{T_j}^{T_{j+1}} Q(t) (1 - a) P Y(t) \, dt$. Substituting for the different holding periods, together with the condition $\lim_{t \to \infty} Q(t) B(t) = 0$, we obtain the constraint on the brokerage account in present value,

$$\sum_{j=1}^{\infty} Q(T_j (s)) [z(T_j (s), s) + P(T_j (s)) \Gamma] \leq \int_0^{\infty} Q(t) (1 - a) P(t) Y(t) + B_0, \tag{7}$$

where $z(T_j (s), s) = M^+ (T_j (s), s) - M^- (T_j (s), s)$.

The problem of agent s is then to obtain $c(t, s)$, $M(t, s)$, and $T_j(s)$ to solve

$$\max \sum_{j=0}^{\infty} \int_{T_j(s)}^{T_{j+1}(s)} e^{-\rho t} u(c(t, s)) \, dt \tag{8}$$

subject to (3)-(7) and to $T_{j+1}(s) \geq T_j(s)$ and $M(t, s) \geq 0$. ρ is the intertemporal rate of discount. The utility function is $u(c) = \frac{c^{1-1/\eta}}{1-1/\eta}$, $\eta \neq 1$, $\eta > 0$; and $u(c) = \log c$, $\eta = 1$. The transfer cost does not enter in the utility function. η is the elasticity of intertemporal substitution.

It is never optimal to set $M^-(T_{j+1}) > 0$, $j \geq 1$. $M^-(T_{j+1}) > 0$ means that the agent maintained money holdings in the bank account during the whole holding period (T_j, T_{j+1}) without receiving interest. The agent is always better off reducing the amount transferred at T_j, $M^+(T_{j+1})$, until $M^-(T_{j+1}) = 0$. As agents cannot change M_0, it can still be the case that $M^-(T_1) > 0$. For the other holding periods, $M^-(T_{j+1}) = 0$. Therefore, using (3), the demand for money at t of agent s is given by $M(t, s) = \int_t^{T_{j+1}(s)} [P(t) c(t, s) - a P(t) Y(t)] dt$, $T_j(s) \leq t < T_{j+1}(s)$, $j \geq 1$.

The transfer cost rules out an equilibrium with a representative agent. In a standard cash-in-advance model, agents have access to bonds and to their income in the end of every period. Here, agents have access to their bonds and to their income deposited in bonds only when they sell bonds for money. At every moment, some agents sell part of their bonds for money and make a transfer while others accumulate bonds and keep using money in the bank account until the next transfer.

To make the budget constraints linear in income, let $\Gamma = \gamma Y(t)$. As preferences are homothetic, this implies that optimal consumption and that the demand for money are linear in income. A demand for money linear in income, that is, income elasticity equal to one, agrees with the empirical evidence as discussed, among others, by Meltzer (1963), Lucas (2000).

Let B_0^G denote the supply of government bonds at $t = 0$. Consider first a situation

with no taxes and no government consumption. In this case, all seigniorage collected by the government is redistributed to agents as initial bonds. The government budget constraint is $B_0^G = \int_0^\infty Q(t) P(t) \frac{\dot{M}(t)}{P(t)} dt$, where $M(t)$ is the aggregate money supply. Higher money growth implies higher B_0^G and more bonds distributed across agents.

The market clearing conditions for money and bonds are $M(t) = \int M(t,s) dF(s)$ and $B_0^G = \int B_0(s) dF(s)$. The market clearing condition for goods takes into account the goods used to pay the transfer cost. Let $A(t,\delta) \equiv \{s : T_j(s) \in [t, t+\delta]\}$ denote the set of agents that make a transfer during $[t, t+\delta]$. The number of goods during $[t, t+\delta]$ to pay the transfer cost is then given by $\int_{A(t,\delta)} \frac{1}{\delta} \Gamma dF(s)$. Taking the limit to obtain the number of goods used at time t yields that the market clearing condition for goods is given by $\int c(t,s) dF(s) + \lim_{\delta \to 0} \int_{A(t,\delta)} \frac{1}{\delta} \Gamma dF(s) = Y$.

An equilibrium is defined as prices $P(t)$, $Q(t)$, allocations $c(t,s)$, $M(t,s)$, transfer times $T_j(s)$, $j = 1, 2, ...$, and a distribution of agents F such that (i) $c(t,s)$, $M(t,s)$, and $T_j(s)$ solve the maximization problem (8) given $P(t)$ and $Q(t)$ for all $t \geq 0$ and s in the support of F; (ii) the government budget constraint holds; and (iii) the market clearing conditions for money, bonds, and goods hold.

Solving the model

Focus on the steady state, an equilibrium in which the nominal interest rate is constant at r, the inflation rate is constant at π, and aggregate consumption grows at the same rate of output. Let output grow at the rate g, $Y(t) = Y_0 e^{gt}$, where $\rho > g(1 - 1/\eta)$. I look for an equilibrium in which all agents have the same consumption pattern within holding periods and the same interval between transfers N. The steady state is interpreted as the allocations and prices of an economy that has not been exposed to shocks for a long time.

Rewrite the maximization problem in terms of the consumption-income ratio $\hat{c}(t,n) \equiv c(t,n)/Y(t)$. $\hat{c}(t,n)$ decreases at a constant rate within holding periods, according to r and η. We can then write $\hat{c}(t,n)$ for the entire holding period as a function of its

value at the beginning of a holding period. With the exception of the short holding period from $t = 0$ to the first transfer, let the steady state be such that all agents begin a holding period with the same consumption-income ratio, $\hat{c}_0$.

At a certain time t, $\hat{c}(t, n)$ varies across agents because each agent is in a different position in the holding period. But all agents look the same within holding periods. They start with the same consumption-income ratio and it decreases at a common rate. As the maximization problem can be written in terms of the consumption-income ratio, having the same $\hat{c}_0$ implies that all agents choose the same interval between transfers N. Let n represent the time of the first transfer, $n \in [0, N)$. Therefore, an agent n makes transfers at n, $n + N$ and so on.

