

All assignment games with the same core have the same nucleolus

Marina Núñez *

Departament de Matemàtica Econòmica, Financera i Actuarial, i CREB
Universitat de Barcelona, Av.Diagonal, 690, 08034 Barcelona, Spain

25th March 2004

*I am grateful to C. Rafels for his comments. Institutional support from research grants BEC 2002-00642 and SGR2001-0029 is also acknowledged

All assignment games with the same core have the same nucleolus

Abstract: There exist coalitional games with transferable utility which have the same core but different nucleoli. We show that this cannot happen in the case of assignment games. Whenever two assignment games have the same core, their nucleoli also coincide. To show this, we prove that the nucleolus of an assignment game coincides with that of its buyer-seller exact representative.

Key words: assignment game, core, kernel, nucleolus

JEL: C71, C78

Resum: Existeixen jocs cooperatius d'utilitat transferible que tot i tenir el mateix core tenen diferent nucleolus. En aquest treball es mostra que això no pot passar amb els jocs d'assignació, és a dir que, en aquests jocs, el nucleolus ve determinat pel core del joc i per tant dos jocs d'assignació amb el mateix core tenen forçosament el mateix nucleolus. Per provar-ho mostrem que el nucleolus d'un joc d'assignació coincideix amb el de l'únic joc que el representa amb la propietat de ser "buyer-seller" exacte.

1 Introduction

In a bilateral assignment market a product that comes in indivisible units is exchanged for money, and each participant either supplies or demands exactly one unit. The units need not be alike and the same unit may have different values for different participants. From these valuations, a matrix can be defined whose entries give the profit that can be obtained by each buyer-seller pair if they trade. Assuming that side payments are allowed, Shapley and Shubik (1972) define the assignment game as a cooperative model for this bilateral market and prove the nonemptiness of its core.

There may exist different assignment matrices which determine assignment games with the same core. In fact, given a matrix A there exists a unique matrix A^r such that (i) the core of the corresponding assignment games coincides and (ii) A^r is maximal in the sense that no matrix entry can be raised without modifying the core of the game (Núñez and Rafels, 2002b).

Several game-theoretic solution concepts for the assignment game have been considered. Among them we should mention the τ -value. The τ -value of a coalitional game was introduced by Tijs (1981) as a compromise between a utopia vector and a minimal rights vector. For the assignment game, the τ -value selects the midpoint between the buyers-optimal core allocation and the sellers-optimal core allocation (Núñez and Rafels, 2002a).

Since the τ -value of an assignment game depends only on its core, the assignment games related to A and to A^r have the same τ -value. The aim of this paper is to prove that this property also holds for two other well-known solution concepts: the kernel and the nucleolus.

2 The assignment model

Let $M = \{1, 2, \dots, m\}$ be a set of buyers and $M' = \{1', 2', \dots, m'\}$ a set of sellers, where we denote the j -th seller by j' to distinguish it from the j -th buyer. Let $A = (a_{ij'})_{(i,j') \in M \times M'}$ be a nonnegative matrix where $a_{ij'}$ represents the profit obtained by the mixed-pair (i, j') if they trade. Let $n = m + m'$ denote the cardinality of $M \cup M'$. An **assignment problem** is a triple (M, M', A) . The goal is to find an optimal matching between the two sides of the market. A **matching** for A is a subset μ of $M \times M'$ such that each $k \in M \cup M'$ belongs at most to one pair in μ . We denote the set of matchings of A by $\mathcal{M}(A)$ or $\mathcal{M}(M, M')$. We say a **matching μ is optimal** if for all $\mu' \in \mathcal{M}(M, M')$, $\sum_{(i,j') \in \mu} a_{ij'} \geq \sum_{(i,j') \in \mu'} a_{ij'}$. We denote the set of optimal matchings by $\mathcal{M}^*(A)$.

A **transferable utility coalitional game** (TU coalitional game) is a

pair (N, v) , where the set $N = \{1, 2, \dots, n\}$ is its finite player set and $v : 2^N \rightarrow \mathbf{R}$ its characteristic function satisfying $v(\emptyset) = 0$. A **payoff vector** will be $x \in \mathbf{R}^n$ and, for every coalition $S \subseteq N$ we write $x(S) := \sum_{i \in S} x_i$ to denote the payoff to coalition S (where $x(\emptyset) = 0$). An **imputation** is a payoff vector that is efficient, $x(N) = v(N)$, and individually rational, which means each player $i \in N$ receives at least the individual worth $v(i)$. We denote the set of imputations by $I(N, v)$. The **core of (N, v)** is defined by $C(N, v) = \{x \in \mathbf{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$. The core is a bounded convex polyhedron. When no confusion can arise regarding the player set, we simply denote the set of imputations by $I(v)$ and the core by $C(v)$.

