Subscription as a price discrimination device

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Introduction

There are several goods and services for which sellers offer to the consumers the possibility of either buying one unit at a time, or buying a subscription giving them the right of consuming these units repeatedly over the year, on a regular (like a newspaper) or irregular basis (like entry tickets to a swimming pool). Other examples would include magazines, theater tickets, entry tickets to recreation parks, airplane or train tickets. On the other hand, there are goods and services for which this opportunity does not arise either because consumers can buy them only one unit at a time, or, on the contrary, only by subscription. Most of foods, – like bread, meat, fish, – fall in the first category; in the second, one finds, for instance, scientific journals, which can be bought only by subscribing for at least one year or access to TV networks.

The practice of sorting consumers into different classes, those who buy a good or a service once at a time, and those who prefer to subscribe, raises the question of why this practice applies to some goods, and not to others, which are as well repeatedly consumed over time. Certainly there are several reasons why it is so. In this paper we shall concentrate on the possibility offered by this practice to increase the profitability of sales via *price discrimination.* When a product is offered simultaneously for purchase once at a time and by subscription, it is as if the seller would supply two different goods, which are of course substitutes. Undoubtedly, it allows the seller to

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discriminate among the buyers, by selling the units of the good at a different price, according to whether the buyer chooses a one-unit-purchase, or buys by subscription. However, it forces the seller "to compete with himself" by offering two substitute products. It is unclear whether the advantage to be gained from price discrimination compensates the loss in surplus resulting from selling simultaneously two products which are close substitutes[1]. Accordingly, it may be that, for some types of goods, the price discrimination effect is advantageous while it is not for others. To get a precise answer to this question, we need to identify the optimal pricing policy for the firm, and compare the profits accruing in both situations: price discrimination and one-unit sales. To this end, we propose a model representing a newspaper producer supplying a product which can be acquired by the readers either on a single purchase basis or by subscription. The population of potential buyers is differentiated according to the frequency at which they want to read the newspaper. First we identify the optimal pricing policy, namely, the optimal price for issues sold at the newsstand, as well as the optimal subscription fee. Second, we study the conditions under which it is more profitable to supply the market with both possibilities, rather than to concentrate on either of these alternatives. In particular, we show that a pure price discrimination argument is sufficient to explain why, in the press industry, the practice of subscription is generally observed.

To the best of our knowledge, we know of only one paper, namely, Glazer and Hassin (1982), which studies the economics of subscription sales in a manner related to the approach proposed here. By contrast with our approach, however, the population of consumers is differentiated in Glazer and Hassin according to the average value each consumer attach to a particular issue of the newspaper. Unfortunately, these authors do not identify the optimal policy for the monopolist. In particular, they do not *prove* that this optimal policy always consists in selling both subscriptions and individual issues, as it is done in the present paper.

The practice of subscription shares some characteristics with other contexts in which price discrimination is observed, like commodity bundling (Adams and Yellen (1976), McAfee *et al.* (1989)) or quality pricing (Mussa and Rosen (1979)). First, subscription can be viewed as a kind of mixed commodity bundling since it consists in selling "en bloc" a given number of units of the same good, which can alternatively be bought separately. In fact, it is a case of mixed bundling where the seller is constrained to set the single purchase prices the same for all products.

Moreover, it seems at first sight that all consumers would prefer to buy their newspaper via subscription rather than separately at the newsstand if the subscription fee divided by the number of issues would be equal to the single purchase price of the newspaper. Indeed, subscription allows consumers to avoid the moving costs inherent to arranging each purchase separately at the newsstand: The higher the moving cost, the more attrac-

[1] An example where this compensation does not occur is provided by Gabszewicz (1983).

tive the subscrption's opportunity. Does it mean that a newspaper obtained via subscription and a newspaper bought at the stand are two goods which are vertically differentiated ? No, simply because subscription deprives the consumer from his freedom to adjust his purchase frequency to the best of his interest : the disadvantage to subscription is that a consumer must purchase more units than he desires. Accordingly, the utility of a newspaper cannot be dissociated from the frequency at which a consumer is willing to read it over the year. The lower the desired frequency, the less attractive the opportunity of subscription. In conclusion, the subscription practice cannot be viewed as a particular case of quality pricing.