As aggregate consumption grows at the same rate of aggregate output, the same number of agents must be starting a new holding period at every time. Otherwise, aggregate consumption would vary over time. As a result, the distribution of agents is uniform along $[0, N)$, with density $1/N$.[1]

The first order condition with respect to consumption implies $c(t, n) = \frac{e^{-(\rho + \pi)\eta t}}{[P_0 Q(T_j)\lambda(n)]^\eta}$, $t \in (T_j, T_{j+1})$, $j = 1, 2, ...$, using $P(t) = P_0 e^{\pi t}$, and where $\lambda(n)$ is the Lagrange multiplier of (7). Set the nominal interest rate in the steady state at $r = \rho + g/\eta + \pi$. The first order condition then implies $\dot{c}(t, n)/c(t, n) = -\eta r + g$ and that $\hat{c}(t, n)$ decreases at the rate ηr.

If η, r or a are high, then agents would consume more in the beginning of holding periods by borrowing against their money receipts within the same holding period. They would consume less than aY in the end of a holding periods. A useful property of the model is that $c > aY$ for the empirically relevant range of η, r, and a. That is, for η between zero and five, r between zero to 16% per year, and $a \leq 0.6$. This is the empirically relevant range of η, r, and a because the usual estimates of η are

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[1]For a proof that the only distribution of agents compatible with a steady state in which agents have the same consumption pattern is the uniform distribution, see Grossman (1985). Grossman (1985, 1987) study the effects of monetary policy changes in a model with N fixed.

below five (Bansal and Yaron 2004 and Bansal 2006 discuss the evidence about η, Bansal and Yaron focus on $\eta = 1.5$); the annual interest rate for the U.S. is below 16% during 1900-1997, using commercial paper rate for r; and because money receipts are interpreted as labor income, implying $a \leq 0.6$ (Khan and Thomas 2010 and Alvarez et al. 2009 also interpret money receipts as labor income; I use the same value for a that they use, $a = 0.6$). We can, therefore, study the properties of the equilibrium without the constraint $c \geq aY$. This property facilitates the analysis and characterization of the equilibrium.

The value of $\hat{c}_0$ is obtained with the market clearing condition for goods. The market clearing condition for goods implies $\frac{1}{N} \int_0^N \hat{c}(t, n) \, dn + \frac{\gamma}{N} = 1$. Write the consumption-income ratio within holding periods as $\hat{c}(t, n) = \hat{c}_0 e^{-\eta r(t - T_j(n))}$, for the highest $j(n)$ such that $T_j(n) \leq t < T_{j+1}(n)$. The market clearing holds for every t. In particular, for $t = N$, $\frac{1}{N} \int_0^N \hat{c}_0 e^{-\eta r(N-n)} dn + \frac{\gamma}{N} = 1$, which implies $\hat{c}_0(N) = \left(1 - \frac{\gamma}{N}\right) \left(\frac{1 - e^{-\eta r N}}{\eta r N}\right)^{-1}$.

The effect of the transfer cost is apparent in the term γ/N. As we must take into account γ/N, the consumption-income ratio can be less than 1 during the entire holding period. With transfer cost in utility terms, γ/N disappears and $\hat{c}_0 > 1$. The effect of γ through the market clearing condition would not be considered. The expression of N, given in proposition 1 below, implies $N > \gamma$. So, $\hat{c}_0 > 0$.

The first order conditions for $T_j(n)$, $j = 2, 3, ...,$ imply

$$(r - \pi - g)\gamma + r \int_{T_j}^{T_{j+1}} e^{(\pi+g)(t-T_j)} \hat{c}(t, n) \, dt + \left[\hat{c}^+(T_j, n) - e^{rN} \hat{c}^-(T_j, n)\right]$$
$$= \frac{\hat{c}^+(T_j, n) - e^{rN} \hat{c}^-(T_j, n)}{1 - 1/\eta} + r \int_{T_j}^{T_{j+1}} a e^{(\pi+g)(t-T_j)} dt + a\left(1 - e^{rN}\right). \quad (9)$$

The left hand side and the right hand side are the marginal gain and loss of delaying T_j. The marginal gain is given first by postponing the transfer cost and second by

11

decreasing balances from T_j to T_{j+1}; the third term is the net effect of increasing $[T_{j-1}, T_j)$ and decreasing $[T_j, T_{j+1})$, this effect is zero when $\eta = 1$. The right hand side is given by the loss in utility caused by the increase in T_j, and by the net effect of the money receipts within the holding period. We obtain N with (9), $r = \rho + g/\eta + \pi$, and the expression of $\hat{c}(t, n)$.

Proposition 1 *The optimal interval between transfers N is the positive root of*

$$\hat{c}_0(N)\, rN \left[\frac{1 - e^{-r(\eta-1)N}}{r(\eta-1)N} - \frac{1 - e^{-[\rho - g(1-1/\eta) + r(\eta-1)]N}}{[\rho - g(1-1/\eta) + r(\eta-1)]N} \right] = [\rho - g(1-1/\eta)]\gamma$$
$$+ arN \left[\frac{e^{rN} - 1}{rN} - \frac{e^{[r - \rho + g(1-1/\eta)]N} - 1}{[r - \rho + g(1-1/\eta)]N} \right], \text{ for } \eta \neq 1, \text{ and } \quad (10)$$

$$\hat{c}_0(N)\, rN \left[1 - \frac{1 - e^{-\rho N}}{\rho N} \right] = \rho\gamma + arN \left[\frac{e^{rN} - 1}{rN} - \frac{e^{(r-\rho)N} - 1}{(r-\rho)N} \right], \text{ for } \eta = 1, \quad (11)$$

where $\rho > g(1 - 1/\eta)$ and $\hat{c}_0(N) = \left(1 - \frac{\gamma}{N}\right) \left(\frac{1 - e^{-\eta rN}}{\eta rN} \right)^{-1}$. N exists and is unique for all positive a that satisfies $\hat{c}_0 e^{-\eta rN} \geq a$ and all positive values of γ, η, ρ, g, and r.