Assignment games were introduced by Shapley and Shubik (1972) to model two-sided markets with transferable utility. Given an assignment problem (M, M', A) , the player set is $M \cup M'$, and the matrix A determines the characteristic function w_A as follows: given $S \subseteq M$ and $T \subseteq M'$, let $\mathcal{M}(S, T)$ be the set of matchings between S and T and let $w_A(S \cup T) = \max\{\sum_{(i, j') \in \mu} a_{ij'} \mid \mu \in \mathcal{M}(S, T)\}$. We assume as usual that a coalition consisting only of sellers or only of buyers has worth zero. A buyer $i \in M$ is not assigned by μ if $(i, j') \notin \mu$ for all $j' \in M'$ (and similarly for sellers).

Shapley and Shubik proved that the core of an assignment game $(M \cup M', w_A)$ is nonempty, and can be represented in terms of any optimal matching μ of $M \cup M'$ by

$$C(w_A) = \left\{ (u, v) \in \mathbf{R}^{M \times M'} \left| \begin{array}{l} u_i \geq 0, \text{ for all } i \in M; v_{j'} \geq 0, \text{ for all } j' \in M' \\ u_i + v_{j'} = a_{ij'} \text{ if } (i, j') \in \mu \\ u_i + v_{j'} \geq a_{ij'} \text{ if } (i, j') \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_{j'} = 0 \text{ if } j' \text{ not assigned by } \mu. \end{array} \right. \right\} \quad (1)$$

Moreover, the core has a lattice structure with two special extreme points: the buyers-optimal core allocation, $(\bar{u}^A, \underline{v}^A)$, where each buyer attains his maximum core payoff, and the sellers-optimal core allocation, $(\underline{u}^A, \bar{v}^A)$, where each seller does.

In Núñez and Rafels (2002b), an assignment game $(M \cup M', w_A)$ is said to be **buyer-seller exact** if for all $i \in M$ and all $j' \in M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_{j'} = a_{ij'}$. Note that when an assignment game is buyer-seller exact, no matrix entry can be raised without changing the core of the game.

As an example, all 3×3 assignment games defined by matrix $\begin{pmatrix} 1 & \alpha & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, with $0 \leq \alpha \leq 1$, have the same core, which is the line segment with endpoints

$(1, 1, 1; 0, 0, 0)$ and $(0, 0, 0; 1, 1, 1)$. But only the one with $\alpha = 1$ is buyer–seller exact. More examples will be found in the paper cited above.

It is proved in that paper that for all $(M \cup M', w_A)$ there exists a unique buyer–seller exact assignment game $(M \cup M', w_{A^r})$ such that $C(w_A) = C(w_{A^r})$, and that both games have at least one optimal matching in common. We then say that w_{A^r} is the **buyer-seller exact representative of w_A** . Then, two assignment games with the same core have the same buyer–seller exact representative. If A is square, this representative matrix A^r can be computed from A in the following way: $a_{ij'}^r = \max\{a_{ij'}, \tilde{a}_{ij'}\}$, where, for all $(i, j') \in M \times M'$,

$$\tilde{a}_{ij'} = \max_{\substack{k_1, k_2, \dots, k_r \in M \setminus \{i, j\} \\ \text{different}}} \{a_{ik'_1} + a_{k_1 k'_2} + \dots + a_{k_r j'} - (a_{k_1 k'_1} + \dots + a_{k_r k'_r})\}. \quad (2)$$

Since w_A and w_{A^r} have the same core, it follows easily that $\tau(w_A) = \tau(w_{A^r})$. The case of another solution concept, the kernel, is similar.

The kernel $\mathcal{K}(N, v)$ of a TU coalitional game (N, v) is a set-solution concept introduced by Davis and Maschler (1965), and we just write $\mathcal{K}(v)$ if no confusion can arise regarding the player set. In the case of a **zero-monotonic game** ($v(S) \geq v(T) + \sum_{i \in S \setminus T} v(i)$, for all $T \subseteq S$), as it is the case of assignment games, the **kernel** is given by

$$\mathcal{K}(v) = \{z \in \mathbf{R}^N \mid \sum_{k \in N} z_k = v(N) \text{ and } s_{ij}^v(z) = s_{ji}^v(z), \text{ for all } i, j \in N, i \neq j, \}$$

where the maximum surplus $s_{ij}^v(z)$ of player i over another player j with respect to the allocation $z \in \mathbf{R}^N$ in the game (N, v) is defined by

$$s_{ij}^v(z) = \max\{v(S) - \sum_{k \in S} z_k \mid S \subseteq N, i \in S, j \notin S\}.$$

Then, the kernel $\mathcal{K}(v)$ can be understood as the set of all efficient allocations for which all pair of players are in equilibrium.