In Section 2 we analyse the optimal pricing policy for the monopolist while in Section 3 we characterize the optimal product-mix. Non zero production costs are introduced in Section 4. Finally, in the conclusion, we discuss the assumptions introduced in Section 2, and propose some possible extensions.

1 The optimal pricing policy

We consider a monopolist selling to a population of potential buyers a newspaper (or a weekly or monthly magazine) which they can buy either at the newsstand at unit price p, or by subscription covering the whole period of reference, typically one year, at a total price r. In this section and the next one, we shall assume zero production costs.

Each consumer selecting the option of buying the newspaper on free sale is characterised by a *reading frequency* expressing the number of times over the year this consumer reads the newspaper divided by the number of issues, say n, produced by the monopolist over the period. We assume that the reading frequency of each consumer is given ex-ante and does not depend on the prices p and r. It can either correspond to the desire of the consumer of not reading a proportion of issues which exceeds his reading frequency. This would be the case for instance if this consumer has developed reading habits which prevent him to devote a share of his free time to reading newspapers which exceeds a given a priori amount. Another possible interpretation is that, due to external constraints, it is not possible for the consumer to buy a proportion of issues in excess of his reading frequency. This would be the case, for instance, if this consumer is abroad during a significant period of the year, and cannot buy the newspaper in the foreign country. We also suppose the distribution of the consumers' reading frequencies to be known by the monopolist. To express the disparities in the reading frequency across the population while keeping the analysis tractable, we assume that the members of this population are uniformly distributed over an interval $[m, v]$, $[m, v]$ included in $[0, 1]$, an element t in $[m, v]$ representing a particular reading frequency of a consumer buying the newspaper on free sale. A consumer of type, $t \in [m, v]$, has a (cumulative) utility of reading newspapers which increases linearly with the number of issues he reads up to the point where the desired reading frequency nt is reached, and remains equal to nt from now on. His utility for reading each particular issue is constant and equal to w, $w > 0$. Furthermore, we assume that consumers buying the newspaper at the newsstand incur at each purchase a *moving cost* equal to c, $c > 0$. We assume that $w > c$. We shall also assume that

$$\frac{w}{w+c} > \frac{m}{v} \tag{1}$$

This condition bears on the domain of reading frequencies which are present in the population. It is of course satisfied when $m = 0$. Given the above assumptions, the utility of a type t-consumer obtained from buying at a unit price p a number ν of issues at the newsstand can be written as equal to $\nu(w - c - p)$ when $\nu \leqslant nt$, and to $nt(w - c - p)$ when $\nu > nt$. Without loss of generality, we assume the number of potential readers of type t be equal to one.

As an alternative to free sale purchase, individuals are offered the option of taking a *subscription* according to which they receive all the issues of the newspaper in their mailing box. In this case, they do not incur any moving cost and have only to pay the subscription fee r. Then a type t-consumer obtains a cumulative utility equal to $ntw - r$. Accordingly, this consumer will prefer to buy from the newsstand rather than taking a subscription if, and only if $nt(w - c - p) \geqslant ntw - r$, or

$$(p + c)tn \leqslant r$$

Denote by $t(p, r)$ the consumer who is just indifferent between the two options when the monopolist quotes the price p at the newsstand and the fee r for the subscription. We obtain

$$t(p, r) = \frac{r}{(p + c)n} \tag{2}$$

at the pair of prices (p, r), all consumers with reading frequency smaller than $t(p, r)$ buy from the newsstand while those with reading frequency higher than $t(p, r)$ prefer to buy a subscription. Accordingly, the receipt $R(p, r)$ of the monopolist is given by

$$R(p, r) = p \int_{m}^{\frac{r}{(p+c)n}} ntdt + r \int_{\frac{r}{(p+c)n}}^{v} dt$$

or

$$R(p, r) = p\left[\frac{1}{2n}\frac{r^2}{(p + c)^2} - \frac{1}{2}nm^2\right] + r\left[v - \frac{r}{[(p + c)n]}\right] \tag{3}$$