With the value of N, we find all optimal trading periods $T_j(n)$, $n \in [0, N)$, as agents trade at n, $n + N$ and so on. With γ in utility terms, $\hat{c}_0$ disappears for $a = 0$ from (10) and (11). As discussed above, $\hat{c}(t) \geq a$ (and so $\hat{c}_0 e^{-\eta rN} \geq a$) for the empirically relevant cases.

The formulas were arranged to facilitate the identification of the terms $\frac{1 - e^{-x}}{x} \approx 1 - \frac{x}{2}$ and $\frac{e^x - 1}{x} \approx 1 + \frac{x}{2}$, where $x \geq 0$ in the formulas if $\eta \geq 1$, $\pi \geq 0$, and $g \geq 0$. In particular, $\rho - g(1 - 1/\eta) + r(\eta - 1) = r\eta - \pi - g$ and $r - \rho + g(1 - 1/\eta) = \pi + g$.

Proposition 2 *N is such that (i) $\frac{\partial N}{\partial r} < 0$ and (ii) $\frac{\partial N}{\partial \gamma} > 0$. Moreover, (iii) $\frac{\partial N}{\partial a} > 0$; (iv) $\frac{\partial N}{\partial \rho} > 0$; (v) $\frac{\partial N}{\partial \eta} > 0$ if $g = 0$; $\frac{\partial N}{\partial \eta} > 0$ if $g > 0$ for $\eta > \eta_g$, where η_g is given in the proof of the proposition; and (vi) $\frac{\partial N}{\partial g} < 0$ if $\eta > 1$ and $\frac{\partial N}{\partial g} > 0$ if $\eta < 1$.*

Proposition 2 shows that N decreases with r and increases with γ. In addition, N increases with a and ρ. When $g = 0$, N increases with η, and when $g > 0$, N increases with η if η is sufficiently high, and decreases with η if η is close to zero. The familiar substitution and income effects are present in the model. They have opposite signs and cancel each other when $\eta = 1$: N increases with g if $\eta < 1$, decreases if $\eta > 1$, and g disappears from the formula of N if $\eta = 1$.

The parameters that affect N the most are r, γ and a. To see this, make a second-order expansion of (10). The result is $N \approx \sqrt{2\gamma/\left(\hat{c}_0\left(N\right) - a\right)r}$. The Baumol-Tobin model assumes $a = 0$ and constant $c = Y$ (so $\hat{c}_0 = 1$). In this case, the square-root formula $\sqrt{2\gamma/r}$ approximates the optimal N. The square-root formula does not approximate N when $a > 0$. With $a = 0.6$, for example, and $\hat{c}_0\left(N\right) = 1$ (a good approximation for $\hat{c}_0$ when r is small such as 3%), we have $N \approx \sqrt{2\gamma/\left(0.4r\right)} = 1.6\sqrt{2\gamma/r}$, 60% higher than the square root approximation. Money receipts within holding periods increase the interval between transfers. For a given a, in any case, the interest elasticity of N is close to -0.5.

With the value of N, the output growth rate g, and the fact that agents consume at the rate $-\eta r + g$ within holding periods, we obtain $M_0\left(n\right)$ and $W_0\left(n\right)$ such that the economy is in the steady state from $t = 0$ and on. The growth rate g is used to write consumption just after a transfer. The consumption-income ratio $\hat{c}_0$ at the beginning of holding periods after $t = T_1\left(n\right)$ is the same for all agents in the steady state. The value of $\hat{c}\left(0, n\right)$ differs across agents because the holding period that initiates at $t = 0$ has different lengths, according to $n \in [0, N)$. We have $c^+\left(T_j\left(n\right), n\right) = \hat{c}_0 Y_0 e^{gT_j\left(n\right)}$: consumption at the beginning of holding periods grows at the rate g. Proposition 3 gives the values of $M_0\left(n\right)$. As we don't need $B_0\left(n\right)$ to discuss the demand for money, the values of $B_0\left(n\right)$ are in the proof of proposition 3, in the appendix.

Proposition 3 *The initial money holdings such that the economy is in a steady state equilibrium for $t \geq 0$ are given by* $M_0\left(n\right) = P_0 Y_0 n \big[e^{\eta r(n-N)} \hat{c}_0 \frac{1 - e^{-[\rho - g(1 - 1/\eta) + r(\eta - 1)]n}}{[\rho - g(1 - 1/\eta) + r(\eta - 1)]n} -$

$a\frac{e^{[r-\rho+g(1-1/\eta)]n}-1}{[r-\rho+g(1-1/\eta)]n}]$, $n \in [0, N)$, *where* $\hat{c}_0(r, N) = \left(1 - \frac{\gamma}{N}\right)\left(\frac{1-e^{-\eta rN}}{\eta rN}\right)^{-1}$ *and* N *is given by proposition (1).*

An agent with $M_0(n)$ makes transfers at $t = n$, $n + N$, and so on. As $\hat{c}_0 e^{-\eta rN} > a$, $M_0(n)$ increases with n. So, agents that make the first transfer later have more initial money holdings. Analogously, the initial value in the brokerage account $B_0(n)$ decreases with n. If an agent makes the first trade of bonds for money soon (n small), then $B_0(n)$ is high.

Although the distribution of agents along $[0, N)$ is uniform, with density $f(n) = \frac{1}{N}$, the distribution of individual money holdings is not uniform. As prices and output grow over time, individual money holdings also grow over time. So, consider the distribution of individual money-income ratios. The distribution of money-income ratios is constant over time.