Proposition 1 *Let $(M \cup M', w_A)$ be an assignment game and $(M \cup M', w_{A^r})$ its buyer–seller exact representative. Then $\mathcal{K}(w_A) = \mathcal{K}(w_{A^r})$.*

PROOF: Given two assignment games with the same core, the intersections of the core and the kernel also coincide (Maschler, Peleg and Shapley, 1979). Thus, $\mathcal{K}(w_A) \cap C(w_A) = \mathcal{K}(w_{A^r}) \cap C(w_{A^r})$. But, since the kernel of an assignment game is always included in the core (Driessen, 1998), the above equality is equivalent to $\mathcal{K}(w_A) = \mathcal{K}(w_{A^r})$. \square

The nucleolus is a single-valued solution for TU coalitional games which is always contained in the kernel, and also in the core whenever the core is nonempty. In the next section we show that all assignment games with the same core have the same nucleolus, since they have the same nucleolus as their buyer–seller exact representative.

3 The nucleolus of the buyer–seller exact representative \mathcal{W}_A^r

The nucleolus is a single-valued solution concept for TU coalitional games. Let us recall the definition, which is due to Schmeidler (1969). For all imputation x of (N, v) , and all coalition $S \subseteq N$, the **excess of coalition S with respect to x** is $e(S, x) = v(S) - x(S)$. Now, for all imputation x , let us define the **vector $\theta(x) \in \mathbf{R}^{2^n - 2}$ of excesses** of all non trivial coalitions at x , arranged in a nonincreasing order. That is to say, for all $k \in \{1, \dots, 2^n - 2\}$, $\theta(x)_k = e(S_k, x)$, where $\{S_1, \dots, S_k, \dots, S_{2^n - 2}\}$ is the set of all nonempty coalitions in N different from N , and $e(S_k, x) \geq e(S_{k+1}, x)$.

Then the **nucleolus** of the game (N, v) is the imputation $\nu(N, v)$ (we just write $\nu(v)$ when no confusion regarding the player set can arise) which minimizes $\theta(x)$ with respect to the lexicographic order over the set of imputations: $\theta(\nu(v)) \leq_{Lex} \theta(x)$ for all $x \in I(v)$. It is easy to see that, whenever the core of a game is nonempty, the nucleolus belongs to it.

One may think that the shape of the core determines the location of the nucleolus. In Maschler, Peleg and Shapley (1979, p. 335) an example is given of two games that have the same core but different nucleoli. This shows that, in the general framework of arbitrary TU coalitional games, the coincidence of the cores of two games does not imply the coincidence of their nucleoli.

An alternative definition of the nucleolus for an arbitrary TU coalitional game was given by Maschler, Peleg and Shapley (1979) as an iterative process that constructs the set of payoffs that lexicographically minimize the vector of ordered excesses. They prove that this set of minimizers is actually a single point, called the **lexicographic center of the game**, which coincides with the nucleolus. Solymosi and Raghavan (1994) present a definition of lexicographic center specialized for assignment games, based on the fact (already pointed out by Huberman, 1980) that for assignment games, only one-player coalitions and mixed-pair coalitions play a role in the computation of the nucleolus.

The definition of lexicographic center of an assignment game we present here is slightly different from that of Solymosi and Raghavan (1994). We use instead that given by Maschler, Peleg and Shapley (1979), while Solymosi and Raghavan replace excess by satisfaction and would therefore interchange min and max in the following definitions. Moreover, we define the initial feasible set X^0 to be the core of the game, while Solymosi and Raghavan begin with a particular subset of imputations that contains all core allocations. However, all steps in the proof of Solymosi and Raghavan (1994) can be followed to prove that our definition of lexicographic center of an assignment game also

consists of only one point and coincides with the nucleolus.¹

Let $(M \cup M', w_A)$ be an assignment game and μ a fixed optimal matching for A . Let us consider the set of one-player coalitions and mixed-pair coalitions, that is $\mathcal{P} = \{\{k\} \mid k \in M \cup M'\} \cup \{\{i, j'\} \mid i \in M, j' \in M'\}$. We iteratively construct a sequence $(\Delta^0, \Sigma^0), \dots, (\Delta^{s+1}, \Sigma^{s+1})$ of partitions of \mathcal{P} , with $\Sigma^0 \supseteq \Sigma^1 \supseteq \dots \supseteq \Sigma^{s+1}$, and a sequence $X^0 \supseteq X^1 \supseteq \dots \supseteq X^{s+1}$ of sets of payoff vectors such that:

Initially $\Delta^0 = \{\{i, j'\} \mid (i, j') \in \mu\} \cup \{\{k\} \mid k \in M \cup M' \text{ not matched by } \mu\}$; $\Sigma^0 = \mathcal{P} \setminus \Delta^0$, and $X^0 = C(w_A) = \{(u, v) \in \mathbf{R}_+^{m+m'} \mid e(S, (u, v)) = 0 \text{ for all } S \in \Delta^0, e(S, (u, v)) \leq 0 \text{ for all } S \in \Sigma^0\}$.