Differentiating (3) with respect to the subscription fee r, we obtain

$$\frac{\partial R}{\partial r} = \frac{p}{n}\frac{r}{(p+c)^2} + \left[v - 2\frac{r}{(p+c)n}\right]$$

The first term of the above derivative $\frac{\partial R}{\partial r}$ expresses the increase in receipts obtained from sales at the newsstand resulting from the increase in the subscription fee r; the second term (between brackets) represents the decrease in receipts originating in the transfer of subscribers who now prefer to buy at the stand due to the increase in r. The optimal subscription fee r^* exactly balances these two effects, and can be easily identified by letting $\frac{\partial R}{\partial r} = 0$, namely

$$r^* = \frac{vn(p+c)^2}{p+2c}$$

Furthermore, we notice that $\frac{\partial^2 R}{\partial r^2}$ is negative, so that r^* is a maximizer of $R(p,r)$ for all values of p. Substituting the value of r^* into $t(p,r)$ as given by (2), we obtain

$$t(p,r^*) = v\frac{(p+c)}{(p+2c)}$$

Now consider the function $R(p,r^*)$ obtained by substituting r^* to r in (3), i.e.

$$R(p,r^*) = \frac{nv^2(p+c)^2}{2(p+2c)} - \frac{nm^2p}{2} \tag{4}$$

It is shown in the appendix (Lemma 1) that this function reaches its maximum in the interval $(0, w-c)$ at the upper boundary $w-c$. Thus, the optimal price p^* of an issue sold at the newsstand obtains as $p^* = w-c$, the reservation price of the consumer minus the moving cost. The corresponding profit $R(p^*,r^*)$ obtains as :

$$R(p^*,r^*) = \frac{n(m^2c^2 + v^2w^2 - m^2w^2)}{2(w+c)}$$

Accordingly, we have the following:

Proposition 1 *When the monopolist decides to sell the newspaper both at the newsstand and by subscription, there exists a unique optimal pricing policy for the monopolist, which is given by*

$$p^* = w - c; \quad r^* = \frac{vnw^2}{w+c} \tag{5}$$

First we notice that both optimal prices, p^* and r^*, increase with w and decrease with c; furthermore, the subscription fee increases with the number of issues, while, divided by the number of issues, it remains constant.

Second, the pricing policy identified in Proposition 1 is meaningful only if some consistency conditions are satisfied. The first condition required is that the marginal consumer who is indifferent between the alternative of buying at the newsstand and taking a subscription, i.e., $t(p^*, r^*)$, has to lie in the interval $[m, v]$. Substituting the optimal price $p^* = w - c$ in the expression for $t(p, r^*)$, we obtain

$$t(p^*, r^*) = \frac{vw}{w + c}, \tag{6}$$

so that the inequality $t(p^*, r^*) \leqslant v$ is always satisfied. We need also $t(p^*, r^*) \geqslant m$, or

$$\frac{w}{w + c} \geqslant \frac{m}{v},$$

as assumed in (1).

Finally, we must check that the receipt of the monopolist is positive when he uses his optimal policy. This follows from the fact that both prices are positive, as well as market shares, when $t(p^*, r^*)$ lies in the interval $[m, v]$. We have just seen that this is the case when condition (1) holds.

It is interesting to examine the behaviour of the market solution as a function of the moving cost c. First, it must be noticed that when c is equal to 0, the model is degenerate. Indeed, performing the above analysis with $c = 0$ leads to an optimal value r^* equal to vnw, so that $t(p, r^*)$ is then equal to v for all values of p. Accordingly, with zero moving costs, the subscription fee is always chosen in a way such as the demand for subscription vanishes. A contrario, when the moving cost is strictly positive, it is worthy for the monopolist to offer to consumers with higher reading frequencies the possibility of avoiding the repetition of moving costs by taking a subscription. Thus, we observe that introducing the possibility of subscription when mailing and production costs are zero, entails a welfare improvement since all consumers buying a subscription avoid thereby the moving costs they would otherwise incur. As we shall see in the next section, this property is no longer necessarily true when production costs are positive[2]. Surprisingly enough, we found that, contrary to what intuition would predict, the equilibrium subscription price actually falls as the moving cost rises ($\frac{dr^*}{dc} < 0$). This is so because a higher moving cost requires a lower single-issue price or else single-issue sales would drop to zero. But the average subscription price must be lower than the single issue price, so it too must fall. However the simple fact that the moving cost is strictly positive does not guarantee that it is always advantageous to offer subscriptions. Indeed, the optimal policy derived above was analysed under the presumption that both alternatives, free sale purchase and subscription, are offered to the consumers. But the