The individual money-income ratio is given by $b(n) = \frac{M_0(n)}{P_0 Y_0}$. The individual money-income ratio is distributed along $[0, m_H)$, where $m_H = \lim_{n \to N} b(n)$. The density $f_b(m)$ of the individual money-income ratios is given by $f(b^{-1}(m))\frac{\partial b^{-1}(m)}{\partial m}$, where $f(n) = \frac{1}{N}$ and $b^{-1}(m)$ is the value of n such that $b(n) = m$ (as $b(n)$ is increasing, we always have one and a unique value of $b^{-1}(m)$). Therefore, $f_b(m) = \frac{1}{N}[\eta rm + e^{[r-\rho+g(1-1/\eta)]b^{-1}(m)}(\hat{c}_0 e^{-\eta rN} - a) + \eta ra\frac{e^{(r-\rho+g(1-1/\eta))b^{-1}(m)}-1}{[r-\rho+g(1-1/\eta)]}]^{-1}$, $m \in [0, m_H)$. The distribution of real money holdings is concentrated on small quantities of money, but it is close to a uniform. The distribution is more concentrated on small quantities of money if η increases.[2]

We obtain the aggregate demand for money with $M_0 = \frac{1}{N}\int_0^N M_0(n)\, dn$. The aggregate money-income ratio $m(r)$, the inverse of velocity, is obtained by dividing

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[2]Berentsen et al. (2004) and Santos (2006) obtain other distributions of individual demands for money.

M_0 by $P_0 Y_0$. The aggregate money-income ratio is then

$$
m\left(r\right) = \frac{\hat{c}_0\left(r, N\right) e^{-\eta r N}}{\rho - g\left(1 - 1/\eta\right) + r\left(\eta - 1\right)} \left[\frac{e^{\eta r N} - 1}{\eta r N} - \frac{e^{\left[r - \rho + g\left(1 - 1/\eta\right)\right]N} - 1}{\left[r - \rho + g\left(1 - 1/\eta\right)\right]N}\right]
$$
$$
- \frac{a}{r - \rho + g\left(1 - 1/\eta\right)} \left[\frac{e^{\left[r - \rho + g\left(1 - 1/\eta\right)\right]N} - 1}{\left[r - \rho + g\left(1 - 1/\eta\right)\right]N} - 1\right], \quad (12)
$$

where N is given by proposition 1 and $\hat{c}_0\left(r, N\right) = \left(1 - \frac{\gamma}{N}\right)\left(\frac{1 - e^{-\eta r N}}{\eta r N}\right)^{-1}$. The first term in the expression of $m\left(r\right)$ is always positive, as $\eta r N > \left[r - \rho + g(1 - 1/\eta)\right]N \Leftrightarrow \rho - g(1 - 1/\eta) + r(\eta - 1) > 0$. Notice that g does not affect the money-income ratio if $\eta = 1$. A fraction a of income received directly as money means, in practice, that the agents need to hold less money to buy goods. So, m decreases with a. The aggregate money-income ratio is a function of the interest rate r and also of preference parameters, financial technology, and output growth. I write $m\left(r\right)$ to emphasize the relation of the money-income ratio to the interest rate.

Figure 1 shows $m\left(r\right)$ and U.S. annual data. I use M1 for money and commercial paper rate for r, as Lucas (2000), Lagos and Wright (2005) and others (there are questions about the choice of the proxies for M and r, as pointed out by Teles and Zhou 2005, I use M1 and commercial paper rate to facilitate comparison with the literature). A second-order approximation of $m\left(r\right)$ yields $(e^{-\eta r N}\hat{c}_0 - a)\frac{N}{2}$. The interest elasticity of m is, therefore, close to the interest elasticity of N, -0.5. Lucas (2000) argues that an interest elasticity of -0.5 provides a good fit to the data. Many empirical studies, however, find smaller interest elasticities in absolute value, especially for the short-run. More recently, on the other hand, Alvarez and Lippi (2009) estimate interest elasticities close to -0.5. The semi-elasticity of m is -12.5, compatible with the findings of Lucas (1988), Stock and Watson (1993), and the long-run elasticities of Guerron-Quintana (2009).

To calibrate the model, I set the conventional values $\rho = 1\%$ p.a., $g = 2\%$ p.a. and
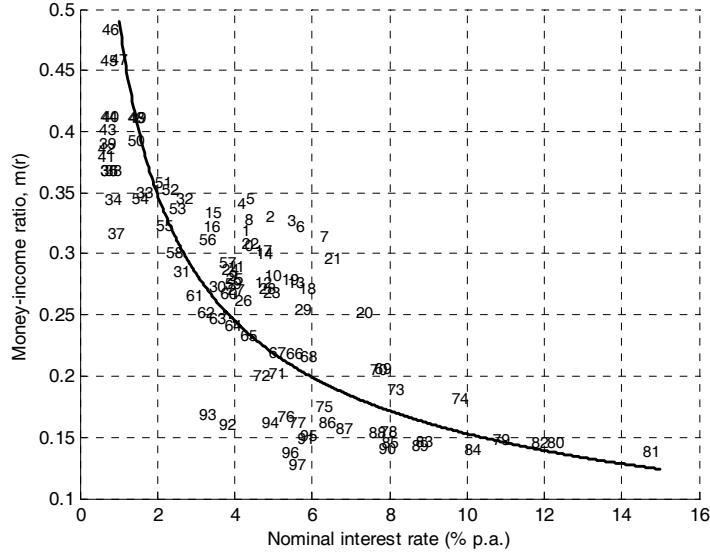
FIG. 1. $m(r)$ and U.S. data, 1900-1997 (M1/(PY) for the money-income ratio and commercial paper rate for the nominal interest rate, the data points indicate years).