For $r \in \{0, 1, \dots, s\}$ define recursively

1. $\alpha^{r+1} = \min_{(u, v) \in X^r} \max_{S \in \Sigma^r} e(S, (u, v))$,
2. $X^{r+1} = \{(u, v) \in X^r \mid \max_{S \in \Sigma^r} e(S, (u, v)) = \alpha^{r+1}\}$,
3. $\Sigma_{r+1} = \{S \in \Sigma^r \mid e(S, (u, v)) \text{ is constant on } X^{r+1}\}$,
4. $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1}$, $\Delta^{r+1} = \Delta^r \cup \Sigma_{r+1}$,

where s is the last index for which $\Sigma^r \neq \emptyset$. The set X^{s+1} is called the **lexicographic center** of $(M \cup M', w_A)$.

Let us now compare the lexicographic center of an assignment game with that of its buyer-seller exact representative. We show that they coincide.

From now on, by adding dummy players on one side of the market (that is to say, null rows or columns in the assignment matrix) we will assume $m = m'$. Note that this does not affect the computation of the nucleolus. If k is one of these added dummy players, then $x_k = 0$ for all $x \in C(w_A)$ which implies $e^A(\{k\}, x) = 0$ for all $x \in C(w_A)$, and this excess does not allow to discriminate between core allocations. Moreover, if $S = \{i, j\}$ is a mixed-pair coalition containing one of these added players, then coalition S is inessential, which means that its worth is at most the sum of the worths of the coalitions of a non-trivial partition of S (in our case $w_A(\{i, j\}) = w_A(\{i\}) + w_A(\{j\})$) and then, from Huberman (1980), S needs not be considered in any computation of the nucleolus.

Theorem 2 *Let $(M \cup M', w_A)$ be an assignment game and $(M \cup M', w_{A^r})$ its buyer-seller exact representative. Then,*

$$\nu(w_A) = \nu(w_{A^r}).$$

¹The complete proof will be provided by the author upon request.

PROOF: Let us fix the same optimal matching μ in both games. Let us assume, without loss of generality, that buyers and sellers have been ordered in such a way that $\mu = \{(i, i') \mid i \in M\}$. We denote by $(\Delta^0, \Sigma^0), \dots, (\Delta^{s+1}, \Sigma^{s+1})$ and X^0, \dots, X^{s+1} the partitions and sets of payoffs which define the lexicographic center of $(M \cup M', w_A)$, and by $(\tilde{\Delta}^0, \tilde{\Sigma}^0), \dots, (\tilde{\Delta}^{s'+1}, \tilde{\Sigma}^{s'+1})$ and $\tilde{X}^0, \dots, \tilde{X}^{s'+1}$ the partitions and sets of payoffs which define the lexicographic center of $(M \cup M', w_{A^r})$. We prove that for all $0 \leq r \leq s+1$, $\Delta^r = \tilde{\Delta}^r$, $\Sigma^r = \tilde{\Sigma}^r$ and $X^r = \tilde{X}^r$ and consequently $\nu(w_A) = \nu(w_{A^r})$. We also write $e^A(S, x) = w_A(S) - x(S)$ and $e^{A^r}(S, x) = w_{A^r}(S) - x(S)$, for all $S \subseteq M \cup M'$ and all $x \in I(w_A) = I(w_{A^r})$.

The proof is by induction on r . By definition, $\Delta^0 = \tilde{\Delta}^0$, $\Sigma^0 = \tilde{\Sigma}^0$ and $X^0 = \tilde{X}^0$. Assume now that, for some $r \geq 0$, $\Delta^r = \tilde{\Delta}^r$, $\Sigma^r = \tilde{\Sigma}^r$ and $X^r = \tilde{X}^r$ and let us show that these equalities hold at step $r+1$. To see this, we prove that, for all $(u, v) \in X^r$,

$$\max_{S \in \Sigma^r} e^A(S, (u, v)) = \max_{S \in \Sigma^r} e^{A^r}(S, (u, v)) = \max_{S \in \tilde{\Sigma}^r} e^A(S, (u, v)),$$

where the last equality follows from the induction assumptions. We then take $(u, v) \in X^r = \tilde{X}^r$ and consider two different cases depending on whether $\max_{S \in \Sigma^r} e^A(S, (u, v))$ is attained at a mixed-pair coalition or at a one-player coalition.

Case 1: $\max_{S \in \Sigma^r} e^A(S, (u, v)) = e^A(\{i, j'\}, (u, v))$ for some $i \in M$ and $j' \in M'$, $\{i, j'\} \in \Sigma^r$.