[2] It would be interesting to calculate the optimal pricing policy when mailing costs for sending subscription are explicitly taken into account. These costs would then countervail the benefits accruing to consumers due to the disappearance of their moving costs when they buy a subscription.

monopolist has always the possibility of either supplying the market with free sale purchase only, or supplying the newspaper only by selling subscriptions, avoiding thereby cannibalization by selling two substitute products. We now examine which of these possibilities is the most advantageous.

2 The optimal product mix

To this end, we must first compare the profit $R(p^*, r^*)$ realised at equilibrium when both products are sold, with the profit obtained when selling the newspaper at the newsstand. Then the optimal price is $w - c$, so that the profit is equal to $\frac{n(v^2-m^2)(w-c)}{2}$. Comparing this profit with the profit realised at equilibrium with two products, we see that the second exceeds the first, whenever

$$G(c) = \frac{n(m^2c^2 + v^2w^2 - m^2w^2)}{2(w+c)} - \frac{n(v^2-m^2)(w-c)}{2} > 0$$

We notice that $\frac{dG}{dc} > 0$ and that $G(c) = 0$, if and only if $c = 0$. Accordingly, the above inequality is always positive, and it is never more advantageous to sell the newspaper at the newsstand only, rather than selling both at the newsstand and by subscription.

On the other hand, it is proved in the appendix (see Lemma 2) that, under Assumption (1), it could never be more profitable to sell by subscription only, compared with the profit realised by selling simultaneously at the newsstand and by subscription. Accordingly, we have

Proposition 2 *Under Assumption (1), the optimal product selection for the monopolist consists in selling the newspaper both at the newsstand and by subscription. Otherwise, the monopolist offers only subscriptions.*

3 Production costs

So far, we have performed the analysis by assuming zero production costs. However, the model can easily be accommodated so as to account for linear production costs. Let d denote the unit production cost of an issue. In view of simplifying notation, we shall henceforth assume that $m = 0$. Under this assumption, the monopolist's profits $R(p, r)$ write as

$$\mathrm{R(p,r)} = \frac{1}{2}\frac{\mathrm{p}}{\mathrm{n}}\frac{\mathrm{r}^2}{(\mathrm{p}+\mathrm{c})^2} + \mathrm{r}\left[\mathrm{v} - \frac{\mathrm{r}}{[(\mathrm{p}+\mathrm{c})\mathrm{n}]}\right] - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{n}}\frac{\mathrm{r}^2}{(\mathrm{p}+\mathrm{c})^2} - \mathrm{dn}\left[\mathrm{v} - \frac{\mathrm{r}}{[(\mathrm{p}+\mathrm{c})\mathrm{n}]}\right]$$

The first-order necessary condition with respect to the variable r implies that

$$r^* = (p+c)n\frac{vp + vc + d}{p + 2c + d}$$

It is easily checked that $\frac{d^2R}{dr^2} < 0$, so that r^* is a maximizer of $R(p, r)$ for all values of p. Substituting the value of r^* into $t(p, r)$, we now obtain

$$t(p, r^*) = \frac{vp + vc + d}{p + 2c + d} \tag{7}$$

Similarly, we get

$$\begin{aligned} R(p, r^*) =& \frac{1}{2}\frac{pn(vp + vc + d)^2}{(p + 2c + d)^2} + \frac{(p + c)n(vp + vc + d)}{p + 2c + d}\left(v - \frac{vp + vc + d}{p + 2c + d}\right) \\ & - \frac{1}{2}dn\frac{(vp + vc + d)^2}{(p + 2c + d)^2} - dn\left(v - \frac{vp + vc + d}{p + 2c + d}\right) \end{aligned} \tag{8}$$