$\eta = 1$ (log utility). I set $a = 0.6$, as Alvarez et al. (2009) and Khan and Thomas (2010), who interpret a as labor income. The only parameter left is γ, which I set to $\gamma = 1.265\%$. γ is set so that $m(r)$ passes through the historical average of the data, that is, $m(\bar{r})$ equals the historical money-income ratio when $\bar{r}$ is the historical interest rate, obtained with their geometric means. This is the same procedure of Lucas (2000). Similarly, Alvarez et al. (2009) and Khan and Thomas (2010) calibrate their models to fit the historical M2 velocity.

$\gamma = 1.265\%$ means that agents in the model spend about 22 minutes per week in financial transfers when inflation is equal to 1%. To get this value, notice that γ/N is the cost of financial transfers per year as a fraction of income. When $\pi = 1\%$, proposition 1 implies that $N = 1.27$. With the average weekly hours from 1957 to 1997 of U.S. workers, equal to 36.5 according to the OECD, the time devoted to financial transfers is then $\frac{\gamma}{N} \times 36.5 \times 60 = 22$ minutes per week.

16

The model implies a large interval between transfers, as common in market segmentation models (Edmond and Weill 2008). With $\pi = 1\%$, $g = 2\%$, and $a = 0$, the model implies N of about 6 months. N increases to 1.27 year when $a = 0.6$. Alvarez et al. (2009), for example, set the transfer interval from 1.5 to 3 years (larger intervals because they use M2). Financial transfers both here and in Alvarez et al. are transfers from high-yielding assets to currency, not ATM withdrawals, which change the allocations of checking deposits and currency, but do not change the quantity of money. Although large, the transfer intervals agree with the low trading frequency of households (Vissing-Jorgensen 2002, Alvarez et al. 2009) and the large cash holdings of firms (Bates et al. 2009). Notice that agents in the model represent households and firms, as 62% of M1 in the U.S. is held by firms (Bover and Watson 2005). Silva (forthcoming) discusses the calibration in more detail and compares $m(r)$ with $a = 0$ or 0.6, N fixed or optimal, and different η's.

I simplified the model to facilitate its application: the objective is to create a framework to study changes in monetary policy taking into account the frictions to manage money holdings and a nondegenerate distribution of money holdings. In particular, $m(r)$ is stable with constant γ and constant financial market participation. Following Reynard (2004) we can obtain a stable $m(r)$ with decreasing γ (financial innovation) together with increasing financial market participation. It simplifies, however, to have constant γ and financial market participation. As figure 1 shows, these assumptions imply a close match with the data. Another simplification is to impose that agents need money to buy goods through a cash-in-advance constraint instead of obtaining a demand for money from matching as in Kiyotaki and Wright (1989), Rocheteau and Wright (2005), and Lagos and Wright (2005). Moreover, Mulligan and Sala-i-Martin (2000) and Ireland (2009) point out that the demand for money changes with low interest rates. The model is intended to be used with moderate interest rates, in the range of the interest rates of figure 1, for which $m(r)$

follow the general pattern of the data.

With taxes and government consumption, the government budget constraint changes to

$$B_0 + \int_0^\infty Q(t) P(t) G dt = \int_0^\infty Q(t) \tau dt + \int_0^\infty Q(t) \dot{M}^S(t) dt, \qquad (13)$$

where G is government consumption and τ is a lump-sum tax. The total supply of government bonds is still given by $B_0 = \frac{1}{N} \int B_0(n) dn$. With lump-sum taxes and a as the fraction in money of gross income, each agent transfers to the brokerage account $\int_0^\infty Q(t) [P(t)(1-a)Y(t) - \tau] dt$ at each time. According to (13), if revenues from seigniorage are zero, for example, then $B_0 = \int_0^\infty Q(t) [\tau - P(t)G] dt$, which means that net government revenues are rebated to agents through government bonds.

Different ways of financing government consumption, therefore, affect the economy in different ways. A higher G financed with an increase in τ decreases consumption to satisfy the market clearing condition for goods, but does not change the frequency of trading bonds for money. According to (13), this is done by making the increase in τ equal to the increase in G so that $\dot{M}^S(t)$ does not change. As $\frac{\dot{M}}{M} = \pi + g$, inflation and the decision on N do not change.

On the other hand, an increase in G financed with seigniorage increases inflation. The change in inflation implies an additional decrease in consumption because the frequency of trading increases and so the resources devoted to financial transactions increase. In a model with fixed N, financing G with taxes or seigniorage would yield similar results. Seigniorage would still increase inflation, but the effect on consumption would be restricted to consumption smoothing within holding periods. Here, the increase in the frequency of trades further affects consumption.

3. Conclusions

This paper introduces a model to study how changes in monetary policy such as changes in the interest rate or in the money supply affect prices and the real demand for money. The distribution of money holdings, prices, interest rates, production and government actions are consistent in equilibrium. That is, they are consistent with market clearing conditions, budget constraints and individual maximization.

The model combines the Baumol-Tobin general equilibrium frameworks of Jovanovic (1982) and Romer (1986) with the market segmentation models of Grossman and Weiss (1983) and Rotemberg (1984). The result is a cash-in-advance model in which the length of the time period is optimal and money holdings are heterogeneous. Some applications of the model are to study how changes in the trading frequency affect the demand for money and the welfare cost of inflation. Taking into account the changes in the trading frequency, Silva (forthcoming) shows that the estimates of the welfare cost of inflation increase substantially. More generally, the model is useful to study how the adjustment of money holdings affects real variables.