This means that $e^A(\{i, j'\}, (u, v)) = a_{ij'} - u_i - v_{j'} \geq a_{i_1 j'_1} - u_{i_1} - v_{j'_1}$, for all $\{i_1, j'_1\} \in \Sigma^r$, $a_{ij'} - u_i - v_{j'} \geq -u_k$, for all $k \in M$, $\{k\} \in \Sigma^r$, and $a_{ij'} - u_i - v_{j'} \geq -v_{k'}$ for all $k' \in M'$, $\{k'\} \in \Sigma^r$.

Let us see that $a_{ij'}^r = a_{ij'}$. Assume on the contrary that $a_{ij'}^r > a_{ij'}$. Then, from (2), there exist $k_1, \dots, k_l \in M \setminus \{i, j\}$ and different such that $a_{ij'}^r = a_{ik'_1} + a_{k_1 k'_2} + \dots + a_{k_{l-1} k'_l} + a_{k_l j'} - a_{k_1 k'_1} - a_{k_2 k'_2} - \dots - a_{k_l k'_l}$.

When computing the excesses of coalition $\{i, j'\}$ in both games we have $e^{A^r}(\{i, j'\}, (u, v)) = a_{ij'}^r - u_i - v_{j'} > a_{ij'} - u_i - v_{j'} = e^A(\{i, j'\}, (u, v))$ and, by (2),

$$\begin{aligned} e^{A^r}(\{i, j'\}, (u, v)) &= a_{ik'_1} + a_{k_1 k'_2} + \dots + a_{k_l j'} - a_{k_1 k'_1} - a_{k_2 k'_2} - \dots - a_{k_l k'_l} - u_i - v_{j'} \\ &= a_{ik'_1} + a_{k_1 k'_2} + \dots + a_{k_l j'} - a_{k_1 k'_1} - a_{k_2 k'_2} - \dots - a_{k_l k'_l} - u_i - v_{j'} \\ &\quad - \sum_{t=1}^l u_{k_t} + \sum_{t=1}^l u_{k_t} - \sum_{t=1}^l v_{k'_t} + \sum_{t=1}^l v_{k'_t} \\ &= (a_{ik'_1} - u_i - v_{k'_1}) + (a_{k_1 k'_2} - u_{k_1} - v_{k'_2}) + \dots + (a_{k_l j'} - u_{k_l} - v_{j'}) \end{aligned}$$

since for all $p \in \{1, \dots, l\}$, $a_{k_p k'_p} - u_{k_p} - v_{k'_p} = 0$ as $(u, v) \in X^r \subseteq C(w_A)$.

We claim that at least one of the coalitions $\{i, k'_1\}, \{k_1, k'_2\}, \dots, \{k_l, j'\}$ belongs to Σ^r . Otherwise, each one of these coalitions would belong to Σ_t

for some $t \in \{1, \dots, r\}$ and would be constant on X^t . Then, as $X^r \subseteq X^{r-1} \subseteq \dots \subseteq X^0$, the excess of all the above coalitions would be constant on X^r . Since the equation

$$e^{A^r}(\{i, j'\}, (u, v)) = (a_{ik'_1} - u_i - v_{k'_1}) + (a_{k_1k'_2} - u_{k_1} - v_{k'_2}) + \dots + (a_{k_lj'} - u_{k_l} - v_{j'}) \quad (3)$$

holds for an arbitrary $(u, v) \in X^r$, this would imply that the excess of $\{i, j'\}$ is also constant on X^r , in contradiction with $\{i, j'\} \in \Sigma^r$.

Let us assume without loss of generality that $\{k_{p-1}, k'_p\} \in \Sigma^r$, for some $p \in \{1, \dots, l\}$ (the cases where $\{i, k'_1\} \in \Sigma^r$ or $\{k_l, j'\} \in \Sigma^r$ are analogous). Then, since all summands in (3) are nonpositive, we obtain $a_{k_{p-1}k'_p} - u_{k_{p-1}} - v_{k'_p} \geq a_{ij'}^r - u_i - v_{j'} > a_{ij'} - u_i - v_{j'}$ which contradicts the assumption of Case 1. Then $a_{ij'}^r = a_{ij'}$ and thus $e^A(\{i, j'\}, (u, v)) = e^{A^r}(\{i, j'\}, (u, v))$.