Tedious calculations (see Lemma 3 in the appendix) show that $\frac{dR(p, r^*)}{dp}$ is strictly positive so that the optimal price p^* of the issue sold at the newsstand is equal to the highest price at which consumers are still willing to buy, i.e. $w - c$. Substituting p^* into r^* and $t(p, r^*)$, we obtain

$$r^* = \frac{wr(vw + d)}{w + c + d} \tag{9}$$

and

$$t(p^*, r^*) = \frac{vw + d}{w + c + d} \tag{10}$$

When we compare the equilibrium value of r^*, without or with production costs, as provided by Proposition 1 and by (9), we obtain that the latter exceeds the former when $vw \leqslant w + c$, a condition which is always satisfied, since $v < 1$.

On the other hand, comparing $t(p^*, r^*)$ without or with production costs, as provided by (6) and (10), we conclude that the marginal consumer indifferent between buying the two options, has a lower reading frequency in the first alternative than in the second one. Accordingly, there are less consumers who subscribe when there are production costs than in the opposite case. This is a direct consequence of the increase in the subscription fee resulting from the introduction of productive costs. Notice that, when production costs are positive, the "unwanted" issues in subscriptions now cause a problem because they are expensive to produce and yet the subscriber is not interested in reading them. In this case, the subscription mechanism is a socially wasteful technology. This also explains why the editor wishes to reduce the number of subscribers : the presence of costs forces him to produce all the unwanted issues while subscribers are not willing to pay for them.

Finally it would be interesting to identify where the firm gets the best return on a cost-reducing investment : is it more profitable to improve distribution capacity and thereby reduce the moving cost c, or is it better to improve production capacity, and reduce the unit production cost d? An answer to the question would need a more elaborate model and is beyond the scope of the present paper.

Conclusion

The analysis above shows that the argument based on price discrimination is sufficient for explaining the rise of the subscription practice in the press industry. Unfortunately, this analysis has been conducted at the cost of severe assumptions. First we have assumed that the reading frequency of a consumer does not depend on the prices p and r. Most probably however, this frequency is influenced by these prices, and should be determined as the solution of an optimization problem balancing an increase in frequency against the increase in expenses resulting from this supplementary purchase of information. Furthermore, one must not expect the reading frequency to be, as implicitely assumed, independent of the events happening over the year. On the contrary, it should vary with the nature and the importance of these events. When there is a presidential election, or when the death of a celebrated actress is announced, peaks in the individual reading frequencies are observed. Our analysis neglects such random elements which influence the reading frequency of the consumers. Finally, the assumption that the reading frequency remains the same, whether the consumer buys at the newsstand or by subscription, is not satisfactory either. Many consumers change their reading habits when they take a subscription. Some of them even buy a subscription in order to constrain themselves to read everyday the newspaper, a practice they would not be able to obey, should they be deprived from receiving every day the newspaper in their mailing box... Nevertheless, the above simplifying assumptions have brought some tractability in the treatment of a problem which seemed at first sight extremely difficult to tackle. But this is clearly a first step, and the analysis should be deepened.

In our opinion, the first extension of the analysis should deal with the problem of randomly fluctuating demand. In other words the assumption according to which reading frequencies are independent from the events happening over the year, should be dropped and replaced by an assumption tolerating some random fluctuations of individual frequencies related to the various possible states of nature which can appear during the period of reference. The profitability of the subscription strategy resulting from smoothing the random fluctuations of demand, could then be identified, and contrasted with the profitability resulting from price discrimination, as analysed in the present paper.

Furhermore, it would be interesting to extend the analysis in view of taking care of the incidence of advertising on the subscription strategies of editorial firms. This approach would be more "industry-specific", but would certainly shed some light on the functioning of a fascinating industry.

Finally, we have considered that the monopolist proposes to the buyers the alternative: "subscribe for one year, or buy at the newsstand". In most cases however, the alternative of subscription is refined since consumers are allowed to subscribe on a three-months, or six-months, or one year-basis, thereby reinforcing the possibility of price discrimination by the monopolist. It should not be difficult to extend the present analysis to such more elaborate pricing schemes.