Appendix - Proofs

I will use the following functions and definitions in propositions 1, 2 and 3: $f(x) = \frac{1-e^{-x}}{x}$, $\hat{\rho} \equiv \rho - g(1 - 1/\eta)$, $x_1 \equiv r(\eta - 1)N$, $x_2 \equiv x_1 + \hat{\rho}N$, $g(y) = \frac{e^y - 1}{y}$, $y_1 = rN$, $y_2 = y_1 - \hat{\rho}N$. $\hat{\rho} > 0$ by assumption to imply a bounded solution for the maximization problem, so $x_2 > x_1$ and $y_1 > y_2$. Moreover, g is increasing and convex and so $[g(y_1) - g(y_2)] > 0$ and $[g'(y_1) - g'(y_2)] > 0$, as $y_1 > y_2$. Similarly, f is decreasing and convex and so $[f(x_1) - f(x_2)] > 0$ and $[f'(x_1) - f'(x_2)] < 0$, as $x_1 < x_2$. Let G be defined by $G(N) = \hat{c}_0(N)rN[f(x_1) - f(x_2)] - \hat{\rho}\gamma - z(N)$, where $z(N) = arN[g(y_1) - g(y_2)]$. The optimal interval N^* is such that $G(N^*) = 0$.

Proposition 1. *Proof.* The first order conditions for T_j, $j = 2, 3$, of agent n are

$e^{-\rho T_j} u\left(c^-\left(T_j\right)\right) - e^{-\rho T_j} u\left(c^+\left(T_j\right)\right) - \lambda[\dot{Q}\left(T_j\right) \int_{T_j}^{T_{j+1}} P\left(t\right) c\left(t\right) dt + Q\left(T_j\right) P\left(T_j\right) c^+\left(T_j\right)$

$-Q\left(T_{j-1}\right) P\left(T_j\right) c^-\left(T_j\right) + \dot{Q}\left(T_j\right) \int_{T_j}^{T_{j+1}} aP\left(t\right) Y e^{gt} dt - Y e^{gT_j} P\left(T_j\right) a(Q\left(T_j\right) - Q\left(T_{j-1}\right))$

$-\gamma Y e^{gT_j}(P\left(T_j\right) \dot{Q}\left(T_j\right) + Q\left(T_j\right) \dot{P}\left(T_j\right)) - Y\gamma g e^{gT_j} P\left(T_j\right) Q\left(T_j\right)] = 0$. The first order

conditions for consumption yield $e^{-\rho T_j} c^-\left(T_j\right)^{-1/\eta} = \lambda Q\left(T_{j-1}\right) P\left(T_j\right)$ and $e^{-\rho T_j} \times$

$c^+\left(T_j\right)^{-1/\eta} = \lambda Q\left(T_j\right) P\left(T_j\right)$. In the steady state, $Q\left(t\right) = e^{-rt}$ and $P\left(t\right) = P_0 e^{\pi t}$,

$N_j = N$, and $c\left(t\right) = \hat{c}\left(t\right) Y_0 e^{gt}$. Substituting and simplifying yields $\gamma(r - \pi - g) +$

$\left[\hat{c}^+\left(T_j\right) - e^{rN}\hat{c}^-\left(T_j\right)\right] = \frac{\hat{c}^+\left(T_j\right) - e^{rN}\hat{c}^-\left(T_j\right)}{1 - 1/\eta} - r\int_{T_j}^{T_{j+1}} \frac{\hat{c}(t)e^{(g+\pi)}}{e^{(g+\pi)T_j}} dt + r\int_{T_j}^{T_{j+1}} \frac{ae^{(g+\pi)t}}{e^{(g+\pi)T_j}} dt + a(1 - $

$e^{rN})$. Moreover, $\hat{c}\left(t\right) = \hat{c}_0 e^{-\eta r(t - T_j)}$, $\hat{c}^+\left(T_j\right) = \hat{c}_0$, and $e^{rN}\hat{c}^-\left(T_j\right) = e^{-r(\eta-1)N}\hat{c}_0$.

Substituting yields $\hat{c}_0 \frac{1 - e^{-r(\eta-1)N}}{\eta - 1} - r\hat{c}_0 \int_{T_j}^{T_{j+1}} e^{(g+\pi-\eta r)(t - T_j)} dt = (r - \pi - g)\gamma + a(e^{rN} - $

$1) - ra \int_{T_j}^{T_{j+1}} e^{(g+\pi)(t - T_j)} dt$. Solving the integrals and rearranging yields (10). Note

that $r = \rho + g/\eta + \pi$. The steps for $\eta = 1$ are analogous.

For existence and uniqueness, the strategy is to show that G is increasing in N, with

$\lim_{N \to \gamma} G\left(N\right) < 0$ and $\lim_{N \to +\infty} G\left(N\right) > 0$. These three properties of G imply that

N^* exists and is unique. Moreover $N^* > \gamma$. We have $G\left(r, N\right) = \hat{c}_0\left(N\right) \left[\frac{1 - e^{-r(\eta-1)N}}{\eta - 1} - $

$r\frac{1 - e^{-[\hat{\rho} + r(\eta-1)]N}}{\hat{\rho} + r(\eta-1)}\right] - \hat{\rho}\gamma - a[e^{rN} - 1 - r\frac{e^{(r-\hat{\rho})N} - 1}{r - \hat{\rho}}]$ and so $G_N = \hat{c}_{0N} rN[f\left(x_1\right) - f\left(x_2\right)] + $

$re^{rN}(1 - e^{-\hat{\rho}N})[\hat{c}_0\left(N\right) e^{-\eta r N} - a]$. If $a = 0$ then, as $\hat{c}_{0N} \equiv \partial \hat{c}_0\left(N\right)/\partial N > 0$, we

have $G_N > 0$. If $a > 0$ then a sufficient condition for $G_N > 0$ is $\hat{c}_0\left(N\right) e^{-\eta r N} \geq a$.

That is, consumption in the end of a holding period is higher than or equal to the

money receipts. This condition is satisfied because of the constraint $c\left(t, n\right) \geq aY\left(t\right)$.