Moreover, for all $S \in \Sigma^r$, $S \neq \{i, j'\}$, we either have $S = \{k\}$ for some $k \in M \cup M'$, and then $e^{A^r}(S, (u, v)) = e^A(S, (u, v)) \leq e^A(\{i, j'\}, (u, v)) = e^{A^r}(\{i, j'\}, (u, v))$, or $S = \{i_1, j'_1\}$ for some $i_1 \in M$ and $j'_1 \in M'$. In this case, if $a_{i_1j'_1}^r = a_{i_1j'_1}$ we are done. Otherwise, by (2) and the same argument as above, there exist distinct $k_1, \dots, k_l \in M \setminus \{i_1, j_1\}$ such that

$$\begin{aligned} e^{A^r}(S, (u, v)) &= a_{i_1j'_1}^r - u_{i_1} - v_{j'_1} \\ &= (a_{i_1k'_1} - u_{i_1} - v_{k'_1}) + (a_{k_1k'_2} - u_{k_1} - v_{k'_2}) + \dots + (a_{k_lj'_1} - u_{k_l} - v_{j'_1}). \end{aligned}$$

Since $\{i_1, j'_1\} \in \Sigma^r$, at least one of these summands corresponds to a coalition in Σ^r . Let $\{k_{p-1}, k'_p\}$ be one such coalition. Then, $e^{A^r}(S, (u, v)) \leq a_{k_{p-1}k'_p} - u_{k_{p-1}} - v_{k'_p} \leq a_{ij'} - u_i - v_{j'} = e^{A^r}(\{i, j'\}, (u, v))$, where the second inequality follows from the assumption of Case 1.

This proves that if $\max_{S \in \Sigma^r} e^A(S, (u, v))$ is attained at a mixed-pair coalition $\{i, j'\}$, then

$$\max_{S \in \Sigma^r} e^A(S, (u, v)) = \max_{S \in \Sigma^r} e^{A^r}(S, (u, v)).$$

Case 2: $\max_{S \in \Sigma^r} e^A(S, (u, v)) = e^A(\{k\}, (u, v))$, for some $k \in M$, $\{k\} \in \Sigma^r$. The proof for the case where the maximum is attained at a coalition $\{k'\} \in \Sigma^r$ with $k' \in M'$ is analogous and left to the reader.

The assumption of Case 2 implies that $e^A(\{k\}, (u, v)) = -u_k \geq a_{ij'} - u_i - v_{j'}$ for all $\{i, j'\} \in \Sigma^r$, $-u_k \geq -u_l$ for all $l \in M$ with $\{l\} \in \Sigma^r$, and $-u_k \geq -v_{l'}$ for all $l' \in M'$ with $\{l'\} \in \Sigma^r$.

Note that $e^{A^r}(\{k\}, (u, v)) = -u_k = e^A(\{k\}, (u, v))$. Take $S \in \Sigma^r$, $S \neq \{k\}$. We either have $S = \{l\}$ for some $l \in M$, or $S = \{l'\}$ for some $l' \in M'$, or $S = \{i, j'\}$ with $i \in M$ and $j' \in M'$. If $S = \{l\}$ for some $l \in M$, then $e^{A^r}(\{l\}, (u, v)) = -u_l \leq -u_k = e^{A^r}(\{k\}, (u, v))$. If $S = \{l'\}$ for some

$l' \in M'$ we similarly obtain $e^{A^r}(\{l'\}, (u, v)) = -v_{l'} \leq -v_k = e^{A^r}(\{k\}, (u, v))$. If finally $S = \{i, j'\}$ with $i \in M$ and $j' \in M'$, and if it holds $a_{ij'}^r = a_{ij'}$, we are done. Otherwise, that is to say if $a_{ij'} < a_{ij'}^r$, from (2) and the same argument as in Case 1, there exist distinct $k_1, \dots, k_l \in M \setminus \{i, j\}$ such that

$$\begin{aligned} e^{A^r}(S, (u, v)) &= a_{ij'}^r - u_i - v_{j'} \\ &= (a_{ik'_1} - u_i - v_{k'_1}) + (a_{k_1k'_2} - u_{k_1} - v_{k'_2}) + \dots + (a_{k_lj'} - u_{k_l} - v_{j'}). \end{aligned}$$

Again, the same argument as in Case 1 leads to

$$e^{A^r}(S, (u, v)) = a_{ij'}^r - u_i - v_{j'} \leq -u_k = e^{A^r}(\{k\}, (u, v)).$$

Thus, also when $\max_{S \in \Sigma^r} e^A(S, (u, v))$ is attained at a one-player coalition $\{k\}$ we obtain $\max_{S \in \Sigma^r} e^A(S, (u, v)) = \max_{S \in \Sigma^r} e^{A^r}(S, (u, v))$.