Appendix

Lemma 1 *The maximizer of $R(p,r^*)$, as defined by (4), is $w-c$ in the domain $[0,w-c]$.*

Proof. It is easy to check that $\frac{\partial^2 R(p,r^*)}{\partial p^2} > 0$, so that any interior solution to the first order condition $\frac{\partial R}{\partial p} = 0$ must be a minimum. Accordingly the optimal value of p is necessarily at one of the two frontiers of the admissible domain $[0,w-c]$. Substituting $m = ar$, we then get that the difference $R(w-c,r^*) - R(0,r^*)$ is equal to $\frac{nr^2(c-w)}{2(w+c)}(-c+2a^2c-2w+2a^2w)$. The final term of this expression is increasing in a and is negative for $a = \frac{m}{r} \leqslant \frac{w}{w+c}$, an inequality which must hold by (1). Since $c-w<0$, the above expression is positive so that $R(w-c,r^*) - R(0,r^*) > 0$, which completes the proof of Lemma 1. □

Lemma 2 *Under Assumption (1), the profit realized when selling only by subscription is smaller than the profit obtained with selling both at the newsstand and by subscription.*

Proof. We start by identifying the optimal subscription fee when the monopoly sells only by subscription. A consumer with reading frequency t will take a subscription at a price r if and only if

$$wtn \geqslant r$$

Accordingly, the set of consumers buying at price r is the interval $\left[\frac{r}{nw}, v\right]$, so that the receipt R writes as

$$\begin{aligned} R(r) &= r\int_{r/nw}^{v} dt \\ &= r\left(v - \frac{r}{nw}\right), \end{aligned}$$

when $m < \frac{r}{nw} \leqslant v$, and

$$R(r) = r \int_m^v dt$$
$$= r(v - m),$$

when $\frac{r}{nw} \leqslant m$. Consequently, at an interior solution, the optimal fee r^* must satisfy the first order condition

$$\frac{dR}{dr} = v - \frac{2r}{nw} = 0,$$

so that $r^* = \frac{vnw}{2}$ and $t(r^*) = \frac{v}{2}$. In order for $t(r^*)$ to exceed m, it must be that $v \geqslant 2m$. Accordingly, when $v \geqslant 2m$, the optimal fee r^* obtains as $r^* = \frac{vnw}{2}$ and $R(r^*) = \frac{v^2nw}{4}$. Now assume that $v < 2m$. Then the optimal price r^* is the highest price at which the market is covered, i.e., r^* solves $m = \frac{r}{nw}$, or

$$r^* = mnw,$$

with $t(r^*) = m$ and $R(r^*) = mnw(v - m)$.

In order to prove the lemma, we shall successively consider the two cases: $v \geqslant 2m$ and $v < 2m$. In the first case, the difference D between the profit when selling the two products – selling at the store and by subscription –, or selling by subscription only, writes as

$$D = \frac{n(-m^2w^2 + m^2c^2 + v^2w^2)}{2(w + c)} - \frac{nv^2w}{4}$$

It is easily checked that $D = 0 \Leftrightarrow c = c_1 = w$, or $c = c_2 = \frac{w(v^2 - 2m^2)}{2m^2}$. Since $v \geqslant 2m$, it follows that $c_2 \geqslant c_1$. On the other hand, we get

$$\frac{dD}{dc} = \frac{nm^2c}{w + c} - \frac{n(-m^2w^2 + m^2c^2 + v^2w^2)}{2(w + c)^2}$$

so that $\frac{dD}{dc} = 0 \Leftrightarrow c = c'_1 = -\frac{w}{m}(v - m)$, or $c = c'_2 = \frac{w}{m}(v - m) > w$, where the last inequality follows from $v \geqslant 2m$. Furthermore

$$\frac{d^2D}{dc^2} = \frac{nv^2w^2}{(w + c)^3}$$

Substituting c'_2 into $\frac{d^2D}{dc^2}$, we obtain

$$\left.\frac{d^2D}{dc^2}\right|_{c=c'_2} = \frac{nw^3}{vw} > 0$$

Accordingly, the difference D, as a function of c, reaches a minimum at c_2', which exceeds w; it follows that D is positive for all values of c in the domain $(0, w)$. We conclude that, when $v \geqslant 2m$, it is more profitable to sell both at the store and by subscription.