As discussed in the text, in any case, we always have $c\left(t, n\right) > aY\left(t\right)$, nonbinding,

for the empirically relevant parameters. For $\eta = 1$, analogously, $G_N > 0$. For the

limits, we have $\lim_{N \to +\gamma} G_N\left(N\right) \leq -\hat{\rho}\gamma$ for all $\eta > 0$, with equality if and only if

$a = 0$. So, $\lim_{N \to +\gamma} G_N\left(N\right) < 0$. Finally, $\lim_{N \to +\infty} G_N\left(N\right) > 0$ for all $\eta > 0$.

Therefore, $\lim_{N \to +\infty} G\left(N\right) = +\infty$. (Eventually, the constraint $\hat{c}\left(t\right) \geq a$ binds, as

$\lim_{N \to +\infty} \hat{c}_0\left(N\right) e^{-\eta r N} = 0$; we still have in this case that $\lim_{N \to +\infty} G\left(N\right) = +\infty$.)

Therefore, G crosses the zero and, as it is increasing, it crosses the zero only once.

The unique N^* is such that $G\left(N^*\right) = 0$.∎

Proposition 2. *Proof.* We have to obtain the sign of $\frac{\partial N}{\partial x} = -\frac{G_x(N^*)}{G_N(N^*)}$ to prove each property, where G is defined above and G_x denotes $\partial G/\partial x$. In proposition 1, we already proved that $G_N > 0$.

(i) $\frac{\partial N}{\partial r} < 0$. We have to show that $G_r(N^*) > 0$. Consider first the case $a = 0$. We have $G_r = \hat{c}_0 N\left[f(x_1) - f(x_2)\right] h(x_1)$, using $\hat{c}_{0r} = -\hat{c}_0 \frac{f'(r\eta N)}{f(r\eta N)}\eta N$, and where $h(x_1) = 1 - r\eta N \frac{f'(r\eta N)}{f(r\eta N)} + r(\eta - 1) N \frac{f'(x_2) - f'(x_1)}{f(x_2) - f(x_1)}$. If $\eta < 1$, all terms in the expression of $h(x_1)$ are positive and so $G_r > 0$ (recall that $f' < 0$, $f'(x_2) - f'(x_1) > 0$ and $f(x_2) - f(x_1) < 0$). The same reasoning applies for $\eta = 1$. For $\eta > 1$, notice that $x_1 \equiv r(\eta - 1) N > 0$ and write $h(x_1)$ as $h(x) = 1 - (x + rN)\frac{f'(x+rN)}{f(x+rN)} + x\frac{f'(x+\rho N) - f'(x)}{f(x+\rho N) - f(x)}$, $x > 0$. The function $x\frac{f'(x)}{f(x)}$ is decreasing (so $-(x + rN)\frac{f'(x+rN)}{f(x+rN)} > -x\frac{f'(x)}{f(x)}$) and we have $\frac{f'(x+\rho N) - f'(x)}{f(x+\rho N) - f(x)} > \frac{f''(x)}{f'(x)}$. Therefore, $h(x) > 1 - x\frac{f'(x)}{f(x)} + x\frac{f''(x)}{f'(x)}$, which is positive for all x. As a result, $G_r > 0$.

When $a > 0$, we have $G_r = \hat{c}_0 N\left[f(x_1) - f(x_2)\right] h(x_1) - z_r(N)$, where $z_r(N) > 0$. Substituting the definition of $z(N)$, we obtain that $G_r > 0$ if and only if $a < \frac{\hat{c}_0[f(x_1) - f(x_2)]h(x_1)}{[g(y_1) - g(y_2)] + rN[g'(y_1) - g'(y_2)]}$. In practice (for $a \leq 0.6$ and the standard values for η and r, for example), this condition is always satisfied and so $G_r > 0$.

(ii) $\frac{\partial N}{\partial \gamma} > 0$. $G_\gamma = -r\frac{f(x_1) - f(x_2)}{f(r\eta N)} - \hat{\rho} < 0$.

(iii) $\frac{\partial N}{\partial a} > 0$. $G_a = -rN[g(y_1) - g(y_2)] < 0$.

(iv) $\frac{\partial N}{\partial \rho} > 0$. It suffices to show that $G_{\hat{\rho}} < 0$, as $\hat{\rho} = \rho - g(1 - 1/\eta)$. We have $G_{\hat{\rho}} = -\hat{c}_0 rN^2 f'(x_2) - \gamma - arN^2 g'(y_2)$. For $a = 0$, $G_{\hat{\rho}}(N^*) < 0$ if $\hat{c}_0 rN^2 f'(x_2) > -\gamma$ at N^*, which is true because $-\gamma = \hat{c}_0 rN^2 \frac{f(x_2) - f(x_1)}{\hat{\rho}N}$ when $N = N^*$, and $f'(x_2) > \frac{f(x_2) - f(x_1)}{\hat{\rho}N}$, as f is convex. Therefore, $G_{\hat{\rho}} < 0$. When a increases, $G_{\hat{\rho}}$ decreases as the first order effect on $G_{\hat{\rho}}$ is $-rN^2 g'(y_2) < 0$ (the effect of N on $G_{\hat{\rho}}$, caused by the increase in a, is small compared with the first order effect of a). Therefore, $G_{\hat{\rho}}(N^*)$ is also negative for $a > 0$.

(v) $\frac{\partial N}{\partial \eta} > 0$. We have to prove that $G_\eta < 0$. For $g = 0$, $G_\eta = \hat{c}_0 (rN)^2 [f(x_1) - f(x_2)]h(x_1)$, using $\hat{c}_{0\eta} = -\hat{c}_0 \frac{f'(r\eta N)}{f(r\eta N)}rN$, and where $h(x_1) = \frac{f'(x_2) - f'(x_1)}{f(x_2) - f(x_1)} - \frac{f'(r\eta N)}{f(r\eta N)}$.