Once proven that for all $(u, v) \in X^r$,

$$\max_{S \in \Sigma^r} e^A(S, (u, v)) = \max_{S \in \Sigma^r} e^{A^r}(S, (u, v)),$$

and taking into account that, from the induction hypothesis, $X^r = \tilde{X}^r$ and $\Sigma^r = \tilde{\Sigma}^r$, it follows that

$$\alpha^{r+1} = \min_{(u,v) \in X^r} \max_{S \in \Sigma^r} e^A(S, (u, v)) = \min_{(u,v) \in \tilde{X}^r} \max_{S \in \tilde{\Sigma}^r} e^{A^r}(S, (u, v)) = \tilde{\alpha}^{r+1}.$$

Consequently

$$\begin{aligned} X^{r+1} &= \{(u, v) \in X^r \mid \max_{S \in \Sigma^r} e^A(S, (u, v)) = \alpha^{r+1}\} = \\ &= \{(u, v) \in \tilde{X}^r \mid \max_{S \in \tilde{\Sigma}^r} e^{A^r}(S, (u, v)) = \tilde{\alpha}^{r+1}\} = \tilde{X}^{r+1}. \end{aligned}$$

If $S = \{k\}$ or $S = \{k'\}$, $e^{A^r}(S, (u, v)) = e^A(S, (u, v))$, while for $S = \{i, j'\}$, $e^{A^r}(S, (u, v)) = e^A(S, (u, v)) + a_{ij'}^r - a_{ij'}$. Then, $e^{A^r}(S, (u, v))$ is constant on X^{r+1} if and only if $e^A(S, (u, v))$ is constant on X^{r+1} . This means $\Sigma_{r+1} = \tilde{\Sigma}_{r+1}$, which implies $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1} = \tilde{\Sigma}^r \setminus \tilde{\Sigma}_{r+1} = \tilde{\Sigma}^{r+1}$ and $\Delta^{r+1} = \Delta^r \cup \Sigma_{r+1} = \tilde{\Delta}^r \cup \tilde{\Sigma}_{r+1} = \tilde{\Delta}^{r+1}$. \square

Corollary 3 *Let $(M \cup M', w_A)$ and $(M \cup M', w_B)$ be two assignment games with the same core. Then, $\nu(w_A) = \nu(w_B)$.*

PROOF: Since $C(w_A) = C(w_B)$, we have $A^r = B^r$ and, by the above theorem, $\nu(w_A) = \nu(w_{A^r}) = \nu(w_{B^r}) = \nu(w_B)$. \square

Note that the above result also holds for assignment games with non-square matrices. Since both games $(M \cup M', w_A)$ and $(M \cup M', w_B)$ have the same player set, the same null rows or columns are necessary in both matrices to make them square. Let \tilde{A} and \tilde{B} be such square matrices. The assignment

games $w_{\tilde{A}}$ and $w_{\tilde{B}}$ also have the same core, which consists in completing all core allocations of $C(w_A) = C(w_B)$ with zero payoffs to the added dummy agents. Then, both \tilde{A} and \tilde{B} have the same buyer-seller exact representative and, by Theorem 2, the same nucleolus. Taking into account the remark that precedes Theorem 2, if we drop the (null) payoffs to the added dummy agents we obtain the nucleolus of both $(M \cup M', w_A)$ and $(M \cup M', w_B)$.

This paper completes a series of studies on the major core-based solution concepts in the assignment game. As a concluding application we consider an assignment game given in Shapley and Shubik (1972). Let $M = \{1, 2, 3\}$ be the set of buyers, $M' = \{1', 2', 3'\}$ the set of sellers, and the assignment matrix be

$$A = \begin{pmatrix} 5 & 8 & 2 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}.$$

As shown in Shapley and Shubik (1972), the core of this game is the convex hull of the points $(3, 5, 0; 2, 5, 1)$, $(3, 6, 0; 2, 5, 0)$, $(4, 6, 1; 1, 4, 0)$, $(5, 6, 1; 1, 3, 0)$, $(5, 6, 0; 2, 3, 0)$, and $(4, 5, 0; 2, 4, 1)$. Among them we can distinguish $(\bar{u}^A, \bar{v}^A) = (5, 6, 1; 1, 3, 0)$ and $(\underline{u}^A, \underline{v}^A) = (3, 5, 0; 2, 5, 1)$. Note that for all $(u, v) \in C(w_A)$, $u_1 + v_{3'} \geq 3 > 2 = a_{13'}$, while each one of the remaining matrix entries is attained in some extreme core allocation. Then, to obtain the buyer-seller exact representative of this matrix we only have to raise $a_{13'}$ from 2 to 3:

$$A^r = \begin{pmatrix} 5 & 8 & 3 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}.$$

In fact, all matrices

$$A(\alpha) = \begin{pmatrix} 5 & 8 & \alpha \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}$$

with $0 \leq \alpha \leq 3$ define assignment games $(M \cup M', w_{A(\alpha)})$ that have the same core as $(M \cup M', w_A)$, $C(w_{A(\alpha)}) = C(w_A)$.

All games in this family have the same τ -value,

$$\tau(w_{A(\alpha)}) = (4, 5.5, 0.5; 1.5, 4, 0.5), \quad \text{for all } \alpha \in [0, 3],$$

which is the midpoint between the buyers-optimal and the sellers-optimal core allocations.