Now consider the case $v < 2m$. In this case, the difference D' between the profits writes as

$$D' = \frac{n(-m^2w^2 + m^2c^2 + v^2w^2)}{2(w+c)} - (v-m)nmw$$

It is easily checked that $D' = 0 \Leftrightarrow c = c'' = \frac{w}{m}(v-m)$. On the other hand, we get

$$\frac{dD'}{dc} = \frac{n(2m^2wc + m^2c^2 + m^2w^2 - v^2w^2)}{(w+c)^2}$$

so that $\frac{dD'}{dc} = 0 \Leftrightarrow c = c'' < w$, where the last inequality follows from $v < 2m$. Furthermore,

$$\frac{d^2D'}{dc^2} = \frac{nvw^2}{(w+c)^3}$$

Substituting c'' into $\frac{d^2D'}{dc^2}$, we obtain

$$\left.\frac{d^2D'}{dc^2}\right|_{c=c''} = \frac{nm^3}{vw} > 0$$

Notice that D' is equal to 0 for the same value of c at which $\frac{dD'}{dc}$ vanishes, namely $c = c''$. Furthermore, $\frac{d^2D'}{dc^2}\big|_{c=c''}$ is positive, so that D' reaches a minimum value at c'', which is smaller than w. It follows from the above that D is positive and decreasing in the interval $(0, c'')$ while it is positive and increasing in the interval (c'', w). In any case, D' is positive for all values of c in the domain $(0, w)$, which completes the proof of the lemma. □

Lemma 3 *The maximizer of $R(p, r^*)$ as defined by (8) is $w - c$.*

Proof. The first derivative of $\frac{1}{n}R(p, r^*)$ w.r.t. p obtains as:

$$\begin{aligned}\frac{1}{n}\frac{dR(p,r^*)}{dp} &= \frac{1}{2}t^2 + \frac{pvt}{p+2c+d} - \frac{pt^2}{p+2c+d} + t(v-t) \\ &+ (p+c)(v-t)\frac{v}{p+2c+d} - (v-t)(p+c)\frac{t}{p+2c+d} - \\ &- \frac{t(p+c)(v-t)}{p+2c+d} - \frac{dtv}{p+2c+d} + d\frac{t^2}{p+2c+d} \\ &+ (v-t)\frac{d}{p+2c+d}\end{aligned}$$

with $t = \frac{vp+vc+d}{p+2c+d}$ (see (10)).
The above expression can be rewritten as

$$\begin{aligned}\frac{1}{n}\frac{dR(p,r^*)}{dp} &= \frac{1}{2}t^2 + \frac{pt(v-t)}{p+2c+d} + t(v-t) + \frac{(p+c)(v-t)^2}{p+2c+d} \\ &\quad - \frac{t(p+c)(v-t)}{p+2c+d} - \frac{dt(v-t)}{p+2c+d} + \frac{d(v-t)}{p+2c+d} \\ &= \frac{1}{2}t^2 + \frac{pt(v-t)}{p+2c+d} + \frac{(p+c)(v-t)^2}{p+2c+d} \\ &\quad + \frac{d(v-t)}{p+2c+d} + t(v-t)\left[1 - \frac{p+c}{p+2c+d} - \frac{d}{p+2c+d}\right]\end{aligned}$$

or

$$\begin{aligned}\frac{1}{n}\frac{dR(p,r^*)}{dp} &= \frac{1}{2}t^2 + \frac{pt(v-t)}{p+2c+d} + \frac{(p+c)(v-t)^2}{p+2c+d} \\ &\quad + \frac{d(v-t)}{p+2c+d} + t(v-t)\left(\frac{c}{p+2c+d}\right) \\ &= \frac{1}{2}t^2 + \frac{v-t}{p+2c+d}[pt + (p+c)(v-t) + d + tc] \\ &= \frac{1}{2}t^2 + (v-t)\left[\frac{vp+vc+d}{p+2c+d}\right] \\ &= t(v - \frac{1}{2}t) > 0,\end{aligned}$$

where the last inequality follows from $t = t(p, r^*) < v$. The proof of the lemma follows. □

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