So $G_\eta < 0$ if $h(x_1) < 0$ (the condition is the same for $a \geq 0$). Write $h(x_1)$ as
$h(x) = \frac{f'(x+\rho N)-f'(x)}{f(x+\rho N)-f(x)} - \frac{f'(x+rN)}{f(x+rN)}$, $x > -rN$. We have $\frac{f'(x+\rho N)-f'(x)}{f(x+\rho N)-f(x)} < \frac{f''(x+\rho N)}{f'(x+\rho N)}$ and
$\frac{f''(x+\rho N)}{f'(x+\rho N)} < \frac{f'(x+\rho N)}{f(x+\rho N)}$. Therefore, $h(x) < \frac{f'(x+\rho N)}{f(x+\rho N)} - \frac{f'(x+rN)}{f(x+rN)}$. If $r > \rho$ then $\frac{f'(x+\rho N)}{f(x+\rho N)} < \frac{f'(x+rN)}{f(x+rN)}$ as $\frac{f'(x)}{f(x)}$ is increasing. So, $h(x) < 0$ and $G_\eta < 0$. If $r < \rho$, use the fact that
$\frac{f'(y)-f'(x)}{f(y)-f(x)}$ is increasing in y and so $\frac{f'(x+\rho N)-f'(x)}{f(x+\rho N)-f(x)} < \lim_{\rho N \to \infty} \frac{f'(x+\rho N)-f'(x)}{f(x+\rho N)-f(x)} = \frac{f'(x)}{f(x)}$.
Therefore, $h(x) < \frac{f'(x)}{f(x)} - \frac{f'(x+rN)}{f(x+rN)} < 0$ as $\frac{f'(x)}{f(x)}$ is increasing. So, $h(x) < 0$ and $G_\eta < 0$.
When $g > 0$, we have $G_\eta = \hat{c}_0 (rN)^2 h(x_1) + rN \frac{gN}{\eta^2} \hat{c}_0 [f'(x_2) - \frac{f(x_2)-f(x_1)}{\hat{\rho}N}]$ for $a = 0$
(the idea is similar for $a > 0$). The second term in the right in positive because
$f'(x_2) > \frac{f(x_2)-f(x_1)}{\hat{\rho}N}$. We have $G_\eta > 0$ for η close to zero when $g > 0$. Let $\eta_g > 0$ be
such that $G_\eta(\eta_g) = 0$. Therefore, $G_\eta < 0$ if and only if $\eta > \eta_g$.

(vi) $\frac{\partial N}{\partial g} < 0$ if $\eta > 1$ and $\frac{\partial N}{\partial g} > 0$ if $\eta < 1$. The only term in which g appears is
$\hat{\rho} = \rho - g(1-1/\eta)$, $\partial N/\partial \hat{\rho} > 0$. As $\partial \hat{\rho}/\partial g = -(\eta-1)/\eta$, $\partial N/\partial g$ has the same sign
of $\partial N/\partial \hat{\rho}$ if $\eta < 1$ and the opposite sign if $\eta > 1$ (g disappears from the formula of
N if $\eta = 1$).$\blacksquare$

Proposition 3. *Proof.* $M_0(n)$ is such that agent n consumes at the steady state
rate in the interval $[0, n)$, $M_0(n) = \int_0^n P(t) c(t, n) dt - \int_0^n aP(t) Y_0 e^{gt} dt$. Agent $n = 0$
consumes c_0 at time $t = 0$. Given that the consumption growth rate within N in the
steady state is equal to $-(\eta r - g)$, and that consumption just after a transfer grows
at the rate g, an arbitrary agent n consumes $c_0 e^{-\eta r(N-n)}$ at $t = 0$. That is, agent n
would consume $c_0 e^{-gN} e^{gn}$ at $t = n - N < 0$ and consumes $(c_0 e^{-gN} e^{gn}) e^{-(\eta r - g)(N-n)} =$
$c_0 e^{-\eta r(N-n)}$ at $t = 0$. Therefore, $c(t, n) = c_0 e^{-\eta r(N-n)} e^{-(\eta r - g)t}$, $0 \leq t < n$. More-
over, $P(t) = P_0 e^{\pi t}$ in the steady state. Substituting above and solving the inte-
grals yields the value of $M_0(n)$ in the body of the text. For W_0, let $W_0(n) \equiv$
$\int_0^\infty Q(t)(1-a)P(t)Y(t) + B_0(n)$. Then, $W_0(n)$ is such that $W_0(n) = \sum_{j=1}^\infty Q(T_j) \times$
$M^+(T_j(n)) + \sum_{j=1}^\infty Q(T_j) P(T_j) \gamma Y(t)$, equal to the present value of future trans-
fers plus transfer costs. The quantity of money needed in each holding period
$M^+(T_j(n))$ is given by $\int_{T_j(n)}^{T_{j+1}(n)} P(t) c(t, n) dt - \int_{T_j(n)}^{T_{j+1}(n)} aP(t) Y(t) dt$, with $c(t, n) =$

$c_0 e^{gT_j(n)} e^{-(\eta r-g)(t-T_j)}$ and $T_j(n) = n + (j-1)N$. Substituting and solving yields $\sum_{j=1}^{\infty} Q(T_j) M^+(T_j(n)) = \frac{P_0 Y_0 N e^{-(\rho-g(1-1/\eta))n}}{1-e^{-(\rho-g(1-1/\eta))N}}[\hat{c}_0 f(x_2) - ag(y_2)]$. Similarly, we have $\sum_{j=1}^{\infty} Q(T_j) P(T_j) \gamma Y(t) = \frac{P_0 Y_0 \gamma e^{-(\rho-g(1-1/\eta))n}}{1-e^{-(\rho-g(1-1/\eta))N}}$. Initial bond holdings $B_0(n)$ can then be obtained with the definition of $W_0(n)$. ∎

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