To compute the kernel of an assignment game, and since it is known that in these games the kernel is included in the core, only the maximal surplus s_{ij} , and s_{ji} , of the pairs $(i, j) \in M \times M'$ belonging to all $\mu \in \mathcal{M}^*(A)$ must be

taken into account (see Rochford, 1984, p.275). Moreover, in an assignment game, for all $x \in C(w_A)$, the maximal surplus $s_{ij}(x)$ is attained either at a one-player coalition or at a mixed-pair coalition.

In our example, since we have a complete description of the core, it is not difficult to check that, for all $x \in C(w_A)$,

$$\begin{aligned} s_{12'}(x) &= 5 - x_1 - x_{1'}, \\ s_{2'1}(x) &= 9 - x_2 - x_{2'}, \\ s_{23'}(x) &= \begin{cases} 7 - x_2 - x_{1'} & \text{if } x_{2'} - x_{1'} \geq 2, \\ 9 - x_2 - x_{2'} & \text{if } x_{2'} - x_{1'} \leq 2, \end{cases} \\ s_{3'2}(x) &= -x_{3'}, \\ s_{31'}(x) &= -x_3, \\ s_{1'3}(x) &= \begin{cases} 5 - x_1 - x_{1'} & \text{if } x_2 - x_1 \geq 2, \\ 7 - x_2 - x_{1'} & \text{if } x_2 - x_1 \leq 2. \end{cases} \end{aligned}$$

Then,

$$\mathcal{K}(w_A) = \{x \in C(w_A) \mid s_{12'}(x) = s_{2'1}(x), s_{23'}(x) = s_{3'2}(x), s_{31'}(x) = s_{1'3}(x)\}. \quad (4)$$

Different cases must be considered, depending on the worth of $x_{2'} - x_{1'}$ and $x_2 - x_1$, but only the point $(4, \frac{17}{3}, \frac{1}{3}; \frac{5}{3}, 4, \frac{1}{3})$ meets the constraints in (4). Note that the excess of the mixed-pair coalition $\{1, 3'\}$ is not relevant for the computation of the kernel of this game, nor for the computation of $\mathcal{K}(w_{A(\alpha)})$, provided $\alpha \in [0, 3]$. Then, as proved in Proposition 1, all the assignment games $(M \cup M', w_{A(\alpha)})$ have the same kernel:

$$\mathcal{K}(w_{A(\alpha)}) = \left\{ \left(4, \frac{17}{3}, \frac{1}{3}; \frac{5}{3}, 4, \frac{1}{3} \right) \right\}, \text{ for all } \alpha \in [0, 3].$$

Since the nucleolus is always contained in the kernel, this point is also the nucleolus of all these games,

$$\nu(w_{A(\alpha)}) = \left(4, \frac{17}{3}, \frac{1}{3}; \frac{5}{3}, 4, \frac{1}{3} \right), \text{ for all } \alpha \in [0, 3],$$

which can also be obtained as the result of the iterative process that defines the lexicographic center of any of the assignment games $(M \cup M', w_{A(\alpha)})$.

References

- [1] Davis M, Maschler M (1965) The kernel of a coperative game, Naval Research Logistics Quarterly 12, 223–259.

- [2] Driessen TSH (1998) A note on the inclusion of the kernel in the core of the bilateral assignment game, *International Journal of Game Theory* 27, 301–303.
- [3] Huberman G (1980) The nucleolus and the essential coalitions, in *Analysis and Optimization of Systems, Lecture Notes in Control and Information Science* 28, 417–422.
- [4] Maschler M, Peleg B, Shapley S (1979) Geometric properties of the kernel, nucleolus, and related solution concepts, *Mathematics of Operations Research* 4, 303–338.
- [5] Núñez M, Rafels C (2002a) The assignment game: the τ -value, *International Journal of Game Theory* 31, 411–422.
- [6] Núñez M, Rafels C (2002b) Buyer–seller exactness in the assignment game, *International Journal of Game Theory* 31, 423–436.
- [7] Rochford SC (1984) Symmetrically Pairwise-Bargained Allocations in an Assignment Market, *Journal of Economic Theory*, 34, 262–281.
- [8] Schmeidler D (1969) The nucleolus of a characteristic function game, *SIAM Journal of Applied Mathematics* 17, 1163–1170
- [9] Shapley LS, Shubik M (1972) The Assignment Game I: The Core, *International Journal of Game Theory* 1, 111–130.
- [10] Solymosi T, Raghavan TES (1994) An algorithm for finding the nucleolus of assignment games, *International Journal of Game Theory* 23, 119–143.
- [11] Tijs SH (1981) Bounds for the core and the τ -value. In: Moeschlin O, Pallaschke D (eds.) *Game Theory and Mathematical Economics*. North-Holland, Amsterdam, 123–132.